Electronic Journal of Differential Equations, Vol. 2008(2008), No. 14, pp. 1–18.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

HOMOGENIZED MODELS FOR A SHORT-TIME FILTRATION IN ELASTIC POROUS MEDIA

ANVARBEK M. MEIRMANOV

Abstract. We consider a linear system of differential equations describing a joint motion of elastic porous body and fluid occupying porous space. The rigorous justification, under various conditions imposed on physical parameters, is fulfilled for homogenization procedures as the dimensionless size of the pores tends to zero, while the porous body is geometrically periodic and a characteristic time of processes is small enough. Such kind of models may describe, for example, hydraulic fracturing or acoustic or seismic waves propagation. As the results, we derive homogenized equations involving non-isotropic Stokes system for fluid velocity coupled with two different types of acoustic equations for the solid component, depending on ratios between physical parameters, or non-isotropic Stokes system for one-velocity continuum. The proofs are based on Nguetseng's two-scale convergence method of homogenization in periodic structures.

1. Introduction

In the present paper we consider a problem of joint motion of a deformable solid (elastic skeleton), perforated by system of pores (pore space) and a fluid, occupying pore space. In dimensionless variables (without primes)

$$\mathbf{x}' = L\mathbf{x}, \quad t' = \tau t, \quad \mathbf{w}' = \frac{L^2}{g\tau^2}\mathbf{w}, \quad \rho_s' = \rho_0\rho_s, \quad \rho_f' = \rho_0\rho_f, \quad \mathbf{F}' = g\mathbf{F},$$

differential equations of the problem in a domain $\Omega \in \mathbb{R}^3$ for the dimensionless displacement vector $\mathbf{w}$ of the continuum medium have the form

$$\bar{\rho}\frac{\partial^2 \mathbf{w}}{\partial t^2} = \operatorname{div}\mathbb{P} + \bar{\rho}\mathbf{F}, \tag{1.1}$$

$$\mathbb{P} = \bar{\chi}\mathbb{P}^f + (1-\bar{\chi})\mathbb{P}^s, \tag{1.2}$$

$$\mathbb{P}^f = \alpha_\mu D\big(x, \frac{\partial \mathbf{w}}{\partial t}\big) - p_f\mathbb{I}, \tag{1.3}$$

$$\mathbb{P}^s = \alpha_\lambda \mathbb{D}(x, \mathbf{w}) + \alpha_\eta(\operatorname{div}\mathbf{w})\mathbb{I}, \tag{1.4}$$

$$p_f + \bar{\chi}\alpha_p \operatorname{div}\mathbf{w} = 0. \tag{1.5}$$

2000 *Mathematics Subject Classification.* 35M20, 74F10, 76S05.
Key words and phrases. Stokes equations; Lamé's equations; hydraulic fracturing; two-scale convergence; homogenization of periodic structures.

Submitted August 27, 2007. Published January 31, 2008.

Hereafter we use notation

$$\mathbb{D}(x,\mathbf{u}) = (1/2)\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right), \quad \bar{\rho} = \bar{\chi}\rho_f + (1-\bar{\chi})\rho_s,$$

The vector $\mathbb{I}$ is a unit tensor, the given function $\bar{\chi}(\mathbf{x})$ is a characteristic function of the pore space, the given function $\mathbf{F}(\mathbf{x},t)$ is a dimensionless vector of distributed mass forces, $\mathbb{P}^f$ is a liquid stress tensor, $\mathbb{P}^s$ is a stress tensor in a solid skeleton and p_f is a liquid pressure.

These differential equations (1.1)–(1.5) mean that the the displacement vector $\mathbf{w}$ satisfies the Stokes equations in the pore space Ω_f and the Lame's equations in the solid skeleton Ω_s.

On the common boundary Γ "solid skeleton-pore space" the displacement vector $\mathbf{w}$ and the liquid pressure p_f satisfy the usual continuity condition

$$[\mathbf{w}](\mathbf{x}_0,t) = 0, \quad \mathbf{x}_0 \in \Gamma,\ t \geq 0 \tag{1.6}$$

and the momentum conservation law in the form

$$[\mathbb{P}\cdot\mathbf{n}](\mathbf{x}_0,t) = 0, \quad \mathbf{x}_0 \in \Gamma,\ t \geq 0, \tag{1.7}$$

where $\mathbf{n}(\mathbf{x}_0)$ is a unit normal to the boundary at the point $\mathbf{x}_0 \in \Gamma$ and

$$\begin{gathered}
[\varphi](\mathbf{x}_0,t) = \varphi_{(s)}(\mathbf{x}_0,t) - \varphi_{(f)}(\mathbf{x}_0,t), \\
\varphi_{(s)}(\mathbf{x}_0,t) = \lim_{\mathbf{x}\to\mathbf{x}_0,\ \mathbf{x}\in\Omega_s} \varphi(\mathbf{x},t), \\
\varphi_{(f)}(\mathbf{x}_0,t) = \lim_{\mathbf{x}\to\mathbf{x}_0,\ \mathbf{x}\in\Omega_f} \varphi(\mathbf{x},t).
\end{gathered}$$

The problem is endowed with homogeneous initial and boundary conditions

$$\mathbf{w}(\mathbf{x},0) = 0, \quad \frac{\partial\mathbf{w}}{\partial t}(\mathbf{x},0) = 0, \quad \mathbf{x} \in \Omega, \tag{1.8}$$

$$\mathbf{w}(\mathbf{x},t) = 0, \quad \mathbf{x} \in S = \partial\Omega, \quad t \geq 0. \tag{1.9}$$

The dimensionless constants α_i $(i = \tau, \nu, \dots)$ are defined by the formulas

$$\alpha_\mu = \frac{2\mu\tau}{L^2\rho_0}, \quad \alpha_\lambda = \frac{2\lambda\tau^2}{L^2\rho_0}, \quad \alpha_p = \rho_f c^2 \frac{\tau^2}{L^2}, \quad \alpha_\eta = \frac{\eta\tau^2}{L^2\rho_0},$$

where μ is the viscosity of fluid, λ and η are elastic Lamé's constants, c is a speed of sound in fluid, L is a characteristic size of the domain in consideration, τ is a characteristic time of the process, ρ_f and ρ_s are respectively mean dimensionless densities of liquid and rigid phases, correlated with mean density of water and g is the value of acceleration of gravity.

The corresponding mathematical model, describing by the system (1.1)–(1.9) is commonly accepted (see [2, 10]) and contains a natural small parameter ε, which is a characteristic size of pores l divided by the characteristic size L of the entire porous body:

$$\varepsilon = \frac{l}{L}.$$

Our aim is to derive all possible limiting regimes (homogenized equations) as $\varepsilon \searrow 0$. Such an approximation significantly simplifies the original problem and at the same time preserves all of its main features. But even this approach is too hard to work out, and some additional simplifying assumptions are necessary. In terms of geometrical properties of the medium, the most appropriate is to simplify the problem postulating that the porous structure is periodic.

We accept the following constraints

Assumption 1.1. The domain $\Omega = (0,1)^3$ is a periodic repetition of an elementary cell $Y^\varepsilon = \varepsilon Y$, where $Y = (0,1)^3$ and quantity $1/\varepsilon$ is integer, so that Ω always contains an integer number of elementary cells Y^ε. Let Y_s be a "solid part" of Y, and the "liquid part" Y_f – is its open complement. We denote as $\gamma = \partial Y_f \cap \partial Y_s$ and γ is a Lipschitz continuous surface. A pore space Ω_f^ε is the periodic repetition of the elementary cell εY_f, and a solid skeleton Ω_s^ε is the periodic repetition of the elementary cell εY_s. A Lipschitz continuous boundary $\Gamma^\varepsilon = \partial\Omega_s^\varepsilon \cap \partial\Omega_f^\varepsilon$ is the periodic repetition in Ω of the boundary $\varepsilon\gamma$. The "solid skeleton" Ω_s^ε and the "pore space" Ω_f^ε are connected domains and an intersection Ω_f^ε with any plane $\{x_i = \text{constant},\ 0 < x_i < 1,\ i = 1,2,3\}$ is an open (in plane topology) set.

In these assumptions

$$\bar{\chi}(\mathbf{x}) = \chi^\varepsilon(\mathbf{x}) = \chi(\mathbf{x}/\varepsilon),$$
$$\bar{\rho} = \rho^\varepsilon(\mathbf{x}) = \chi^\varepsilon(\mathbf{x})\rho_f + (1 - \chi^\varepsilon(\mathbf{x}))\rho_s,$$

where $\chi(\mathbf{y})$ is a characteristic function of Y_f in Y.

Suppose that all dimensionless parameters depend on the small parameter ε and there exist limits (finite or infinite)

$$\lim_{\varepsilon\searrow 0}\alpha_\mu(\varepsilon) = \mu_0, \quad \lim_{\varepsilon\searrow 0}\alpha_\lambda(\varepsilon) = \lambda_0,$$
$$\lim_{\varepsilon\searrow 0}\alpha_\eta(\varepsilon) = \eta_0, \quad \lim_{\varepsilon\searrow 0}\alpha_p(\varepsilon) = p_*,$$
$$\lim_{\varepsilon\searrow 0}\frac{\alpha_\mu}{\varepsilon^2} = \mu_1, \quad \lim_{\varepsilon\searrow 0}\frac{\alpha_\lambda}{\varepsilon^2} = \lambda_1.$$

The first research with the aim of finding limiting regimes in the case when the skeleton was assumed to be an absolutely rigid body was carried out by Sanchez-Palencia and Tartar. Sanchez-Palencia [10, Sec. 7.2] formally obtained Darcy's law of filtration using the method of two-scale asymptotic expansions, and Tartar [10, Appendix] rigorously justified the homogenization procedure. Using the same method of two-scale expansions Keller and Burridge [2] derived formally the system of Biot's equations from the problem (1.1)–(1.9) in the case when the parameter α_μ was of order ε^2, and the rest of the coefficients were fixed independent of ε. Under the same assumptions as in the article [2], the rigorous justification of Biot's model was given by Nguetseng [9] and later by Clopeaut *et al.* [3]. The most general case of the problem (1.1)–(1.9) when

$$\mu_0,\ \lambda_0^{-1},\ p_*^{-1},\ \eta_0^{-1} < \infty$$

has been studied in [7]. All these authors have used Nguetseng's two-scale convergence method [8, 6].

In the present work by means of the same method we investigate all possible limiting regimes in the problem (1.1)–(1.9) in the cases, when

$$0 < \mu_0 < \infty; \quad \lambda_0 = 0; \quad 0 < p_*,\ \eta_0. \tag{1.10}$$

These cases correspond, for example, to the hydraulic fracturing, when all processes end during the seconds ($\tau \searrow 0$).

We show that for the case $\lambda_1 < \infty$ the homogenized equations describe two velocity continuum and consist of non-isotropic Stokes equations for fluid velocity coupled with acoustic equations for the solid component and for the case $\lambda_1 = \infty$ the homogenized equations describe one-velocity continuum and consist of non-isotropic Stokes system for the limiting displacements of the continuum.

This property of the mathematical model, which initially describes one-velocity continuum and becomes a model describing two-velocity continuum after homogenization procedure, appears as a result of different smoothness of the solution in the solid and in the liquid components:

$$\int_\Omega \alpha_\mu(\varepsilon)\chi^\varepsilon|\nabla\mathbf{w}^\varepsilon|^2dx \le C_0, \quad \int_\Omega \alpha_\lambda(\varepsilon)(1-\chi^\varepsilon)|\nabla\mathbf{w}^\varepsilon|^2)dx \le C_0,$$

where C_0 is a constant independent of the small parameter ε. To preserve the best properties of the solution we must use the well-known extension lemma [1, 4] and extend the solution from the solid part to the liquid one and vice-versa. On this stage criterion λ_1 becomes crucial. Namely, let $\mathbf{w}^\varepsilon_s$ be an extension of the solid displacements to the liquid part and $\lambda_1 = \infty$. Then the limiting (homogenized) system describes one-velocity continuum. It takes place because each of sequences $\{\mathbf{w}^\varepsilon\}$ and $\{\mathbf{w}^\varepsilon_s\}$ two –scale converges to the function independent of the fast variable. This statement easily follows from Nguetseng's theorem.

2. Formulation of the main results

There are various equivalent in the sense of distributions forms of representation of equation (1.1) in each domain Ω^ε_f and Ω^ε_s and boundary conditions (1.6)–(1.7) on the common boundary Γ^ε between pore space Ω^ε_f and solid skeleton Ω^ε_s. In what follows, it is convenient to write them in the form of the integral equalities.

We say that functions $\big(\mathbf{w}^\varepsilon,\, p^\varepsilon_f,\, p^\varepsilon_s\big)$ are called a generalized solution of the problem (1.1)–(1.9), if they satisfy the regularity conditions

$$\mathbf{w}^\varepsilon,\ \mathbb{D}(x,\mathbf{w}^\varepsilon),\ p^\varepsilon_f,\ p^\varepsilon_s \in L^2(\Omega_T) \tag{2.1}$$

in the domain $\Omega_T = \Omega\times(0,T)$, boundary condition (1.9) on the outer boundary S in the trace sense, equations

$$\frac{1}{\alpha_p}p^\varepsilon_f = -\chi^\varepsilon\big(\operatorname{div}\mathbf{w}^\varepsilon - \frac{\beta^\varepsilon}{m}\big), \tag{2.2}$$

$$\frac{1}{\alpha_\eta}p^\varepsilon_s = -(1-\chi^\varepsilon)\big(\operatorname{div}\mathbf{w}^\varepsilon + \frac{\beta^\varepsilon}{1-m}\big) \tag{2.3}$$

a.e. in Ω_T and, finally, integral identity

$$\begin{aligned}\int_{\Omega_T}\Big(\rho^\varepsilon\mathbf{w}^\varepsilon\cdot\frac{\partial^2\varphi}{\partial t^2} - \chi^\varepsilon\alpha_\mu D(\mathbf{x},\mathbf{w}^\varepsilon):\mathbb{D}(x,\frac{\partial\varphi}{\partial t}) - \rho^\varepsilon\mathbf{F}\cdot\varphi\\ +\{(1-\chi^\varepsilon)\alpha_\lambda\mathbb{D}(x,\mathbf{w}^\varepsilon) - (p^\varepsilon_f+p^\varepsilon_s)\mathbb{I}\}:\mathbb{D}(x,\varphi)\Big)\,dx\,dt = 0\end{aligned} \tag{2.4}$$

for all smooth vector-functions $\varphi = \varphi(\mathbf{x},t)$ such that

$$\varphi(\mathbf{x},t) = 0,\ \mathbf{x}\in S,\ t>0; \quad \varphi(\mathbf{x},T) = \frac{\partial\varphi}{\partial t}(\mathbf{x},T) = 0,\ \mathbf{x}\in\Omega.$$

In this definition we changed the form of representation of the stress tensor $\mathbb{P}$ in the integral identity (2.4) by introducing new unknown function p^ε_s, which in a certain way has a sense of pressure. In what follows we call this function p^ε_s as a solid pressure and equations (2.2) and (2.3)– as continuity equations. We also introduced functionals

$$\beta^\varepsilon = \int_\Omega \chi^\varepsilon\operatorname{div}\mathbf{w}^\varepsilon dx \text{ if } p_*+\eta_0 = \infty \quad\text{and}\quad \beta^\varepsilon = 0 \text{ if } p_*+\eta_0 < \infty,$$

which have been chosen from the conditions

$$\int_\Omega p_f^\varepsilon dx = \int_\Omega p_s^\varepsilon dx = 0, \tag{2.5}$$

if $p_* + \eta_0 = \infty$. This special choice of continuity equations permits to estimate pressures, even if $p_* = \infty$ (incompressible liquid) or $\eta_0 = \infty$ (incompressible solid) and simplifies the use of homogenization procedure.

In (2.3) by $A : B$ we denote the convolution (or, equivalently, the inner tensor product) of two second-rank tensors along the both indexes, i.e., $A : B = \mathrm{tr}\,(B^* \circ A) = \sum_{i,j=1}^3 A_{ij}B_{ji}$.

The following two theorems are the main results of the paper.

Theorem 2.1. *Let* $\mathbf{F}$ *and* $\partial\mathbf{F}/\partial t$ *are bounded in* $L^2(\Omega_T)$. *Then for all* $\varepsilon > 0$ *on the arbitrary time interval* $[0,T]$ *there exists a unique generalized solution of the problem* (1.1)–(1.9) *and*

$$\max_{0\le t\le T} \|\frac{\partial^2 \mathbf{w}^\varepsilon}{\partial t^2}(t)\|_{2,\Omega} + \|\chi^\varepsilon \sqrt{\alpha_\mu}\nabla \frac{\partial^2 \mathbf{w}^\varepsilon}{\partial t^2}|\|_{2,\Omega_T} \le C_0, \tag{2.6}$$

$$\max_{0\le t\le T} \|\chi^\varepsilon \sqrt{\alpha_\mu}|\nabla_x \frac{\partial \mathbf{w}^\varepsilon}{\partial t}(t)| + (1-\chi^\varepsilon)\sqrt{\alpha_\lambda}|\nabla_x \frac{\partial \mathbf{w}^\varepsilon}{\partial t}(t)|\|_{2,\Omega} \le C_0, \tag{2.7}$$

$$\max_{0\le t\le T} \||p_f^\varepsilon(t)| + |p_s^\varepsilon(t)|\|_{2,\Omega} \le C_0, \tag{2.8}$$

where C_0 *does not depend on the small parameter* ε.

Theorem 2.2. *Assume that the hypotheses in theorem 2.1 and restrictions* (1.10) *hold. Then functions* $\partial\mathbf{w}^\varepsilon/\partial t$ *admit an extension* $\mathbf{v}^\varepsilon$ *from* $\Omega_f^\varepsilon \times (0,T)$ *into* Ω_T *such that sequence* $\{\mathbf{v}^\varepsilon\}$ *converges strongly in* $L^2(\Omega_T)$ *and weakly in* $L^2((0,T); W_2^1(\Omega))$ *to the function* $\mathbf{v}$. *At the same time, sequences* $\{\mathbf{w}^\varepsilon\}$, $\{(1-\chi^\varepsilon)\mathbf{w}^\varepsilon\}$, $\{p_f^\varepsilon\}$ *and* $\{p_s^\varepsilon\}$ *converge weakly in* $L^2(\Omega_T)$ *to* $\mathbf{w}$, $\mathbf{u}_s$, p_f *and* p_s, *respectively.*

(I) If $\lambda_1 = \infty$, *then* $\partial\mathbf{u}_s/\partial t = (1-m)\mathbf{v} = (1-m)\partial\mathbf{w}/\partial t$ *and weak and strong limits* p_f, p_s *and* $\mathbf{v}$ *satisfy in* Ω_T *the initial-boundary value problem*

$$\begin{aligned} &\hat{\rho}\frac{\partial \mathbf{v}}{\partial t} + \nabla(p_f + p_s) - \hat{\rho}\mathbf{F} \\ &= \mathrm{div}\{\mu_0 \mathbb{A}_0^f : \mathbb{D}(x,\mathbf{v}) + \mathbb{B}_0^f p_s + \mathbb{B}_1^f \,\mathrm{div}\,\mathbf{v} + \int_0^t \mathbb{B}_2^f(t-\tau)\,\mathrm{div}\,\mathbf{v}(\mathbf{x},\tau)d\tau\}, \end{aligned} \tag{2.9}$$

$$\begin{aligned} &p_*^{-1}\partial p_f/\partial t + \mathbb{C}_0^f : \mathbb{D}(x,\mathbf{v}) + a_0^f p_s \\ &+ (a_1^f + m)\,\mathrm{div}\,\mathbf{v} + \int_0^t a_2^f(t-\tau)\,\mathrm{div}\,\mathbf{v}(\mathbf{x},\tau)d\tau = 0, \end{aligned} \tag{2.10}$$

$$\frac{1}{p_*}\frac{\partial p_f}{\partial t} + \frac{1}{\eta_0}\frac{\partial p_s}{\partial t} + \mathrm{div}\,\mathbf{v} = 0, \tag{2.11}$$

where $\hat{\rho} = m\rho_f + (1-m)\rho_s$ *is the average density of the mixture,* $m = \int_Y \chi dy$ *is a porosity and the symmetric strictly positively defined constant fourth-rank tensor* $\mathbb{A}_0^f$, *matrices* $\mathbb{C}_0^f, \mathbb{B}_0^f$, $\mathbb{B}_1^f$ *and* $\mathbb{B}_2^f(t)$ *and scalars* a_0^f, a_1^f *and* $a_2^f(t)$ *are defined below by formulas* (1.10), (5.33) *and* (5.35), *where* $\mathbb{B}_1^f = 0$, $a_1^f = 0$ *if* $p_* < \infty$, *and* $\mathbb{B}_2^f = 0$, $a_2^f = 0$ *if* $p_* = \infty$.

Differential equations (2.9) *are endowed with homogeneous initial and boundary conditions*

$$\mathbf{v}(\mathbf{x},0) = 0, \quad \mathbf{x} \in \Omega; \quad \mathbf{v}(\mathbf{x},t) = 0, \quad \mathbf{x} \in S, \quad t > 0. \tag{2.12}$$

(II) If $\lambda_1 < \infty$, then weak and strong limits $\mathbf{u}_s$, p_f, p_s and $\mathbf{v}$ satisfy in Ω_T the initial-boundary value problem, which consists of Stokes like system

$$\begin{aligned} &\rho_f m \frac{\partial \mathbf{v}}{\partial t} + \rho_s \frac{\partial^2 \mathbf{u}_s}{\partial t^2} + \nabla(p_f + p_s) - \hat{\rho}\mathbf{F} \\ &= \operatorname{div}\{\mathbb{B}_0^f p_s + \mu_0 \mathbb{A}_0^f : \mathbb{D}(x, \mathbf{v}) + \mathbb{B}_1^f \operatorname{div} \mathbf{v} + \int_0^t \mathbb{B}_2^f(t-\tau) \operatorname{div} \mathbf{v}(\mathbf{x}, \tau) d\tau\}, \end{aligned} \tag{2.13}$$

$$\begin{aligned} &p_*^{-1} \partial p_f / \partial t + \mathbb{C}_0^f : \mathbb{D}(x, \mathbf{v}) + a_0^f p_s \\ &+ (a_1^f + m) \operatorname{div} \mathbf{v} + \int_0^t a_2^f(t-\tau) \operatorname{div} \mathbf{v}(\mathbf{x}, \tau) d\tau = 0, \end{aligned} \tag{2.14}$$

for the liquid component, coupled with the continuity equation

$$\frac{1}{p_*} \frac{\partial p_f}{\partial t} + \frac{1}{\eta_0} \frac{\partial p_s}{\partial t} + \operatorname{div} \frac{\partial \mathbf{u}_s}{\partial t} + m \operatorname{div} \mathbf{v} = 0, \tag{2.15}$$

the relation

$$\begin{aligned} \frac{\partial \mathbf{u}_s}{\partial t} &= (1-m)\mathbf{v}(\mathbf{x}, t) + \int_0^t \mathbb{B}_1^s(t-\tau) \cdot \mathbf{z}(\mathbf{x}, \tau) d\tau, \\ \mathbf{z}(\mathbf{x}, t) &= -\frac{1}{1-m} \nabla p_s(\mathbf{x}, t) + \rho_s \mathbf{F}(\mathbf{x}, t) - \rho_s \frac{\partial \mathbf{v}}{\partial t}(\mathbf{x}, t) \end{aligned} \tag{2.16}$$

in the case of $\lambda_1 > 0$, or the balance of momentum equation in the form

$$\rho_s \frac{\partial^2 \mathbf{u}_s}{\partial t^2} = \rho_s \mathbb{B}_2^s \cdot \frac{\partial \mathbf{v}}{\partial t} + ((1-m)I - \mathbb{B}_2^s) \cdot (-\frac{1}{1-m} \nabla p_s + \rho_s \mathbf{F}) \tag{2.17}$$

in the case of $\lambda_1 = 0$ for the solid component. The problem is supplemented by boundary and initial conditions (2.12) for the velocity $\mathbf{v}$ of the liquid component and by the homogeneous initial conditions

$$\mathbf{u}_s(\mathbf{x}, 0) = \frac{\partial \mathbf{u}_s}{\partial t}(\mathbf{x}, 0) = 0, \quad (\mathbf{x}, t) \in \Omega \tag{2.18}$$

and homogeneous boundary condition

$$\mathbf{u}_s(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}) = 0, \quad (\mathbf{x}, t) \in S, \quad t > 0, \tag{2.19}$$

for the displacements $\mathbf{u}_s$ of the solid component. In (2.16)–(2.19) $\mathbf{n}(\mathbf{x})$ *is the unit normal vector to S at a point $\mathbf{x} \in S$, and matrices $\mathbb{B}_1^s(t)$ and $\mathbb{B}_2^s$ are given below by* (5.40) *and* (5.42)*, where the matrix $((1-m)\mathbb{I} - \mathbb{B}_2^s)$ is symmetric and strictly positively definite.*

3. Preliminaries

3.1. **Nguetseng's theorem.** Justification of theorem 2.2 relies on systematic use of the method of two-scale convergence, which had been proposed by G. Nguetseng [8] and has been applied recently to a wide range of homogenization problems (see, for example, the survey [6]).

Definition 3.1. A sequence $\{w^\varepsilon\} \subset L^2(\Omega_T)$ is said to be *two-scale convergent* to a 1- periodic in $\mathbf{y}$ function $W(\mathbf{x}, \mathbf{y}, t) \in L^2(\Omega_T \times Y)$, if and only if for any 1-periodic in $\mathbf{y}$ function $\sigma = \sigma(\mathbf{x}, t, \mathbf{y})$

$$\int_{\Omega_T} w^\varepsilon(\mathbf{x}, t) \sigma\big(\mathbf{x}, t, \frac{\mathbf{x}}{\varepsilon}\big)\, dx\, dt \to \int_{\Omega_T} \int_Y W(\mathbf{x}, t, \mathbf{y}) \sigma(\mathbf{x}, t, \mathbf{y}) dy\, dx\, dt \tag{3.1}$$

as $\varepsilon \to 0$.

Existence and main properties of weakly convergent sequences are established by the following fundamental theorem [8, 6].

Theorem 3.2 (Nguetseng's theorem). (1) *Any bounded in $L^2(\Omega_T)$ sequence contains a subsequence, two-scale convergent to some limit $W \in L^2(\Omega_T \times Y)$.*

(2) *Let sequences $\{w^\varepsilon\}$ and $\{\varepsilon\nabla_x w^\varepsilon\}$ be uniformly bounded in $L^2(\Omega_T)$. Then there exist a 1-periodic in $\mathbf{y}$ function $W = W(\mathbf{x}, t, \mathbf{y})$ and a subsequence $\{w^\varepsilon\}$ such that W, $\nabla_y W \in L^2(\Omega_T \times Y)$, and the subsequences $\{w^\varepsilon\}$ and $\{\varepsilon\nabla w^\varepsilon\}$ two-scale converge to W and $\nabla_y W$, respectively.*

(3) *Let sequences $\{w^\varepsilon\}$ and $\{\nabla w^\varepsilon\}$ be bounded in $L^2(Q)$. Then there exist functions $w \in L^2(\Omega_T)$ and $W \in L^2(\Omega_T \times Y)$ and a subsequence from $\{\nabla w^\varepsilon\}$ such that the function W is 1-periodic in $\mathbf{y}$, $\nabla w \in L^2(\Omega_T)$, $\nabla_y W \in L^2(\Omega_T \times Y)$, and the subsequence $\{\nabla w^\varepsilon\}$ two-scale converge to the function $(\nabla w(\mathbf{x}, t) + \nabla_y W(\mathbf{x}, t, \mathbf{y}))$.*

Corollary 3.3. *Let $\sigma \in L^2(Y)$ and $\sigma^\varepsilon(\mathbf{x}) = \sigma(\mathbf{x}/\varepsilon)$. Assume that a sequence $\{w^\varepsilon\} \subset L^2(\Omega_T)$ two-scale converges to $W \in L^2(\Omega_T \times Y)$. Then the sequence $\{\sigma^\varepsilon w^\varepsilon\}$ two-scale converges to the function σW.*

3.2. **An extension lemma.** The typical difficulty in homogenization problems, like problem (1.1)–(1.9), while passing to a limit as $\varepsilon \searrow 0$ arises because of the fact that the bounds on the gradient of displacement $\nabla_x \mathbf{w}^\varepsilon$ may be distinct in liquid and rigid components. The classical approach in overcoming this difficulty consists of constructing of extension to the whole Ω of the displacement field defined merely on Ω_s or Ω_f. The following lemma is valid due to the well-known results from [1, 4]. We formulate it in appropriate for us form:

Lemma 3.4. *Suppose that assumption 1.1 on geometry of periodic structure holds, $w^\varepsilon \in W^1_2(\Omega^\varepsilon_f)$ and $w^\varepsilon = 0$ on $S^\varepsilon_f = \partial\Omega^\varepsilon_f \cap \partial\Omega$ in the trace sense. Then there exists a function $w^\varepsilon_f \in W^1_2(\Omega)$ such that its restriction on the sub-domain Ω^ε_f coincide with w^ε, i.e.,*

$$\chi^\varepsilon(\mathbf{x})(w^\varepsilon_f(\mathbf{x}) - w^\varepsilon(\mathbf{x})) = 0, \quad \mathbf{x} \in \Omega, \tag{3.2}$$

and, moreover, the estimate

$$\|w^\varepsilon_f\|_{2,\Omega} \le C\|w^\varepsilon\|_{2,\Omega^\varepsilon_f}, \quad \|\nabla w^\varepsilon_f\|_{2,\Omega} \le C\|\nabla w^\varepsilon\|_{2,\Omega^\varepsilon_f} \tag{3.3}$$

hold true, where the constant C depends only on geometry Y and does not depend on ε.

3.3. **Friedrichs–Poincaré's inequality in periodic structure.** The following lemma was proved by Tartar in [10, Appendix]. It specifies Friedrichs–Poincaré's inequality for ε-periodic structure. We formulate this lemma for our particular case just to estimate functions in the ε–layer Q^ε of the boundary S. This domain Q^ε consists of all elementary cells εY touching the boundary $\partial\Omega$. We consider special class of functions w^ε_f, which are extensions of functions $w^\varepsilon \in W^1_2(\Omega^\varepsilon_f)$, vanishing on the part $S^\varepsilon_f = \partial\Omega^\varepsilon_f \cap \partial\Omega$ of the boundary $S = \partial\Omega$, from subdomain Ω^ε_f onto whole domain Ω (see lemma 3.4). Due to supposition on the structure of the pore space, the intersection of the boundary of the "liquid part" Y_f with each sides of the boundary ∂Y is a set with nonempty interior and strictly positive measure. Therefore on the each side of the boundary S the function w^ε_f is equal to zero on some set with nonempty interior, periodic structure and strictly positive measure, independent of ε.

Lemma 3.5. *Suppose that assumptions on the geometry of Ω^{ε}_f hold true. Then for any function $w^{\varepsilon}_f \in W^1_2(\Omega)$ such that $w^{\varepsilon}_f = 0$ on the part $S^{\varepsilon}_f = \partial\Omega^{\varepsilon}_f \cap \partial\Omega$ of the boundary S, the inequality*

$$\int_{Q^{\varepsilon}} |w^{\varepsilon}_f|^2 dx \leq C\varepsilon^2 \int_{Q^{\varepsilon}} |\nabla w^{\varepsilon}_f|^2 dx \tag{3.4}$$

holds true with some constant C independent of the small parameter ε.

3.4. **Some notation.** Further we denote (1)

$$\langle\Phi\rangle_Y = \int_Y \Phi dy, \quad \langle\Phi\rangle_{Y_f} = \int_Y \chi\Phi dy, \quad \langle\Phi\rangle_{Y_s} = \int_Y (1-\chi)\Phi dy,$$
$$\langle\varphi\rangle_{\Omega} = \int_{\Omega} \varphi dx, \quad \langle\varphi\rangle_{\Omega_T} = \int_{\Omega_T} \varphi \, dx \, dt.$$

(2) If $\mathbf{a}$ and $\mathbf{b}$ are two vectors then the matrix $\mathbf{a} \otimes \mathbf{b}$ is defined by the formula

$$(\mathbf{a} \otimes \mathbf{b}) \cdot \mathbf{c} = \mathbf{a}(\mathbf{b} \cdot \mathbf{c})$$

for any vector $\mathbf{c}$.

(3) If $\mathbb{B}$ and $\mathbb{C}$ are two matrices, then $\mathbb{B} \otimes \mathbb{C}$ is a forth-rank tensor such that its convolution with any matrix $\mathbb{A}$ is defined by the formula

$$(\mathbb{B} \otimes \mathbb{C}) : \mathbb{A} = \mathbb{B}(\mathbb{C} : \mathbb{A}).$$

(4) By $\mathbb{I}^{ij} = \mathbf{e}_i \otimes \mathbf{e}_j$ we denote the 3×3-matrix with just one non-vanishing entry, which is equal to one and stands in the i-th row and the j-th column.

(5) We also introduce

$$\mathbb{J}^{ij} = \frac{1}{2}(\mathbb{I}^{ij} + \mathbb{I}^{ji}) = \frac{1}{2}(\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i), \quad \mathbb{J} = \sum_{i,j=1}^{3} \mathbb{J}^{ij} \otimes \mathbb{J}^{ij},$$

where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ are the standard Cartesian basis vectors.

4. Proof of theorem 2.2

Estimates (2.6)-(2.7) follow from the energy equality in the form

$$\begin{aligned}
&\frac{d}{dt}\{\int_{\Omega} \rho^{\varepsilon}(\frac{\partial^2 \mathbf{w}^{\varepsilon}}{\partial t^2})^2 dx + \alpha_{\lambda}\int_{\Omega}(1-\chi^{\varepsilon})\mathbb{D}(x, \frac{\partial \mathbf{w}^{\varepsilon}}{\partial t}) : \mathbb{D}(x, \frac{\partial \mathbf{w}^{\varepsilon}}{\partial t})dx \\
&+ \alpha_p \int_{\Omega} \chi^{\varepsilon}(\operatorname{div} \frac{\partial \mathbf{w}^{\varepsilon}}{\partial t})^2 dx + \alpha_{\eta}\int_{\Omega}(1-\chi^{\varepsilon})(\operatorname{div} \frac{\partial \mathbf{w}^{\varepsilon}}{\partial t})^2 dx\} \\
&+ \alpha_{\mu}\int_{\Omega} \chi^{\varepsilon}\mathbb{D}(x, \frac{\partial^2 \mathbf{w}^{\varepsilon}}{\partial t^2}) : \mathbb{D}(x, \frac{\partial^2 \mathbf{w}^{\varepsilon}}{\partial t^2})dx \\
&= \int_{\Omega} \frac{\partial \mathbf{F}}{\partial t} \cdot \frac{\partial^2 \mathbf{w}^{\varepsilon}}{\partial t^2} dx \\
&\quad + \frac{\partial \beta^{\varepsilon}}{\partial t}\big(\frac{\alpha_p}{m}\int_{\Omega} \chi^{\varepsilon} \operatorname{div} \frac{\partial^2 \mathbf{w}^{\varepsilon}}{\partial t^2} dx + \frac{\alpha_{\eta}}{(1-m)}\int_{\Omega}(1-\chi^{\varepsilon}) \operatorname{div} \frac{\partial^2 \mathbf{w}^{\varepsilon}}{\partial t^2} dx\big).
\end{aligned} \tag{4.1}$$

We obtain this equality if we differentiate equation for $\mathbf{w}^{\varepsilon}$ with respect to time, multiply the result by $\partial^2 \mathbf{w}^{\varepsilon}/\partial t^2$ and integrate the product by parts using continuity equations (2.2) and (2.3). Note, that all terms on the common interface Γ^{ε} "solid skeleton–pore space" disappear due to boundary conditions (1.6)–(1.7).

In fact, if $p_*+\eta_0<\infty$ ($\beta^\varepsilon=0$), then we just use Hölder and Gronwall inequalities in (4.1) and get

$$\begin{aligned} &\max_{0<t<T}\big(\||\frac{\partial^2\mathbf{w}^\varepsilon}{\partial t^2}(t)|+(1-\chi^\varepsilon)(\sqrt{\alpha_\lambda}|\nabla\frac{\partial\mathbf{w}^\varepsilon}{\partial t}(t)|+\sqrt{\alpha_\eta}|\operatorname{div}\frac{\partial\mathbf{w}^\varepsilon}{\partial t}(t)|)\|_{2,\Omega}\\ &+\sqrt{\alpha_p}\|\chi^\varepsilon\operatorname{div}\frac{\partial\mathbf{w}^\varepsilon}{\partial t}(t)\|_{2,\Omega}\big)+\|\chi^\varepsilon\nabla\frac{\partial^2\mathbf{w}^\varepsilon}{\partial t^2}|\|_{2,\Omega_T}\leq C_0,\end{aligned}\tag{4.2}$$

where C_0 is independent of the small parameter ε.

Estimates (2.8) for pressures follow from estimates (4.2) and continuity equations.

For the case $p_*+\eta_0=\infty$ estimates (2.6) and (2.7) follow again from energy identity (4.1) in the same way as before, if we additionally use inequalities

$$\begin{gathered}\frac{1}{m}(\int_\Omega\chi^\varepsilon\operatorname{div}\frac{\partial\mathbf{w}^\varepsilon}{\partial t}dx)^2\leq\int_\Omega\chi^\varepsilon(\operatorname{div}\frac{\partial\mathbf{w}^\varepsilon}{\partial t})^2dx,\\ \frac{1}{(1-m)}(\int_\Omega(1-\chi^\varepsilon)\operatorname{div}\frac{\partial\mathbf{w}^\varepsilon}{\partial t}dx)^2\leq\int_\Omega(1-\chi^\varepsilon)(\operatorname{div}\frac{\partial\mathbf{w}^\varepsilon}{\partial t})^2dx.\end{gathered}$$

To estimate pressures we use estimates (2.6) and (2.7) and integral identity (2.4) in the form

$$\begin{aligned}&\int_\Omega\big(p_f^\varepsilon+p_s^\varepsilon\big)\operatorname{div}\psi dx\\ &=\int_\Omega\Big(\rho^\varepsilon\big(\frac{\partial^2\mathbf{w}^\varepsilon}{\partial t^2}-\mathbf{F}\big)\cdot\psi+\{\chi^\varepsilon\alpha_\mu\mathbb{D}(x,\frac{\partial\mathbf{w}^\varepsilon}{\partial t})+(1-\chi^\varepsilon)\alpha_\lambda\mathbb{D}(x,\mathbf{w}^\varepsilon)\}:\mathbb{D}(x,\psi)\Big)dx,\end{aligned}$$

Considering the sum of pressures $q=p_f+p_s$ as a linear functional on the space $\overset{\circ}{W}{}_2^1(\Omega)$ we get

$$|\int_\Omega q\operatorname{div}\psi dx|\leq C_0\max_{0\leq t\leq T}\|\psi(t)\|_{W_2^1(\Omega)},$$

where C_0 is independent of the small parameter ε.

Choosing now ψ such that $\operatorname{div}\psi=q$ we arrive at

$$\max_{0\leq t\leq T}\|\operatorname{div}\psi(t)\|_\Omega^2=\max_{0\leq t\leq T}\|q(t)\|_\Omega^2\leq C_0\max_{0\leq t\leq T}\|\psi(t)\|_{W_2^1(\Omega)}.\tag{4.3}$$

Such a choice of the function ψ is always possible (see [5]), if we put

$$\psi=\nabla\varphi+\psi_{\mathbf{0}},$$

where

$$\Delta\varphi=q,\quad\mathbf{x}\in\Omega,\quad\varphi=0,\quad\mathbf{x}\in\partial\Omega,\tag{4.4}$$

$$\operatorname{div}\psi_0=0,\quad\mathbf{x}\in\Omega,\quad\psi_{\mathbf{0}}=-\nabla\varphi,\quad\mathbf{x}\in\partial\Omega.\tag{4.5}$$

In fact, extending the solution φ of the problem (4.4) as odd function over boundaries $\{x_i=0,1;\ i=1,2,3\}$ we conclude that

$$\varphi\in\overset{\circ}{W}{}_2^2(\Omega),\quad\text{and}\quad\max_{0\leq t\leq T}\|\nabla\varphi(t)\|_{W_2^1(\Omega)}\leq C\max_{0\leq t\leq T}\|q(t)\|_\Omega.$$

Now we look for the solution ψ_0 of the problem (4.5) as a solution of the Stokes system

$$\Delta\psi_0+\nabla p=0,\ \operatorname{div}\psi_{\mathbf{0}}=0,\quad\mathbf{x}\in\Omega$$

with non-homogeneous boundary condition

$$\psi_{\mathbf{0}} = -\nabla\varphi, \quad \mathbf{x} \in \partial\Omega.$$

The above problem has unique solution, such that

$$\max_{0\le t\le T} \|\psi_0(t)\|_{W_2^1(\Omega)} \le C \max_{0\le t\le T} \|\nabla\varphi(t)\|_{W_2^1(\Omega)},$$

if and only if

$$\int_\Omega \operatorname{div}(\nabla\varphi)dx \equiv \int_\Omega \Delta\varphi dx = \int_\Omega q dx = 0.$$

This solvability condition follows from conditions (2.5). Thus, gathering all estimates together we obtain the desired estimates for the sum of pressures $(p_f^\varepsilon + p_s^\varepsilon)$. Finally, thanks to the property that the product of these two functions is equal to zero, we get bounds for each of pressures p_f^ε and p_s^ε.

5. Proof of theorem 2.2

Weak and two-scale limits of sequences of displacement and pressures. On the strength of theorem 2.1, the sequences $\{p_f^\varepsilon\}$, $\{p_s^\varepsilon\}$ and $\{\mathbf{w}^\varepsilon\}$ are uniformly in ε bounded in $L^2(\Omega_T)$. Hence there exist a subsequence of small parameters $\{\varepsilon > 0\}$ and functions p_f, p_s and $\mathbf{w}$ such that

$$p_f^\varepsilon \to p_f, \quad p_s^\varepsilon \to p_s, \quad \mathbf{w}^\varepsilon \to \mathbf{w}$$

weakly in $L^2(\Omega_T)$ as $\varepsilon \searrow 0$. Relabeling if necessary, we assume that the sequences converge themselves. At the same time

$$(1-\chi^\varepsilon)\alpha_\lambda \mathbb{D}(x, \mathbf{w}^\varepsilon) \to 0. \tag{5.1}$$

strongly in $L^2(\Omega_T)$ and the sequence $\{\operatorname{div}\mathbf{w}^\varepsilon\}$ converges weakly in $L^2(\Omega_T)$ to $\operatorname{div}\mathbf{w}$ as $\varepsilon \searrow 0$.

Moreover, due to extension lemma 3.4 there are functions

$$\mathbf{v}^\varepsilon \in L^\infty(0,T; W_2^1(\Omega))$$

such that $\mathbf{v}^\varepsilon = \partial\mathbf{w}^\varepsilon/\partial t$ in $\Omega_f \times (0,T)$, $v^\varepsilon = 0$ on the part S_f^ε of the boundary S and

$$\|\frac{\partial\mathbf{v}^\varepsilon}{\partial t}\|_{2,\Omega_T} + \|\nabla\frac{\partial\mathbf{v}^\varepsilon}{\partial t}\|_{2,\Omega_T} \le C_0, \tag{5.2}$$

$$\max_{0\le t\le T}\big(\|\mathbf{v}^\varepsilon(t)\|_{2,\Omega} + \|\nabla\mathbf{v}^\varepsilon(t)\|_{2,\Omega}\big) \le C_0, \tag{5.3}$$

where C_0 does not depend on the small parameter ε.

Lemma 5.1. *There exist a subsequence of* $\{\varepsilon > 0\}$ *and function*

$$\mathbf{v} \in L^\infty\big(0,T; W_2^1(\Omega)\big),$$

such that

1. $\mathbf{v}^\varepsilon(\,,t) \to \mathbf{v}(\,,t)$ *weakly in* $W_2^1(\Omega)$ *as* $\varepsilon \searrow 0$ *for all* $t \in [0,T]$, *and*
2. $\mathbf{v}(\,,t) \in \overset{\circ}{W}{}_2^1(\Omega)$ *for all* $t \in [0,T]$.

Proof. First of all note, that there are a subsequence of small parameters $\{\varepsilon > 0\}$ and function $\mathbf{v}$, such that

$$\mathbf{v}, \frac{\partial\mathbf{v}}{\partial t} \in L^2\big(0,T; W_2^1(\Omega)\big),$$

and $\mathbf{v}^\varepsilon(\,,t) \to \mathbf{v}(\,,t)$ weakly in $L^2\big(0,T; W_2^1(\Omega)\big)$ as $\varepsilon \searrow 0$.

Now, let $\varphi(\mathbf{x})$ be an arbitrary smooth function, and

$$J^{\varepsilon}_{\varphi}(t)=\int_{\Omega}\Big(\big(\mathbf{v}^{\varepsilon}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)\big)\cdot\varphi(\mathbf{x})+\nabla\big(\mathbf{v}^{\varepsilon}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)\big)\cdot\nabla\varphi(\mathbf{x})\Big)dx.$$

By construction

$$\int_0^T J^{\varepsilon}_{\varphi}(t)\psi(t)dt\to 0$$

as $\varepsilon\searrow 0$ for any $\psi\in L^2(0,T)$. The first statement of the lemma means that

$$J^{\varepsilon}_{\varphi}(t)\to 0$$

as $\varepsilon\searrow 0$ for all $t\in[0,T]$. Estimates (5.2) and (5.3) imply

$$\int_0^T |\frac{dJ^{\varepsilon}_{\varphi}}{dt}(t)|^2dt\le C_0^2.$$

Using this estimate, the initial condition $J^{\varepsilon}_{\varphi}(0)=0$, and the weak convergence in $L^2(0,T)$ of the sequence $\{J^{\varepsilon}_{\varphi}\}$ to zero, one may easily prove that

$$J^{\varepsilon}_{\varphi}(t)\to 0\quad\text{in } C[0,T],$$

which proves the first part of the lemma.

To prove the second part of the lemma note, that

$$\mathbf{v}^{\varepsilon}(\,,t)\to\mathbf{v}(\,,t)\quad\text{strongly in}\quad L^2(S)\quad\text{as}\quad\varepsilon\searrow 0\quad\text{for all}\quad t\in[0,T].$$

This fact follows from the well-known imbedding theorem, which states that any weakly convergent sequence in $W^1_2(\Omega)$ converges strongly in $L^2(S)$.

Now we use lemma 3.5 and estimate (3.4) to conclude that

$$\max_{0\le t\le T}\|\mathbf{v}^{\varepsilon}(t)\|^2_{2,S}\le\varepsilon C_0. \tag{5.4}$$

In fact, we may prove it for each facet separately. Considering, for example, the facet $S_{3,0}=\{x_3=0,\,x^{'}=(x_1,x_2)\in(0,1)\times(0,1)\}$ one has

$$\begin{aligned}&|\mathbf{v}^{\varepsilon}(x^{'},0,t)|^2\\&=|\mathbf{v}^{\varepsilon}(x^{'},x_3,t)|^2+2\int_0^{x_3}\mathbf{v}^{\varepsilon}(x^{'},y_3,t)\frac{\partial\mathbf{v}^{\varepsilon}}{\partial y_3}(x^{'},y_3,t)dy_3\\&\le|\mathbf{v}^{\varepsilon}(x^{'},x_3,t)|^2+2\Big(\int_0^{\varepsilon}|\mathbf{v}^{\varepsilon}(x^{'},y_3,t)|^2dy_3\Big)^{1/2}\Big(\int_0^{\varepsilon}|\frac{\partial\mathbf{v}^{\varepsilon}}{\partial y_3}(x^{'},y_3,t)|^2dy_3\Big)^{1/2}\end{aligned}$$

and consequently, after integration over $S_{3,0}$ and interval $x_3\in(0,\varepsilon)$,

$$\varepsilon\int_{S_{3,0}}|\mathbf{v}^{\varepsilon}|^2dx^{'}\le\int_{Q^{\varepsilon}}|\mathbf{v}^{\varepsilon}|^2dx+2\varepsilon\Big(\int_{Q^{\varepsilon}}|\mathbf{v}^{\varepsilon}|^2dx\Big)^{1/2}\Big(\int_{\Omega}|\nabla\mathbf{v}^{\varepsilon}|^2dx\Big)^{1/2}.$$

Using estimates (3.4) and (5.3) we finally get estimate (5.4), which means that

$$\mathbf{v}^{\varepsilon}(\,,t)\to 0\quad\text{strongly in}\ \ L^2(S)$$

as $\varepsilon\searrow 0$ for all $t\in[0,T]$ and that $\mathbf{v}=0$ on the boundary S. □

On the strength of Nguetseng's theorem, there exist 1-periodic in $\mathbf{y}$ functions $P_f(\mathbf{x},t,\mathbf{y})$, $P_s(\mathbf{x},t,\mathbf{y})$, $\mathbf{W}(\mathbf{x},t,\mathbf{y})$ and $\mathbf{V}(\mathbf{x},t,\mathbf{y})$ such that the sequences $\{p^{\varepsilon}_f\}$, $\{p^{\varepsilon}_s\}$, $\{\mathbf{w}^{\varepsilon}\}$ and $\{\nabla\mathbf{v}^{\varepsilon}\}$ two-scale converge to $P_f(\mathbf{x},t,\mathbf{y})$, $P_s(\mathbf{x},t,\mathbf{y})$, $\mathbf{W}(\mathbf{x},t,\mathbf{y})$ and $\nabla\mathbf{v}(\mathbf{x},t)+\nabla_y\mathbf{V}(\mathbf{x},t,\mathbf{y})$, respectively.

5.1. **Micro- and macroscopic equations I.**

Lemma 5.2. *For all* $\mathbf{x}\in\Omega$ *and* $\mathbf{y}\in Y$ *weak and two-scale limits of the sequences* $\{p_f^\varepsilon\}$, $\{p_s^\varepsilon\}$, $\{\mathbf{w}^\varepsilon\}$, *and* $\{\mathbf{v}^\varepsilon\}$ *satisfy the relations*

$$P_s = p_s\frac{(1-\chi)}{(1-m)}, \quad P_f = \chi P_f, \tag{5.5}$$

$$\frac{1}{p_*}\frac{\partial p_f}{\partial t} + m\operatorname{div}\mathbf{v} + \langle\operatorname{div}_y\mathbf{V}\rangle_{Y_f} = \frac{\partial\beta}{\partial t}, \tag{5.6}$$

$$\frac{1}{p_*}\frac{\partial P_f}{\partial t} + \chi(\operatorname{div}\mathbf{v} + \operatorname{div}_y\mathbf{V}) = \frac{\chi}{m}\frac{\partial\beta}{\partial t}, \tag{5.7}$$

$$\frac{1}{p_*}p_f + \frac{1}{\eta_0}p_s + \operatorname{div}\mathbf{w} = 0, \tag{5.8}$$

$$\mathbf{w}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x}) = 0, \quad \mathbf{x}\in S, \tag{5.9}$$

$$\operatorname{div}_y\mathbf{W} = 0, \tag{5.10}$$

$$\frac{\partial\mathbf{W}}{\partial t} = \chi\mathbf{v} + (1-\chi)\frac{\partial\mathbf{W}}{\partial t}, \tag{5.11}$$

where $\partial\beta/\partial t = \langle\langle\operatorname{div}_y\mathbf{V}\rangle_{Y_f}\rangle_\Omega$, *if* $p_*+\eta_0=\infty$ *and* $\beta=0$, *if* $p_*+\eta_0<\infty$ *and* $\mathbf{n}(\mathbf{x})$ *is the unit normal vector to* S *at a point* $\mathbf{x}\in S$.

Proof. To prove (5.5), into (2.4) we insert a test function $\psi^\varepsilon = \varepsilon\psi(\mathbf{x},t,\mathbf{x}/\varepsilon)$, where $\psi(\mathbf{x},t,\mathbf{y})$ is an arbitrary 1-periodic and finite on Y_s function in $\mathbf{y}$. Passing to the limit as $\varepsilon\searrow 0$, we get

$$\nabla_y P_s(\mathbf{x},t,\mathbf{y}) = 0, \quad \mathbf{y}\in Y_s. \tag{5.12}$$

Next, fulfilling the two-scale limiting passage in equality

$$\chi^\varepsilon p_s^\varepsilon = 0$$

we arrive at $\chi P_s = 0$ which along with (5.12) justifies (5.5).

Equations (5.6)–(5.9) appear as the results of two-scale limiting passages in (2.2)–(2.3) with the proper test functions being involved. Thus, for example, (5.8) and (5.9) arise, if we consider the sum of (2.2) and (2.3),

$$\frac{1}{\alpha_p}p_f^\varepsilon + \frac{1}{\alpha_\eta}p_s^\varepsilon + \operatorname{div}\mathbf{w}^\varepsilon = \frac{1}{m(1-m)}\beta^\varepsilon(\chi^\varepsilon - m); \tag{5.13}$$

multiply by an arbitrary function, independent of the "fast" variable $\mathbf{x}/\varepsilon$, and then pass to the limit as $\varepsilon\searrow 0$. In order to prove (5.10), it is sufficient to consider the two-scale limiting relations in (5.13) as $\varepsilon\searrow 0$ with the test functions $\varepsilon\psi(\mathbf{x}/\varepsilon)\,h(\mathbf{x},t)$, where ψ and h are arbitrary smooth functions. In order to prove (5.11) it is sufficient to consider the two-scale limiting relations in

$$\chi^\varepsilon\Big(\frac{\partial\mathbf{w}^\varepsilon}{\partial t} - \mathbf{v}^\varepsilon\Big) = 0.$$

□

Corollary 5.3. *If* $p_*+\eta_0=\infty$, *then weak limits* p_f *and* p_s *satisfy relations*

$$\langle p_f\rangle_\Omega = \langle p_s\rangle_\Omega = 0. \tag{5.14}$$

Lemma 5.4. *For all* $(\mathbf{x},t)\in\Omega_T$ *the relations*

$$\operatorname{div}_y\{\mu_0\chi(\mathbb{D}(y,\mathbf{V}) + \mathbb{D}(x,\mathbf{v})) - (P_f + \frac{(1-\chi)}{(1-m)}p_s)\cdot\mathbb{I}\} = 0, \tag{5.15}$$

holds true.

Proof. Substituting a test function of the form $\psi^\varepsilon = \varepsilon\psi(\mathbf{x}, t, \mathbf{x}/\varepsilon)$, where $\psi(\mathbf{x}, t, \mathbf{y})$ is an arbitrary 1-periodic in $\mathbf{y}$ function vanishing on the boundary S, into integral identity (2.4), and passing to the limit as $\varepsilon \searrow 0$, we arrive at (5.15). $\square$

Lemma 5.5. *Let $\hat{\rho} = m\rho_f + (1-m)\rho_s$. Then functions $\mathbf{u}_s = \langle \mathbf{W} \rangle_{Y_s}$, $\mathbf{v}$, p_f and p_s satisfy in Ω_T the system of macroscopic equations*

$$\rho_f m \frac{\partial \mathbf{v}}{\partial t} + \rho_s \frac{\partial^2 \mathbf{u}_s}{\partial t^2} - \hat{\rho}\mathbf{F} = \operatorname{div}\{\mu_0 (m\mathbb{D}(x, \mathbf{v}) + \langle \mathbb{D}(y, \mathbf{V}) \rangle_{Y_f}) - (p_f + p_s) \cdot \mathbb{I}\}, \tag{5.16}$$

and the homogeneous initial conditions

$$\mathbf{u}_s(\mathbf{x}, 0) = \rho_f m \mathbf{v}(\mathbf{x}, 0) + \rho_s \frac{\partial \mathbf{u}_s}{\partial t}(\mathbf{x}, 0) = 0, \quad \mathbf{x} \in \Omega. \tag{5.17}$$

Proof. Equations (5.16) and initial conditions (5.17) arise as the limit of (2.4) with test functions being independent of ε in Ω_T. $\square$

Micro- and macroscopic equations II.

Lemma 5.6. *If $\lambda_1 = \infty$, then the weak limits of $\{\mathbf{v}^\varepsilon\}$ and $\{\partial \mathbf{w}^\varepsilon / \partial t\}$ coincide and*

$$(1-m)\mathbf{v} = \frac{\partial \mathbf{u}_s}{\partial t}.$$

Proof. Let $\Psi(\mathbf{x}, t, \mathbf{y})$ be an arbitrary smooth function periodic in $\mathbf{y}$. The sequence $\{\sigma_{ij}^\varepsilon\}$, where

$$\sigma_{ij}^\varepsilon = \int_\Omega \sqrt{\alpha_\lambda} \frac{\partial w_i^\varepsilon}{\partial x_j}(\mathbf{x}, t)\Psi(\mathbf{x}, t, \mathbf{x}/\varepsilon)dx, \quad \mathbf{w}^\varepsilon = (w_1^\varepsilon, w_2^\varepsilon, w_3^\varepsilon)$$

is uniformly bounded in ε. Therefore,

$$\int_\Omega \varepsilon \frac{\partial w_i^\varepsilon}{\partial x_j}(\mathbf{x}, t)\Psi(\mathbf{x}, t, \mathbf{x}/\varepsilon)dx = \frac{\varepsilon}{\sqrt{\alpha_\lambda}}\sigma_{ij}^\varepsilon \to 0$$

as $\varepsilon \searrow 0$, which is equivalent to

$$\int_\Omega \int_Y W_i(\mathbf{x}, t, \mathbf{y}) \frac{\partial \Psi}{\partial y_j}(\mathbf{x}, t, \mathbf{y})dxdy = 0, \quad \mathbf{W} = (W_1, W_2, W_3),$$

or $\mathbf{W}(\mathbf{x}, t, \mathbf{y}) = \mathbf{w}(\mathbf{x}, t)$. Therefore, taking the two-scale limit as $\varepsilon \searrow 0$ in the equality

$$\chi^\varepsilon(\mathbf{v}^\varepsilon - \frac{\partial \mathbf{w}^\varepsilon}{\partial t}) = 0$$

we arrive at the first statement of the lemma. The last statement follows from the definition of $\mathbf{u}_s$. $\square$

Lemma 5.7. *Let $\lambda_1 < \infty$. Then the weak and two-scale limits p_s and $\mathbf{W}$ satisfy the microscopic relations*

$$\rho_s \frac{\partial^2 \mathbf{W}}{\partial t^2} = \lambda_1 \Delta_y \mathbf{W} - \nabla_y R - \frac{1}{1-m}\nabla p_s + \rho_s \mathbf{F}, \quad \mathbf{y} \in Y_s, \tag{5.18}$$

$$\frac{\partial \mathbf{W}}{\partial t} = \mathbf{v}, \quad \mathbf{y} \in \gamma \tag{5.19}$$

in the case $\lambda_1 > 0$, and relations

$$\rho_s \frac{\partial^2 \mathbf{W}}{\partial t^2} = -\nabla_y R - \frac{1}{1-m}\nabla p_s + \rho_s \mathbf{F}, \quad \mathbf{y} \in Y_s, \tag{5.20}$$

$$\Big(\frac{\partial \mathbf{W}}{\partial t} - \mathbf{v}\Big) \cdot \mathbf{n} = 0, \quad \mathbf{y} \in \gamma \tag{5.21}$$

in the case $\lambda_1 = 0$.

Differential equations (5.18) and (5.20) are endowed with homogeneous initial conditions

$$\mathbf{W}(\mathbf{y}, 0) = \frac{\partial \mathbf{W}}{\partial t}(\mathbf{y}, 0) = 0, \quad \mathbf{y} \in Y_s. \tag{5.22}$$

In (5.21), $\mathbf{n}$ is the unit normal to γ.

Proof. Differential equations (5.18), (5.20) and initial conditions (5.22) follow as $\varepsilon \searrow 0$ from integral equality (2.4) with the test function $\psi = \varphi(x\varepsilon^{-1}) \cdot h(\mathbf{x}, t)$, where φ is solenoidal and finite in Y_s.

Boundary condition (5.19) is a consequence of the two-scale convergence of $\{\sqrt{\alpha_\lambda}\nabla \mathbf{w}^\varepsilon\}$ to the function $\sqrt{\lambda_1}\nabla_y \mathbf{W}(\mathbf{x}, t, \mathbf{y})$. On the strength of this convergence, the function $\nabla_y \mathbf{W}(\mathbf{x}, t, \mathbf{y})$ is L^2-integrable in Y. The boundary condition (5.21) follows from Eqs. (5.10)-(5.11). $\square$

5.2. **Homogenized equations I.** In this section we derive homogenized equations for the liquid component.

Lemma 5.8. *If $\lambda_1 = \infty$ then $\partial \mathbf{w}/\partial t = \mathbf{v}$ and the weak limits $\mathbf{v}$, p_f and p_s satisfy in Ω_T the initial-boundary value problem*

$$\begin{aligned} &\hat{\rho}\frac{\partial \mathbf{v}}{\partial t} + \nabla(p_f + p_s) - \hat{\rho}\mathbf{F} \\ &= \operatorname{div}\{\mu_0 \mathbb{A}_0^f : \mathbb{D}(x, \mathbf{v}) + \mathbb{B}_0^f p_s + \mathbb{B}_1^f \operatorname{div}\mathbf{v} + \int_0^t \mathbb{B}_2^f(t-\tau)\operatorname{div}\mathbf{v}(\mathbf{x}, \tau)d\tau\}, \end{aligned} \tag{5.23}$$

$$\begin{aligned} &p_*^{-1}\partial p_f/\partial t + \mathbb{C}_0^f : \mathbb{D}(x, \mathbf{v}) + a_0^f p_s + (a_1^f + m)\operatorname{div}\mathbf{v} \\ &+ \int_0^t a_2^f(t-\tau)\operatorname{div}\mathbf{v}(\mathbf{x}, \tau)d\tau = 0, \end{aligned} \tag{5.24}$$

$$\frac{1}{p_*}\frac{\partial p_f}{\partial t} + \frac{1}{\eta_0}\frac{\partial p_s}{\partial t} + \operatorname{div}\mathbf{v} = 0, \tag{5.25}$$

where the symmetric strictly positively defined constant fourth-rank tensor $\mathbb{A}_0^f$, matrices $\mathbb{C}_0^f, \mathbb{B}_0^f$, $\mathbb{B}_1^f$ and $\mathbb{B}_2^f(t)$ and scalars a_0^f, a_1^f and $a_2^f(t)$ are defined below by formulas (1.10), (5.33) *and* (5.35), *where $\mathbb{B}_1^f = 0$, $a_1^f = 0$ if $p_* < \infty$, and $\mathbb{B}_2^f = 0$, $a_2^f = 0$ if $p_* = \infty$.*

Differential equations (5.23) are endowed with homogeneous initial and boundary conditions

$$\mathbf{v}(\mathbf{x}, 0) = 0, \quad \mathbf{x} \in \Omega, \quad \mathbf{v}(\mathbf{x}, t) = 0, \quad \mathbf{x} \in S, \quad t > 0. \tag{5.26}$$

Proof. First note that $\mathbf{v} = \partial \mathbf{w}/\partial t$ due to lemma 5.6.

The homogenized equations (5.23) follow from the macroscopic equations (5.16), after we insert in them the expression

$$\mu_0\langle \mathbb{D}(y, \mathbf{V})\rangle_{Y_f} = \mu_0 \mathbb{A}_1^f : \mathbb{D}(x, \mathbf{v}) + \mathbb{B}_0^f p_s + \mathbb{B}_1^f \operatorname{div}\mathbf{v} + \int_0^t \mathbb{B}_2^f(t-\tau)\operatorname{div}\mathbf{v}(\mathbf{x}, \tau)d\tau + \mathbb{A}(t).$$

In turn, this expression follows by virtue of solutions of (5.7) and (5.15) on the pattern cell Y_f. In fact, if $p_* < \infty$, then setting

$$\mathbf{V} = \sum_{i,j=1}^{3} \mathbf{V}^{(ij)}(\mathbf{y}) D_{ij} + \mathbf{V}^{(0)}(\mathbf{y}) p_s + \int_0^t \mathbf{V}^{(2)}(\mathbf{y}, t-\tau) \operatorname{div} \mathbf{v}(\mathbf{x}, \tau) d\tau,$$

$$P_f = \mu_0 \sum_{i,j=1}^{3} P^{ij}(\mathbf{y}) D_{ij} + P^0(\mathbf{y}) p_s + \int_0^t P^{(2)}(\mathbf{y}, t-\tau) \operatorname{div} \mathbf{v}(\mathbf{x}, \tau) d\tau,$$

where

$$D_{ij}(\mathbf{x}, t) = \frac{1}{2}(\frac{\partial v_i}{\partial x_j}(\mathbf{x}, t) + \frac{\partial v_j}{\partial x_i}(\mathbf{x}, t)),$$

we arrive at the following periodic-boundary value problems in Y:

$$\operatorname{div}_y\{\chi \mathbb{D}(y, \mathbf{V}^{(ij)}) - \chi P^{(ij)} \mathbb{I} + \chi J^{ij}\} = 0, \quad \chi \operatorname{div}_y \mathbf{V}^{(ij)} = 0; \tag{5.27}$$

$$\operatorname{div}_y\{\mu_0 \chi \mathbb{D}(y, \mathbf{V}^{(0)}) - \big(\chi P^{(0)} + \frac{1-\chi}{1-m}\big)\mathbb{I}\} = 0, \quad \chi \operatorname{div}_y \mathbf{V}^{(0)} = 0; \tag{5.28}$$

$$\operatorname{div}_y\{\mu_0 \chi \mathbb{D}(y, \mathbf{V}^{(2)}) - \chi P^{(2)} \mathbb{I}\} = 0, \tag{5.29}$$

$$\frac{1}{p_*}\frac{\partial P^{(2)}}{\partial t} + \chi \operatorname{div}_y \mathbf{V}^{(2)} = 0, \ \frac{1}{p_*} P^{(2)}(\mathbf{y}, 0) = -\chi(\mathbf{y}). \tag{5.30}$$

For the case $p_* = \infty$ we put

$$\mathbf{V} = \sum_{i,j=1}^{3} \mathbf{V}^{(ij)}(\mathbf{y}) D_{ij} + \mathbf{V}^{(0)}(\mathbf{y}) p_s + \mathbf{V}^{(1)}(\mathbf{y}) \operatorname{div} \mathbf{v},$$

$$P_f = \sum_{i,j=1}^{3} P^{ij}(\mathbf{y}) D_{ij} + P^0(\mathbf{y}) p_s + P^{(1)}(\mathbf{y}) \operatorname{div} \mathbf{v},$$

where functions $\mathbf{V}^{(1)}$ and $P^{(1)}$ satisfy in Y the following periodic-boundary value problem in Y:

$$\operatorname{div}_y\{\mu_0 \chi \mathbb{D}(y, \mathbf{V}^{(1)}) - \chi P^{(1)} \mathbb{I}\} = 0, \ \chi(\operatorname{div}_y \mathbf{V}^{(1)} + 1) = 0. \tag{5.31}$$

Note, that for all cases the functional β is equal to zero due to the special choice of the function $\mathbf{V}$, boundary condition (5.26) for the function $\mathbf{v}$ and conditions (5.14).

On the strength of the assumptions on the geometry of the pattern "liquid" cell Y_f, problems (5.27)–(5.31) have unique solution, up to an arbitrary constant vector. In order to discard the arbitrary constant vectors we demand

$$\langle \mathbf{V}^{(ij)} \rangle_{Y_f} = \langle \mathbf{V}^{(0)} \rangle_{Y_f} = \langle \mathbf{V}^{(1)} \rangle_{Y_f} = \langle \mathbf{V}^{(2)} \rangle_{Y_f} = 0.$$

Thus

$$\mathbb{A}_0^f = m\mathbb{J} + \mathbb{A}_1^f, \quad \mathbb{A}_1^f = \sum_{i,j=1}^{3} \langle \mathbb{D}(y, \mathbf{V}^{(ij)}) \rangle_{Y_f} \otimes \mathbb{J}^{ij}, \tag{5.32}$$

$$\mathbb{B}_i^f = \mu_0 \langle \mathbb{D}(y, \mathbf{V}^{(i)}) \rangle_{Y_f}, \quad i = 0, 1, 2. \tag{5.33}$$

Symmetry of the tensor $\mathbb{A}_0^f$ follows from symmetry of the tensor $\mathbb{A}_1^f$. And symmetry of the latter one follows from the equality

$$\langle \mathbb{D}(y, \mathbf{V}^{(ij)}) \rangle_{Y_f} : J^{kl} = -\langle \mathbb{D}(y, \mathbf{V}^{(ij)}) : \mathbb{D}(y, \mathbf{V}^{(kl)}) \rangle_{Y_f} \tag{5.34}$$

which appears by means of multiplication of (5.27) for $\mathbf{V}^{(ij)}$ by $\mathbf{V}^{(kl)}$ and by integration by parts.

This equality also implies positive definiteness of the tensor $\mathbb{A}^f_0$. Indeed, let $\mathbb{Z} = (Z_{ij})$ be an arbitrary symmetric matrix. Setting $\mathbf{Z} = \sum_{i,j=1}^3 \mathbf{V}^{(ij)} Z_{ij}$ and taking into account (5.34) we get

$$\langle \mathbb{D}(y, \mathbf{Z})\rangle_{Y_f} : \mathbb{Z} = -\langle \mathbb{D}(y, \mathbf{Z}) : \mathbb{D}(y, \mathbf{Z})\rangle_{Y_f}.$$

This equality and the definition of the tensor A^f_0 give

$$(\mathbb{A}^f_0 : \mathbb{Z}) : \mathbb{Z} = \langle (\mathbb{D}(y, \mathbf{Z}) + \mathbb{Z}) : (\mathbb{D}(y, \mathbf{Z}) + \mathbb{Z})\rangle_{Y_f}.$$

Now the strict positive definiteness of the tensor $\mathbb{A}^f_0$ follows immediately from the equality above and the geometry of the elementary cell Y_f. Namely, suppose that $(\mathbb{A}^s_0 : \mathbb{Z}) : \mathbb{Z} = 0$ for some matrix $\mathbb{Z}$, such that $\mathbb{Z} : \mathbb{Z} = 1$. Then $(\mathbb{D}(y, \mathbf{Z}) + \mathbb{Z}) = 0$, which is possible if and only if $\mathbf{Z}$ is a linear function in $\mathbf{y}$. On the other hand, all linear periodic functions on Y_f are constant. Finally, the normalization condition $\langle \mathbf{V}^{(ij)}\rangle_{Y_f} = 0$ yields that $\mathbf{Z} = 0$. However, this is impossible because the functions $\mathbf{V}^{(ij)}$ are linearly independent.

Equations (5.24) and (5.25) for the pressures follow from (5.6), (5.8) and equality

$$\langle \operatorname{div}_y \mathbf{V}\rangle_{Y_f} = \mathbb{C}^f_0 : \mathbb{D}(x, \mathbf{v}) + a^f_0 p_s + a^f_1 \operatorname{div} \mathbf{v} + \int_0^t a^f_2(t-\tau) \operatorname{div} \mathbf{v}(\mathbf{x}, \tau) d\tau$$

with

$$\mathbb{C}^f_0 = \sum_{i,j=1}^3 \langle \operatorname{div}_y \mathbf{V}^{(ij)}\rangle_{Y_f} \mathbb{J}^{ij}, \quad a^f_i = \langle \operatorname{div}_y \mathbf{V}^{(i)}\rangle_{Y_f}, \quad i = 0, 1, 2. \tag{5.35}$$

Finally note, that initial conditions (5.26) follow from initial conditions (5.17) and lemma 5.6. □

5.3. **Homogenized equations II.** We complete the proof of theorem 2.2 with homogenized equations for the solid component.

Let $\lambda_1 < \infty$. In the same manner as above, we verify that the limit $\mathbf{v}$ of the sequence $\{\mathbf{v}^\varepsilon\}$ satisfies the initial-boundary value problem likes (5.23)– (5.25). The main difference here that, in general, the weak limit $\partial \mathbf{w}/\partial t$ of the sequence $\{\partial \mathbf{w}^\varepsilon/\partial t\}$ differs from $\mathbf{v}$. More precisely, the following statement is true.

Lemma 5.9. *Let $\lambda_1 < \infty$. Then the weak limits $\mathbf{v}$, $\mathbf{u}_s$, p_f, and p_s of the sequences $\{\mathbf{v}^\varepsilon\}$, $\{(1-\chi^\varepsilon)\mathbf{w}^\varepsilon\}$, $\{p^\varepsilon_f\}$, and $\{p^\varepsilon_s\}$ satisfy the initial-boundary value problem in Ω_T, consisting of the balance of momentum equation*

$$\begin{aligned} &\rho_f m \frac{\partial \mathbf{v}}{\partial t} + \rho_s \frac{\partial^2 \mathbf{u}_s}{\partial t^2} + \nabla(p_f + p_s) - \hat{\rho}\mathbf{F} \\ &= \operatorname{div}\{\mu_0 \mathbb{A}^f_0 : \mathbb{D}(x, \mathbf{v}) + \mathbb{B}^f_0 p_s + \mathbb{B}^f_1 \operatorname{div} \mathbf{v} + \int_0^t \mathbb{B}^f_2(t-\tau) \operatorname{div} \mathbf{v}(\mathbf{x}, \tau) d\tau\}, \end{aligned} \tag{5.36}$$

and the continuity equation (5.24) for the liquid component, where $\mathbb{A}^f_0$, $\mathbb{B}^f_0$ – $\mathbb{B}^f_2$ are the same as in (5.23), the continuity equation

$$\frac{1}{p_*} \frac{\partial p_f}{\partial t} + \frac{1}{\eta_0} \frac{\partial p_s}{\partial t} + \operatorname{div} \frac{\partial \mathbf{u}_s}{\partial t} + m\, div \mathbf{v} = 0, \tag{5.37}$$

the relation

$$\frac{\partial \mathbf{u}_s}{\partial t} = (1-m)\mathbf{v}(\mathbf{x},t) + \int_0^t \mathbb{B}_1^s(t-\tau)\cdot \mathbf{z}(\mathbf{x},\tau)d\tau, \tag{5.38}$$

$$\mathbf{z}(\mathbf{x},t) = -\frac{1}{1-m}\nabla_x p_s(\mathbf{x},t) + \rho_s \mathbf{F}(\mathbf{x},t) - \rho_s \frac{\partial \mathbf{v}}{\partial t}(\mathbf{x},t) \tag{5.39}$$

in the case $\lambda_1 > 0$, or the balance of momentum equation in the form

$$\rho_s \frac{\partial^2 \mathbf{u}_s}{\partial t^2} = \rho_s \mathbb{B}_2^s \cdot \frac{\partial \mathbf{v}}{\partial t} + ((1-m)I - \mathbb{B}_2^s)\cdot(-\frac{1}{1-m}\nabla p_s + \rho_s \mathbf{F}) \tag{5.40}$$

in the case of $\lambda_1 = 0$ for the solid component. The problem is supplemented by boundary and initial conditions (5.26) for the velocity $\mathbf{v}$ of the liquid component and by homogeneous initial conditions and the boundary condition

$$\mathbf{u}_s(\mathbf{x},t)\cdot \mathbf{n}(\mathbf{x}) = 0, \quad (\mathbf{x},t)\in S, \quad t>0, \tag{5.41}$$

for the displacement $\mathbf{u}_s$ of the solid component. In Eqs. (5.38)–(5.41) $\mathbf{n}(\mathbf{x})$ is the unit normal vector to S at a point $\mathbf{x}\in S$, and matrices $\mathbb{B}_1^s(t)$ and $\mathbb{B}_2^s$ are given below by Eqs. (5.43) and (5.45).

Proof. The boundary condition (5.41) follows from (5.9), the equality

$$\frac{\partial \mathbf{w}}{\partial t} = \frac{\partial \mathbf{u}_s}{\partial t} + m\mathbf{v},$$

and the homogeneous boundary condition for $\mathbf{v}$.

The same equality and (5.8) imply (5.37). The homogenized equations of balance of momentum (5.36) derives exactly as before. Therefore we omit the relevant proofs now and focus ourself only on derivation of homogenized equation of the balance of momentum for the solid displacements $\mathbf{u}_s$.

(a) If $\lambda_1 > 0$, then the solution of the system of microscopic equations (5.10), (5.18), and (5.19), provided with the homogeneous initial data (5.22), is given by formula

$$\mathbf{W} = \int_0^t (\mathbf{v}(\mathbf{x},\tau) + \sum_{i=1}^3 \mathbf{W}^i(\mathbf{y},t-\tau)\otimes \mathbf{e}_i \cdot \mathbf{z}(\mathbf{x},\tau))d\tau,$$

$$R = \int_0^t \sum_{i=1}^3 R^i(\mathbf{y},t-\tau)\mathbf{e}_i\cdot \mathbf{z}(\mathbf{x},\tau)d\tau,$$

in which functions $\mathbf{W}^i(\mathbf{y},t)$ and $R^i(\mathbf{y},t)$ are defined by virtue of the periodic initial-boundary value problem

$$\begin{gathered} \rho_s \frac{\partial^2 \mathbf{W}^i}{\partial t^2} - \lambda_1 \Delta \mathbf{W}^i + \nabla R^i = 0, \quad \operatorname{div}_y \mathbf{W}^i = 0, \quad \mathbf{y}\in Y_s,\ t>0, \\ \mathbf{W}^i = 0, \quad \mathbf{y}\in\gamma,\ t>0, \\ \mathbf{W}^i(y,0) = 0, \quad \rho_s \frac{\partial \mathbf{W}^i}{\partial t}(y,0) = \mathbf{e}_i, \quad \mathbf{y}\in Y_s. \end{gathered} \tag{5.42}$$

In (5.42), $\mathbf{e}_i$ is the standard Cartesian basis vector. Therefore,

$$B_1^s(t) = \sum_{i=1}^3 \langle \frac{\partial \mathbf{W}^i}{\partial t}\rangle_{Y_s} \otimes \mathbf{e}_i(t). \tag{5.43}$$

Note, that differential equations in (5.42) are understood in the sense of distributions (the compatibility conditions on the boundary γ at $t = 0$ have no place) and therefore the functions $\partial \mathbf{W}^i/\partial t$ have no time derivative at $t = 0$.

(b) If $\lambda_1 = 0$ then in the process of solving the system (5.10), (5.18), and (5.19) we firstly find the pressure $R(\mathbf{x}, t, \mathbf{y})$ by virtue of solving the Neumann problem for Laplace's equation in Y_s in the form

$$R(\mathbf{x}, t, \mathbf{y}) = \sum_{i=1}^{3} R_i(\mathbf{y})\mathbf{e}_i \cdot \mathbf{z}(\mathbf{x}, t),$$

where $R^i(\mathbf{y})$ is the solution of the problem

$$\Delta_y R_i = 0, \quad \mathbf{y} \in Y_s; \quad \nabla_y R_i \cdot \mathbf{n} = \mathbf{n} \cdot \mathbf{e}_i, \quad \mathbf{y} \in \gamma. \tag{5.44}$$

Formula (5.36) appears as the result of homogenization of (5.18) and

$$B_2^s = \sum_{i=1}^{3} \langle \nabla R_i(\mathbf{y}) \rangle_{Y_s} \otimes \mathbf{e}_i, \tag{5.45}$$

where the matrix $((1 - m)I - B_2^s)$ is symmetric and positively definite. In fact, let $\tilde{R} = \sum_{i=1}^{3} R_i \mathbf{x} i_i$ for any unit vector ξ. Then

$$(B \cdot \xi) \cdot \xi = \langle (\xi - \nabla \tilde{R})^2 \rangle_{Y_f} > 0$$

due to the same reasons as in lemma 5.7. On the strength of the assumptions on the geometry of the pattern "solid" cell Y_s, problem (5.42) has unique solution and problem (5.44) has unique solution up to an arbitrary constant. □

References

[1] E. Acerbi, V. Chiado Piat, G. Dal Maso and D. Percivale; *An extension theorem from connected sets and homogenization in general periodic domains*, Nonlinear Anal., 18 (1992), pp. 481–496.

[2] R. Burridge R and J. B. Keller; *Poroelasticity equations derived from microstructure*, J. Acoust. Soc. Am.,70, N4 (1981), pp. 1140–1146.

[3] Th. Clopeau, J. L. Ferrin, R. P. Gilbert and A. Mikelić; *Homogenizing the acoustic properties of the seabed: Part II*, Mathematical and Computer Modelling, 33 (2001), pp. 821–841.

[4] V. V. Jikov, S. M. Kozlov and O. A. Oleinik; *Homogenization of Differential Operators and Integral Functionals*, Springer-Verlag, New York, 1994.

[5] O. A. Ladyzhenskaya; *The mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, New York, 1969.

[6] D. Lukkassen, G. Nguetseng and P. Wall; *Two-scale convergence*, Int. J. Pure and Appl. Math., 2, N1 (2002, pp. 35–86.

[7] A. Meirmanov; *Nguetseng's two-scale convergence Method for filtration and seismic acoustic problems in elastic porous media*, Siberian Math. Journal, 48, N 3 (2007), pp. 519–538.

[8] G. Nguetseng; *A general convergence result for a functional related to the theory of homogenization*, SIAM J. Math. Anal., 20 (1989), pp. 608–623.

[9] G. Nguetseng; *Asymptotic analysis for a stiff variational problem arising in mechanics*, SIAM J. Math. Anal., 21 (1990), pp. 1394–1414.

[10] E. Sanchez-Palencia; *Non-Homogeneous Media and Vibration Theory*, Lecture Notes in Physics, Vol.129, Springer, Berlin, 1980.

Anvarbek M. Meirmanov

Department of mahtematics, Belgorod State University, ul. Pobedi 85, 308015 Belgorod, Russia

E-mail address: meirmanov@bsu.edu.ru