Electronic Journal of Differential Equations, Vol. 2008(2008), No. 144, pp. 1–16.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
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MONOTONE SOLUTIONS FOR A NONCONVEX FUNCTIONAL DIFFERENTIAL INCLUSIONS OF SECOND ORDER

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Abstract. We give sufficient conditions for the existence of a monotone solution for second-order functional differential inclusions. No convexity condition, on the values of the multifunction defining the inclusion, is involved in this construction.

1. Introduction

Let K be a closed subset of $\mathbb{R}^n$, Ω an open subset of $\mathbb{R}^n$, and P a lower semicontinuous set-valued map (multifunction) from K to the family of all non-empty subsets of K, with closed graph satisfying the following two conditions:

(i) for all $x \in K$, $x \in P(x)$
(ii) for all $x, y \in K$, $y \in P(x) \Rightarrow P(y) \subseteq P(x)$.

Under these conditions, a preorder (reflexive and transitive relation) on K is defined as

$$x \preceq y \Leftrightarrow y \in P(x)\,.$$

Let $\sigma > 0$ and $C([-\sigma, 0], \mathbb{R}^n)$ be the space of continuous functions from $[-\sigma, 0]$ to $\mathbb{R}^n$ with the uniform norm $\|x\|_\sigma = \sup\{\|x(t)\| : t \in [-\sigma, 0]\}$. For each $t \in [0, T]; T > 0$, we define the operator $\tau(t)$ from $C([-\sigma, T], \mathbb{R}^n)$ to $C([-\sigma, 0], \mathbb{R}^n)$ as

$$(\tau(t)x)(s) = x(t+s), \quad \text{for all } s \in [-\sigma, 0].$$

Here, $\tau(t)x$ represents the history of the state from the time $t - \sigma$ to the present time t.

Let $K_0 = \{\varphi \in C([-\sigma, T], \mathbb{R}^n) : \varphi(0) \in K\}$ and F be a set-valued map (multifunction) defined from $K_0 \times \Omega$ to the family of non-empty compact subsets (not necessarily convex) in $\mathbb{R}^n$ and (φ_0, y_0) be a given element in $K_0 \times \Omega$. We consider

2000 *Mathematics Subject Classification.* 34A60, 49K24.
Key words and phrases. Functional differential inclusion; monotone solution; second-order contingent cone.

Submitted June 10, 2008. Published October 24, 2008.

the second-order functional differential inclusion

$$\begin{gathered} x''(t) \in F(\tau(t)x, x'(t)), \quad \text{a.e. on } [0,T] \\ x(t) = \varphi_0(t), \quad \forall t \in [-\sigma, 0] \\ x'(0) = y_0 \\ x(t) \in P(x(t)) \subset K, \quad \forall t \in [0,T] \\ x(s) \preceq x(t) \quad \text{whenever } 0 \le s \le t \le T \end{gathered} \tag{1.1}$$

In the present work, we prove under reasonable conditions that there are a positive real number T and a continuous function $x : [-\sigma, T] \to \mathbb{R}^n$ such that

(1) the function x is absolutely continuous on $[0,T]$ with absolutely continuous derivative
(2) $\tau(t)x \in K_0$, for all $t \in [0,T]$
(3) $x'(t) \in \Omega$, a.e. on $[0,T]$
(4) the functions x, x', x'' satisfy (1.1).

Ibrahim and Alkulaibi [13] proved the existence of a monotone solution for (1.1) without delay. They consider the problem

$$\begin{gathered} x''(t) \in F(x(t), x'(t)), \quad \text{a.e. on } [0,T] \\ x(0) = x_0, \quad x'(0) = y_0 \\ x(t) \in K, \quad \forall t \in [0,T] \\ x(s) \preceq x(t), \quad \text{whenever } 0 \le s \le t \le T\,. \end{gathered}$$

Further, Lupulescu [17] proved the existence of a local solution, not necessarily monotone, for (1.1) in the particular case $P(x) = K$, for all $x \in K$. Thus, the result, we are going to prove, generalizes the results of Ibrahim and Alkulaibi [13] and Lupulescu [17].

We mention, among others the works, [9, 11, 12, 13, 18] for the proof of existence of monotone solutions for differential inclusions or functional differential inclusions and the works [3, 4, 7, 8, 10, 14, 15, 16, 17, 19] for solutions not necessarily monotone. Note that the case where the solutions are not necessarily monotone has been widely investigated compared with that of monotone solutions which has been rarely investigated.

The present paper is organized as follows: In section 2, some definitions and facts to be used later are introduced. In section 3, the main result is proved.

2. Preliminaries

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with norm $\|\cdot\|$ and scalar product $\langle \cdot, \cdot \rangle$. For $x \in \mathbb{R}^n$ and $r > 0$ let $B(x,r) = \{y \in \mathbb{R}^n : \|y - x\| < r\}$ denote the open ball centered at x of radius r, and $\overline{B(x,r)}$ its closure.

For $\varphi \in C([-\sigma, 0], \mathbb{R}^n)$ let $B_\sigma(\varphi, r) = \{\psi \in C([-\sigma, 0], \mathbb{R}^n) : \|\psi - \varphi\|_\sigma < r\}$ and $\overline{B}_\sigma(\varphi, r) = \{\psi \in C([-\sigma, 0], \mathbb{R}^n) : \|\psi - \varphi\|_\sigma \le r\}$.

We also, denote by $d(x, A) = \inf\{\|x - y\| : y \in A\}$ the distance from $x \in \mathbb{R}^n$ to a closed subset $A \subseteq \mathbb{R}^n$.

A function $V : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is said to be proper if its effective domain $D(V) = \{x \in \mathbb{R}^n : V(x) < \infty\}$ is non-empty.

The subdifferential of a proper convex lower semicontinuous function $V : \mathbb{R}^n \to \mathbb{R}$ at a point $x \in \mathbb{R}^n$ is defined (in the sense of convex analysis) by

$$\partial V(x) = \big\{\xi \in \mathbb{R}^n : V(y) - V(x) \geq \langle \xi, y - x\rangle,\ \forall y \in \mathbb{R}^n\big\}$$

The second-order contingent cone of a non-empty closed subset $C \subset \mathbb{R}^n$ at a point $(x, y) \in C \times \mathbb{R}^n$ is defined by

$$T_C^2(x, y) = \big\{z \in \mathbb{R}^n : \liminf_{t\to 0^+} \frac{d(x + ty + \frac{t^2}{2}z, C)}{t^2} = 0\big\}.$$

For the properties of the second-order contingent cone see for example [2, 3, 7, 14].

A multifunction $F : K_0 \times \Omega \to 2^{\mathbb{R}^n}$ is said to be upper semicontinuous at a point $(\varphi, y) \in K_0 \times \Omega$ if for every $\varepsilon > 0$ there exists $\delta > 0$, such that

$$F(\psi, z) \subset F(\varphi, y) + B(0, \varepsilon),$$

for all $(\psi, z) \in B_\sigma(\varphi, \delta) \times B(y, \delta)$. For more information about the continuity properties for multifunctions we refer the reader to [1, 2, 6].

3. Main Result

Lemma 3.1. *Let K be a non-empty closed subset of $\mathbb{R}^n$, Ω a non-empty open subset of $\mathbb{R}^n$, P a set-valued map from K to the family of non-empty closed subsets of K and $K_0 = \{\varphi \in C([-\sigma, 0], \mathbb{R}^n), \varphi(0) \in K\}$. Let F be an upper semicontinuous set-valued map from $K_0 \times \Omega$ to the family of non-empty compact subsets of $\mathbb{R}^n$. Assume also the following conditions:*

- (H1) *For all $x \in K$, $x \in P(x)$*
- (H2) *There exists a proper convex lower semicontinuous function $V : \mathbb{R}^n \to \mathbb{R}$ such that $F(\varphi, y) \subseteq \partial V(y)$, for every $(\varphi, y) \in K_0 \times \Omega$*
- (H3) *For $(\varphi, y) \in K_0 \times \Omega$, $F(\varphi, y) \subseteq T^2_{P(\varphi(0))}(\varphi(0), y)$, where $T^2_{P(\varphi(0))}(\varphi(0), y)$ is the second order contingent cone of the closed subset $P(\varphi(0))$ at the point $(\varphi(0), y)$.*

Let (φ_0, y_0) be a fixed element in $K_0 \times \Omega$. Then there are two positive numbers r and T such that for each positive integer m there are:

1. *A positive integer ν_m.*
2. *A set of points*
$$P_m = \{t_0^m = 0 < t_1^m < \dots < t^m_{\nu_m - 1} \leq T < t^m_{\nu_m}\}$$
3. *Three sets of elements in $\mathbb{R}^n$:*
$$X_m = \{x_p^m : p = 0, 1, \dots, \nu_m - 1\},$$
$$Y_m = \{y_p^m : p = 0, 1, \dots, \nu_m - 1\},$$
$$Z_m = \{z_p^m : p = 0, 1, \dots, \nu_m - 1\},$$
with $x_0^m = \varphi_0(0)$ and $y_0^m = y_0$
4. *A continuous function $x_m : [-\sigma, T] \to \mathbb{R}^n$ with $x_m(t) = \varphi_0(t)$, for all $t \in [-\sigma, 0]$, such that for each $p = 0, 1, \dots, \nu_m - 1$, the following properties are satisfied:*
   - (i) $h^m_{p+1} = t^m_{p+1} - t^m_p < \frac{1}{m}$
   - (ii) $z_p^m = u_p^m + w_p^m$ *where* $u_p^m \in F(\tau(t_p^m)x_m, y_p^m)$ *and* $w_p^m \in \frac{1}{m}B(0, 1)$
   - (iii) $x_m(t) = x_p^m + (t - t_p^m)y_p^m + \frac{1}{2}(t - t_p^m)^2 z_p^m$, *for all* $t \in [t_p^m, t^m_{p+1}]$
   - (iv) $x^m_{p+1} = x_p^m + h^m_{p+1} y_p^m + \frac{1}{2}(h^m_{p+1})^2\, z_p^m = x_m(t^m_{p+1})$

(v) $x_{p+1}^m \in P(x_p^m) \cap B(\varphi_0(0), r) \subseteq K$ *and*
$y_{p+1}^m = y_p^m + h_{p+1}^m z_p^m \in \overline{B(y_0, r)} \subseteq \Omega$

(vi) $x_m(t) \in B(\varphi_0(0), r)$, *for all* $t \in [t_p^m, t_{p+1}^m]$.

(vii) $\tau(t_{p+1}^m)x_m \in B_\sigma(\varphi_0, r) \cap K_0$.

Proof. We follow the techniques developed in [18]. From [6, Prop. I.26], for each $y \in \mathbb{R}^n$, the subset $\partial V(y)$ is closed, convex and bounded. Moreover, by [1, Thm. 0.7.2] the multifunction $y \to \partial V(y)$ is upper semicontinuous. So, by [1, Prop. 1.1.3] there are two positive real numbers r and M such that

$$\sup\{\|z\| : z \in \partial V(y)\} \le M\,,$$

for all $y \in \overline{B(y_0, r)}$. Using condition (H2), we get

$$\sup\{\|z\| : z \in F(\psi, y)\} < M\,, \tag{3.1}$$

for all $(\psi, y) \in (K_0 \cap B_\sigma(\varphi_0, r)) \times \overline{B(y_0, r)}$. Since Ω is open we can choose r such that $\overline{B(y_0, r)} \subseteq \Omega$. It is obvious that the closedness of K implies the closedness of K_0 in $C([-\sigma, 0], \mathbb{R}^n)$. From the continuity of φ_0 on $[-\sigma, 0]$, there is $\mu > 0$ such that for all $t, s \in [-\sigma, 0]$ we have

$$|t - s| < \mu \Longrightarrow \|\varphi_0(t) - \varphi_0(s)\| < \frac{r}{4}. \tag{3.2}$$

Put

$$T = \min\Big\{\mu, \frac{r}{4(M+1)}, \frac{r}{8(\|y_0\|+1)}, \sqrt{\frac{r}{4(M+1)}}\Big\} \tag{3.3}$$

Thus the numbers r and T are well defined. Now let m be a fixed positive integer. We put $t_0^m = 0$, $x_0^m = \varphi_0(0)$ and $y_0^m = y_0$. The sets P_m, X_m, Y_m and Z_m will be defined by induction. We first define x_1^m, t_1^m, y_1^m, z_0^m and x_m on $[0, t_1^m]$ such that the properties (i)–(vii) are satisfied for $p = 0$.

Using condition (H3), there is $u_0^m \in F(\varphi_0, y_0)$ such that

$$\liminf_{h\downarrow 0} \frac{1}{h^2} d(\varphi_0(0) + h\, y_0^m + \frac{h^2}{2} u_0^m, P(\varphi_0(0))) = 0.$$

So, a positive number h_1^m is found such that $h_1^m \le \min\{\frac{1}{m}, T\}$ and

$$d(\varphi_0(0) + h_1^m\, y_0^m + \frac{(h_1^m)^2}{2} u_0^m, P(\varphi_0(0))) \le \frac{(h_1^m)^2}{4m}.$$

Since $P(\varphi_0(0))$ is closed, there is $x_1^m \in P(\varphi_0(0))$ with

$$\|\varphi_0(0) + h_1^m y_0^m + \frac{(h_1^m)^2}{2} u_0^m - x_1^m\| \le \frac{(h_1^m)^2}{4m}.$$

Consequently there is $w_0^m \in \mathbb{R}^n$ such that $\|w_0^m\| \le \frac{1}{2m}$ and

$$x_1^m = \varphi_0(0) + h_1^m y_0^m + \frac{(h_1^m)^2}{2} u_0^m + \frac{(h_1^m)^2}{2} w_0^m.$$

Now we define $z_0^m = u_0^m + w_0^m$, therefore $z_0^m \in F(\varphi_0, y_0) + \frac{1}{2m} B(0, 1)$ and $x_1^m = \varphi_0(0) + h_1^m y_0^m + \frac{(h_1^m)^2}{2} z_0^m$. We put $y_1^m = y_0^m + h_1^m z_0^m$ and $t_1^m = t_0^m + h_1^m$ and for $t \in [t_0^m, t_1^m]$ we define

$$x_m(t) = \varphi_0(0) + (t - t_0^m) y_0^m + \tfrac{(t-t_0^m)^2}{2} z_0^m.$$

Thus, the properties (i)–(iv) are clearly satisfied for $p = 0$.

Since $\tau(t_0^m)x_m = \varphi_0$, using relation (3.1), we obtain

$$\sup\{\|v\| : v \in F(\tau(t_0^m)x_m, y_0)\} \le M.$$

Therefore, $\|z_0^m\| \le M + \frac{1}{2m} < M+1$. We get from the definition of y_1^m, $\|y_1^m - y_0^m\| \le h_1^m\|z_0^m\| \le T(M+1) \le r$. Thus $y_1^m \in \overline{B(y_0, r)}$. Since, $x_1^m \in P(x_0^m) \subseteq K$ then to prove property (v) for $p = 0$, it is sufficient to show that

$$\|x_1^m - \varphi_0(0)\| < r.$$

We get using (3.1), (3.3)

$$\begin{aligned}
\|x_1^m - \varphi_0(0)\| &= h_1^m\|y_0^m\| + \frac{(h_1^m)^2}{2}\|z_0^m\| \\
&\le T\|y_0^m\| + \frac{T^2}{2}(M+1) \\
&\le \frac{r}{8(\|y_0\|+1)}\|y_0\| + \frac{r}{8(M+1)}(M+1) \\
&< \frac{r}{8} + \frac{r}{8} < r
\end{aligned}$$

and hence (v) is satisfied for $p = 0$. To prove (vi) for $p = 0$, we note that, for $t \in [t_0^m, t_1^m]$,

$$\begin{aligned}
\|x_m(t) - \varphi_0(0)\| &\le (t - t_0^m)\|y_0^m\| + \frac{(t-t_0^m)^2}{2}\|z_0^m\| \\
&\le h_1^m\|y_0^m\| + \frac{(h_1^m)^2}{2}\|z_0^m\| \\
&\le T\|y_0\| + \frac{T^2}{2}(M+1) \\
&\le \frac{r}{8(\|y_0\|+1)}\|y_0\| + \frac{1}{2}\frac{r}{4(M+1)}(M+1) \\
&< \frac{r}{8} + \frac{r}{8} < r,
\end{aligned}$$

which proves (vi) for $p = 0$.

To prove property (vii) for $p = 0$, we note that if $-\sigma \le s \le -t_1^m$, then $t_1^m + s \le 0$ and by (3.2), (3.3), we get

$$\|\tau(t_1^m)x_m - \varphi\|_\sigma = \sup_{-\sigma\le s\le 0}\|x_m(t_1^m+s) - \varphi_0(s)\| = \sup_{-\sigma\le s\le 0}\|\varphi_0(t_1^m+s) - \varphi_0(s)\| < \frac{r}{4}.$$

while if $-t_1^m \le s \le 0$, then $0 \le t_1^m + s \le t_1^m$ and hence by (3.3) we get

$$\begin{aligned}
\|x_m(t_1^m+s) - \varphi_0(s)\| &\le \|x_m(t_1^m+s) - \varphi_0(0)\| + \|\varphi_0(0) - \varphi_0(s)\| \\
&\le (h_1^m + s)\|y_0^m\| + \frac{(h_1^m+s)^2}{2}\|z_0^m\| + \frac{r}{4} \\
&\le T\|y_0^m\| + \frac{T^2}{2}\|z_0^m\| + \frac{r}{4} \\
&\le \frac{r}{8(\|y_0\|+1)}\|y_0\| + \frac{r}{8(M+1)}(M+1) + \frac{r}{4} \\
&< \frac{r}{8} + \frac{r}{8} + \frac{r}{4} = \frac{r}{2},
\end{aligned}$$

which shows that $\tau(t_1^m)x_m \in B_\sigma(\varphi_0, r)$ and hence (vii) is proved.

Now we suppose that t^m_{p+1}, x^m_{p+1}, y^m_{p+1}, z^m_p are well defined for $p = 0, 1, \dots, (q-1)$ and x_m is defined on the interval $[-\sigma, t^m_q]$ such that all the properties (i)–(vii) are satisfied for $p = 0, 1, \dots, (q-1)$.

We define t^m_{q+1}, x^m_{q+1}, y^m_{q+1}, z^m_q and x_m on $[t^m_q, t^m_{q+1}]$ such that the properties (i)–(vii) are satisfied for $p = q$. We denote by H^m_q the set of all $h \in]0, \frac{1}{m}[$ for which the following conditions are satisfied:

(a) $h < T - t^m_q$.

(b) there exists $u^m_q \in F(\tau(t^m_q)x_m, y^m_q)$ such that

$$d(x^m_q + hy^m_q + \frac{h^2}{2}u^m_q, p(x^m_q)) \le \frac{h^2}{4m}.$$

From the fact that (v) and (vii) are true for $p = q-1$, we get $y^m_q \in \Omega$ and $\tau(t^m_q)x_m \in K_0$. Moreover, since (iv) is true for $p = q-1$, then

$$\tau(t^m_q)x_m(0) = x_m(t^m_q) = x^m_q.$$

So, the condition (H3) gives:

$$F(\tau(t^m_q)x_m, y^m_q) \subseteq T^2_{P(x^m_q)}(x^m_q, y^m_q).$$

Therefore there is $u^m_q \in F(\tau(t^m_q)x_m, y^m_q)$ such that

$$\liminf_{h\downarrow 0} \frac{1}{h^2} d(x^m_q + h\, y^m_q + \frac{h^2}{2}u^m_q, P(x^m_q)) = 0,$$

which shows that there is a positive number h such that $h < \min\{\frac{1}{m}, T - t^m_q\}$ and

$$d(x^m_q + hy^m_q + \frac{h^2}{2}u^m_q, P(x^m_q)) \le \frac{h^2}{4m}.$$

Hence $h \in H^m_q$. Since H^m_q is bounded by the number T, there is a number d^m_q such that $d^m_q = \sup\{\alpha : \alpha \in H^m_q\}$. Since $H^m_q \cap [\frac{d^m_q}{2}, d^m_q] \ne \phi$, an element $h^m_{q+1} \in H^m_q \cap [\frac{d^m_q}{2}, d^m_q]$ is found such that

$$d(x^m_q + h^m_{q+1}y^m_q + \frac{(h^m_{q+1})^2}{2}u^m_q, P(x^m_q)) \le \frac{(h^m_{q+1})^2}{4m}.$$

From the closedness of $P(x^m_q)$, there is $x^m_{q+1} \in P(x^m_q) \subseteq K$ with

$$\|x^m_q + h^m_{q+1}y^m_q + \frac{(h^m_{q+1})^2}{2}u^m_q - x^m_{q+1}\| \le \frac{(h^m_{q+1})^2}{4m}.$$

Consequently, there is $w^m_q \in \mathbb{R}^n$ with $\|w^m_q\| \le \frac{1}{2m} < \frac{1}{m}$ such that

$$\begin{aligned} x^m_{q+1} &= x^m_q + h^m_{q+1}y^m_q + \frac{(h^m_{q+1})^2}{2}u^m_q + \frac{(h^m_{q+1})^2}{2}w^m_q \\ &= x^m_q + h^m_{q+1}y^m_q + \frac{(h^m_{q+1})^2}{2}(u^m_q + w^m_q). \end{aligned}$$

We define $z^m_q = u^m_q + w^m_q$. So that

$$\begin{gathered} z^m_q \in F(\tau(t^m_q)x_m, y^m_q) + \frac{1}{m}B(0,1), \\ x^m_{q+1} = x^m_q + h^m_{q+1}y^m_q + \frac{(h^m_{q+1})^2}{2}z^m_q. \end{gathered}$$

We put $y_{q+1}^m = y_q^m + h_{q+1}^m z_q^m$ and $t_{q+1}^m = t_q^m + h_{q+1}^m$ and for $t \in [t_q^m, t_{q+1}^m]$, we define

$$x_m(t) = x_q^m + (t - t_q^m) y_q^m + \frac{(t - t_q^m)^2}{2} z_q^m.$$

Obviously the relations (i)–(iv) are satisfied for $p = q$.

Now we prove that (v) is true for $p = q$. Since (v) and (vii) are true for $p = q-1$, then $\tau(t_q^m) x_m \in B_\sigma(\varphi_0, r)$ and $y_q^m \in \overline{B(y_0, r)}$, and hence by (3.1) we get $\|z_q^m\| \le M + 1$.

Let us prove that $\|y_{q+1}^m - y_0\| < r$. We note that $y_{q+1}^m = y_q^m + h_{q+1}^m z_q^m = y_{q-1}^m + h_q^m z_{q-1}^m + h_{q+1}^m z_q^m$. By iterating we get

$$y_{q+1}^m = y_0^m + \sum_{s=0}^{q} h_{s+1}^m z_s^m. \tag{3.4}$$

Thus,

$$\|y_{q+1}^m - y_0^m\| \le \sum_{s=0}^{q} h_{s+1}^m \|z_s^m\| \le (M+1) \sum_{s=0}^{q} h_{s+1}^m \le (M+1)T < \frac{r}{4} < r.$$

To prove that $x_{q+1}^m \in B_\sigma(\varphi_0(0), r)$ we first use the induction technique to prove the relation

$$x_{p+1}^m = \varphi_0(0) + \big(\sum_{j=0}^{p} h_{j+1}^m\big) y_0 + \frac{1}{2} \sum_{j=0}^{p} (h_{j+1}^m)^2 \, z_j^m + \sum_{i=0}^{p-1} \sum_{j=i+1}^{p} h_{i+1}^m h_{j+1}^m z_i^m, \tag{3.5}$$

for $p = 1, \dots, q$. For $p = 1$ we note that

$$\begin{aligned}
x_2^m &= x_1^m + h_2^m y_1^m + \frac{1}{2} (h_2^m)^2 z_1^m \\
&= x_1^m + h_2^m (y_0^m + h_1^m z_0^m) + \frac{1}{2} (h_2^m)^2 z_1^m \\
&= (x_0^m + h_1^m y_0^m + \frac{1}{2} (h_1^m)^2 z_0^m) + h_2^m (y_0^m + h_1^m z_0^m) + \frac{1}{2} (h_2^m)^2 \, z_1^m \\
&= x_0^m + (h_1^m + h_2^m) y_0^m + \frac{1}{2} ((h_1^m)^2 z_0^m + (h_2^m)^2 z_1^m) + h_1^m h_2^m z_0^m \\
&= \varphi_0(0) + (\sum_{j=0}^{1} h_{j+1}^m) y_0^m + \frac{1}{2} \sum_{j=0}^{1} (h_{j+1}^m)^2 \, z_j^m + \sum_{j=1}^{1} h_1^m h_{j+1}^m z_0^m.
\end{aligned}$$

Then relation (3.5) is true for $p = 1$. Suppose that (3.5) is true for $p = q - 1$. This gives us

$$x_q^m = \varphi_0(0) + \sum_{j=0}^{q-1} h_{j+1}^m) y_0 + \frac{1}{2} \sum_{j=0}^{q-1} (h_{j+1}^m)^2 \, z_j^m + \sum_{i=0}^{q-2} \sum_{j=i+1}^{q-1} h_{i+1}^m h_{j+1}^m z_i^m.$$

So, according to the definition of x_{q+1}^m we have

$$\begin{aligned}
x_{q+1}^m &= x_q^m + h_{q+1}^m y_q^m + \frac{1}{2} (h_{q+1}^m)^2 z_q^m \\
&= \varphi_0(0) + (\sum_{j=0}^{q-1} h_{j+1}^m) y_0^m + \frac{1}{2} \sum_{j=0}^{q-1} (h_{j+1}^m)^2 \, z_j^m + \sum_{i=0}^{q-2} \sum_{j=i+1}^{q-1} h_{i+1}^m h_{j+1}^m z_i^m
\end{aligned}$$

$$
\begin{aligned}
&+ h_{q+1}^m (y_0 + \sum_{s=0}^{q-1} h_{s+1}^m z_s^m) + \frac{1}{2}(h_{q+1}^m) z_q^m \\
&= \varphi_0(0) + (\sum_{j=0}^{q} h_{j+1}^m) y_0^m + \frac{1}{2}\sum_{j=0}^{q} (h_{j+1}^m)^2\, z_j^m + \sum_{i=0}^{q-2}\sum_{j=i+1}^{q-1} h_{i+1}^m h_{j+1}^m z_i^m \\
&+ h_{q+1}^m (h_1^m z_0^m + h_2^m z_1^m + \dots + h_q^m z_{q-1}^m) \\
&= \varphi_0(0) + \sum_{j=0}^{q} h_{j+1}^m) y_0^m + \frac{1}{2}\sum_{j=0}^{q} (h_{j+1}^m)^2 z_j^m + \sum_{i=0}^{q-1}\sum_{j=i+1}^{q} h_{i+1}^m h_{j+1}^m z_i^m .
\end{aligned}
$$

This implies that the relation (3.5) is true for $p = q$. Now, from the fact that $\|z_p^m\| \le M + 1$, for all $p = 0, 1, \dots, q$ we get

$$
\begin{aligned}
&\|x_{q+1}^m - \varphi_0(0)\| \\
&\le \|y_0\| (\sum_{j=0}^{q} h_{j+1}^m) + \frac{1}{2}\sum_{j=0}^{q} (h_{j+1}^m)^2 (M+1) + (M+1)\sum_{i=0}^{q-1}\sum_{j=i+1}^{q} h_{i+1}^m h_{j+1}^m \\
&\le \|y_0\| T + \frac{1}{2}(M+1)T^2 + (M+1)\Big[h_1^m \sum_{j=1}^{q} h_{j+1}^m + h_2^m \sum_{j=2}^{q} h_{j+1}^m + \dots \\
&\quad + h_q^m \sum_{j=q}^{q} h_{j+1}^m\Big] \\
&\le \|y_0\| T + \frac{1}{2}(M+1)T^2 + (M+1)T^2 \\
&= \|y_0\| T + \frac{3}{2}(M+1)T^2 \\
&< \frac{r}{8} + \frac{3r}{8} = \frac{r}{2}.
\end{aligned}
$$

Thus (v) is true for $p = q$.

Let us prove (vi) for $p = q$, namely $\|x_m(t) - \varphi_0(0)\| < r$, for all $t \in [t_q^m, t_{q+1}^m]$. Let $t \in [t_q^m, t_{q+1}^m]$. We have

$$
\begin{aligned}
x_m(t) &= x_q^m + (t - t_q^m) y_q^m + \frac{1}{2}(t - t_q^m)^2\, z_q^m \\
&= \varphi_0(0) + (\sum_{j=0}^{q-1} h_{j+1}^m) y_0 + \frac{1}{2}\sum_{j=0}^{q-1} (h_{j+1}^m)^2\, z_j^m + \sum_{i=0}^{q-2}\sum_{j=i+1}^{q-1} h_{i+1}^m h_{j+1}^m z_i^m \\
&\quad + (t - t_q^m)(y_0 + \sum_{j=0}^{q-1} h_{j+1}^m z_j^m) + \frac{1}{2}(t - t_q^m)^2 z_q^m .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\|x_m(t) - \varphi_0(0)\| \\
&\le \|y_0\| (\sum_{j=0}^{q-1} h_{j+1}^m) + \frac{1}{2}\sum_{j=0}^{q-1} (h_{j+1}^m)^2 \|z_j^m\|
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=0}^{q-2}\sum_{j=i+1}^{q-1} h_{i+1}^m h_{j+1}^m \|z_i^m\| + h_{q+1}^m (y_0 + \sum_{j=0}^{q-1} h_{j+1}^m \|z_j^m\|) + \frac{1}{2}(h_{q+1}^m)^2 \|z_q^m\| \\
&\le \|y_0\| (\sum_{j=0}^{q} h_{j+1}^m) + \frac{1}{2}\sum_{j=0}^{q} (h_{j+1}^m)^2 \|z_j^m\| + \sum_{i=0}^{q-1}\sum_{j=i+1}^{q} h_{i+1}^m h_{j+1}^m \|z_i^m\| \\
&\le \|y_0\| T + \frac{1}{2} T^2 (M+1) + T^2 (M+1) \\
&\le \|y_0\| T + \frac{3(M+1)}{2} T^2 \\
&< \frac{r}{8} + \frac{r}{8} = \frac{r}{4}.
\end{aligned}
$$

We prove (vii) for $p = q$.

$$
\begin{aligned}
&\|\tau(t_{q+1}^m) x_m - \varphi_0\|_\sigma \\
&= \sup_{-\sigma \le s \le 0} \|\tau(t_{q+1}^m) x_m(s) - \varphi_0(s)\|. \\
&= \sup_{-\sigma \le s \le 0} \|x_m(t_{q+1}^m + s) - \varphi_0(s)\|. \\
&\le \sup_{-\sigma \le s \le -t_{q+1}^m} \|\tau(t_{q+1}^m) x_m(s) - \varphi_0(s)\| + \sup_{-t_{q+1}^m \le s \le 0} \|\tau(t_{q+1}^m) x_m(s) - \varphi_0(s)\|. \\
&\le \sup_{-\sigma \le s \le -t_{q+1}^m} \|\varphi_0(t_{q+1}^m + s) - \varphi_0(s)\| + \sup_{-t_{q+1}^m \le s \le 0} \|x_m(t_{q+1}^m + s) - \varphi_0(s)\| \\
&\quad + \sup_{-t_{q+1}^m \le s \le 0} \|\varphi_0(0) - \varphi_0(s)\|. \\
&\le \frac{r}{4} + \frac{r}{4} + \frac{r}{4} < r.
\end{aligned}
$$

It remains to show that there is a positive number ν_m such that $t_{\nu_m - 1}^m \le T < t_{\nu_m}^m$. Therefore, we have to prove that the iterative process is finite. For this purpose suppose that the iterative process is not finite. So, for each non negative integer number p, there are $t_p^m \in [0, T[$, x_p^m, y_p^m, z_p^m such that the relations (i)–(vii) are satisfied. Since the sequence $\{t_p^m\}_{p\ge 1}$ is bounded and increasing, there is $t_\alpha^m \in]0, T]$ such that $\lim_{p\to\infty} t_p^m = t_\alpha^m$. Let us show that $\{x_p^m\}_{p\ge1}, \{y_p^m\}_{p\ge1}$ are Cauchy sequences. Let p and q be two positive integers such that $p > q$. From the relation (3.5) we have

$$
\begin{aligned}
&\|x_p^m - x_q^m\| \\
&= \|(\sum_{j=0}^{p-1} h_{j+1}^m) y_0 + \frac{1}{2}\sum_{j=0}^{p-1} (h_{j+1}^m)^2\, z_j^m + \sum_{i=0}^{p-2}\sum_{j=i+1}^{p-1} h_{i+1}^m h_{j+1}^m z_i^m \\
&\quad - (\sum_{j=0}^{q-1} h_{j+1}^m) y_0 - \frac{1}{2}\sum_{j=0}^{q-1} (h_{j+1}^m)^2\, z_j^m - \sum_{i=0}^{q-2}\sum_{j=i+1}^{q-1} h_{i+1}^m h_{j+1}^m z_i^m \|. \\
&= \|(\sum_{j=q}^{p-1} h_{j+1}^m) y_0 + \frac{1}{2}\sum_{j=q}^{p-1} (h_{j+1}^m)^2 z_j^m + \sum_{i=0}^{q-1}\sum_{j=q}^{p-1} h_{i+1}^m h_{j+1}^m z_i^m \|. \\
&\le \sum_{j=q}^{p-1} h_{j+1}^m) \|y_0\| + \frac{1}{2}\sum_{j=q}^{p-1} (h_{j+1}^m)^2 \|z_j^m\| + \sum_{i=q-1}^{p-2}\sum_{j=i+1}^{p-1} h_{i+1}^m h_{j+1}^m \|z_i^m\|.
\end{aligned}
$$

$$
\begin{aligned}
&\leq \|y_0\|(t_p^m - t_q^m) + \frac{1}{2}(M+1)(t_p^m - t_q^m)^2 + (M+1)\sum_{j=q}^{p-1} h_q^m h_{j+1}^m \\
&\quad + (M+1)\sum_{j=q+1}^{p-1} h_{q+1}^m h_{j+1}^m + \dots + (M+1)\sum_{j=p-2}^{p-1} h_{p-2}^m h_{j+1}^m. \\
&= \|y_0\|(t_p^m - t_q^m) + \frac{1}{2}(M+1)(t_p^m - t_q^m)^2 + (M+1)h_q^m(t_p^m - t_q^m) \\
&\quad + (M+1)h_{q+1}^m(t_p^m - t_q^m) + \dots + (M+1)h_{p-2}^m(t_p^m - t_q^m). \\
&= \|y_0\|(t_p^m - t_q^m) + \frac{1}{2}(M+1)(t_p^m - t_q^m)^2 \\
&\quad + (M+1)(t_p^m - t_q^m)(h_q^m + h_{q+1}^m + \dots + h_{p-2}^m). \\
&= \|y_0\|(t_p^m - t_q^m) + \frac{1}{2}(M+1)(t_p^m - t_q^m)^2 + (M+1)(t_p^m - t_q^m)^2.
\end{aligned}
$$

Since the sequence $\{t_p^m\}_{p\geq 1}$ is convergent, the sequence $\{x_p^m\}_{p\geq 1}$ is Cauchy. Then there is $x_\alpha^m \in \mathbb{R}^n$ such that $\lim_{p\to\infty} x_p^m = x_\alpha^m$. Also,

$$
\|y_p^m - y_q^m\| = \|\sum_{s=q}^{p-1} h_{s+1}^m z_s^m\| \leq (M+1)(t_p^m - t_q^m).
$$

Thus the sequence $\{y_p^m\}_{p\geq 1}$ is a Cauchy sequence in $\mathbb{R}^n$. Hence there is $y_\alpha^m \in \mathbb{R}^n$ such that $y_\alpha^m = \lim_{p\to\infty} y_p^m$.

From property (v) we note that

$$
x_p^m \in P(x_p^m) \cap \overline{B(\varphi_0(0), r)} \subseteq K, \tag{3.6}
$$

and

$$
y_p^m \in \overline{B(y_0, r)} \subset \Omega.
$$

Thus $x_\alpha^m \in K$ and $y_\alpha^m \in \overline{B(y_0, r)} \subset \Omega$.

Now we put $x_m(t_\alpha^m) = x_\alpha^m$. To show that x_m is continuous at t_α^m let $\{s_p^m : p \geq 1\}$ be a sequence in $[0, t_\alpha^m[$ such that $\lim_{p\to\infty} s_p^m = t_\alpha^m$ and $t_p^m \leq s_p^m \leq t_{p+1}^m$ for every $p \geq 1$. We have

$$
\begin{aligned}
&\|x_m(s_p^m) - x_m(t_\alpha^m)\| \leq \|x_m(s_p^m) - x_m(t_p^m)\| + \|x_m(t_p^m) - x_\alpha^m\| \\
&\qquad \leq (s_p^m - t_p^m)\|y_p^m\| + \frac{1}{2}(s_p^m - t_p^m)^2(M+1) + \|x_p^m - x_\alpha^m\|.
\end{aligned}
$$

By taking the limit as $p \to \infty$, we obtain

$$
\lim_{p\to\infty} \|x_m(s_p^m) - x_m(t_\alpha^m)\| = 0
$$

which prove that x_m is continuous at t_α^m. Hence x_m is continuous on $[-\sigma, t_\alpha^m]$. Consequently,

$$
\lim_{p\to\infty} \tau(t_p^m)x_m = \tau(t_\alpha^m)x_m.
$$

Note that from (vii), $\tau(t_p^m)x_m \in K_0 \cap \overline{B_\sigma(\varphi_0, r)}$. Since the subset $K_0 \cap \overline{B_\sigma(\varphi_0, r)}$, is closed, we obtain

$$
\tau(t_\alpha^m)x_m \in K_0 \cap \overline{B_\sigma(\varphi_0, r)}\,.
$$

Furthermore, by (ii) and the relation (3.1), the sequences $\{z_p\}_{p\geq 1}$ and $\{u_p\}_{p\geq 1}$ are bounded in $\mathbb{R}^n$. So, there are two convergent subsequences, denoted again by, $\{z_p\}_{p\geq 1}$, $\{u_p\}_{p\geq 1}$. Thus there are two elements z_α^m, u_α^m of $\mathbb{R}^n$ such that $\lim_{p\to\infty} z_p^m = z_\alpha^m$, $\lim_{p\to\infty} u_p^m = u_\alpha^m$.

Now since F is upper semicontinuous on $K_0\times\Omega$ with compact values and since $u_p^m \in F(\tau(t_p^m)x_m, y_p^m)$, for all $p\geq 1$, it follows that $u_\alpha^m \in F(\tau(t_\alpha^m)x_m, y_\alpha^m)$. Applying condition (H3),

$$\lim_{h\to 0^+} d(x_m(t_\alpha^m) + hy_\alpha^m + \frac{h^2}{2}u_\alpha^m, P(x_m(t_\alpha^m))) = 0.$$

Hence, there is $h\in]0, T-t_\alpha^m[$ such that

$$d(x_\alpha^m + hy_\alpha^m + \frac{h^2}{2}u_\alpha^m, P(x_\alpha^m)) \leq \frac{h^2}{16m}. \tag{3.7}$$

We prove that h belongs to H_p^m for every p sufficient large. Since $\{t_p^m\}_p$ is an increasing sequence to t_α^m and since $\lim_{p\to\infty} x_p^m = x_\alpha^m$, $\lim_{p\to\infty} y_p^m = y_\alpha^m$ and $\lim_{p\to\infty} u_p^m = u_\alpha^m$. Then we can find a natural number p_1 such that for every $p > p_1$ we have $t_p^m < t_\alpha^m < t_p^m + h < t_\alpha^m + h$,

$$\|x_p^m - x_\alpha^m\| \leq \frac{h^2}{24m}, \tag{3.8}$$

$$\|y_p^m - y_\alpha^m\| \leq \frac{h}{24m}, \tag{3.9}$$

$$\|u_p^m - u_\alpha^m\| \leq \frac{1}{12m}\,. \tag{3.10}$$

From the lower semicontinuity of P at x_p^m, there is a natural number p_2 such that $P(x_\alpha^m)\subseteq P(x_p^m) + \frac{h^2}{16m}B(0,1)$, for all $p>p_2$. This gives that if $z\in\mathbb{R}^n$, then

$$d(z, P(x_p^m)) \leq d(z, P(x_\alpha^m)) + \frac{h^2}{16m}, \forall p > p_2. \tag{3.11}$$

Now let $p > \max(p_1, p_2)$. By (3.7)–(3.11), we have

$$\begin{aligned}
&d(x_p^m + hy_p^m + \frac{h^2}{2}u_p^m, P(x_p^m))\\
&\leq d(x_p^m + hy_p^m + \frac{h^2}{2}u_p^m, x_\alpha^m + hy_\alpha^m + \frac{h^2}{2}u_\alpha^m)\\
&\quad + d\big(x_\alpha^m + hy_\alpha^m + \frac{h^2}{2}u_\alpha^m, P(x_\alpha^m)\big) + \frac{h^2}{16m}\\
&\leq \|x_p^m - x_\alpha^m\| + h\|y_p^m - y_\alpha^m\| + \frac{h^2}{2}\|u_p^m - u_\alpha^m\| + \frac{h^2}{8m} + \frac{h^2}{16m}\\
&\leq \frac{h^2}{24m} + \frac{h^2}{24m} + \frac{h^2}{24m} + \frac{h^2}{16m} + \frac{h^2}{16m}\\
&= \frac{h^2}{8m} + \frac{h^2}{8m} = \frac{h^2}{4m}.
\end{aligned}$$

Thus $h\in H_p^m$, for all $p\geq \max(p_1,p_2)$. From the choice of h_p^m we have

$$\frac{1}{2}\sup H_p^m \leq h_p^m \leq \sup H_p^m.$$

Hence, $h_p^m \geq \frac{h}{2} > \frac{h}{4}$ for all $p\geq\max(p_1,p_2)$. This means that $\lim_{p\to\infty} h_p^m = \lim_{p\to\infty}(t_{p+1}^m - t_p^m)$ can not equal to zero, which contradicts with the fact that the sequence $\{t_p^m\}_{p\geq 1}$ is convergent. So, the process must be finite. $\square$

Theorem 3.2. *In addition to the assumptions of lemma 3.1 we suppose that the graph of P is closed and the following condition is satisfied.*

(H4) *for all $x \in K$ and all $y \in P(x)$ we have $P(y) \subseteq P(x)$.*

Then for all $(\varphi_0, y_0) \in K_0 \times \Omega$ there exist $T > 0$ and an absolutely continuous function $x : [0,T] \to K$, with absolutely continuous derivative such that

$$\begin{gathered} x''(t) \in F(\tau(t)x, x'(t)) \quad \text{a.e. on } [0,T] \\ x(t) = \varphi_0(t), \quad \forall t \in [-\sigma, 0] \\ x'(0) = y_0. \end{gathered}$$

Moreover, x is monotone with respect to P in the sense that for all $t \in [0,T]$ and all $s \in [t,T]$ we have $x(s) \in P(x(t))$; i.e., $0 \le t \le s \le T \Rightarrow x(t) \preceq x(s)$.

Proof. According to the definition of x_m, for all $m \ge 1$, all $p = 0, 1, 2, \dots, \nu_m - 1$ and all $t \in [t_p^m, t_{p+1}^m]$ we have

$$x'_m(t) = y_p^m + (t - t_p^m) z_p^m, \quad x''_m(t) = z_p^m \in F(\tau(t_p^m)x_m, y_p^m) + \frac{1}{m} B(0,1).$$

Then from (ii) and (v) of lemma 3.1 we get

$$\begin{aligned} \|x'_m(t)\| &\le \|y_p^m\| + h_{p+1}^m \|z_p^m\| \le \|y_0\| + r + T(M+1) \\ &\le \|y_0\| + r + \frac{r}{4}, \quad \forall t \in [0,T] \end{aligned} \tag{3.12}$$

and

$$\|x''_m(t)\| \le M + \frac{1}{m} \le M + 1, \quad \forall t \in [0,T]. \tag{3.13}$$

Then the sequences (x_m) and (x'_m) are equicontinuous in $C([0,T], \mathbb{R}^n)$. Applying Ascoli-Arzela theorem, there is a subsequence of (x_m),denoted again by (x_m), and an absolutely continuous function $x : [0,T] \to \mathbb{R}^n$ with absolutely continuous derivative x' such that (x_m) converges uniformly to x on $[0,T]$ and (x'_m) converges uniformly to x' on $[0,T]$ and (x''_m) converges weakly to x'' in $L^2([0,T], \mathbb{R}^n)$. Furthermore, since all the functions x_m equal φ_0 on $[-\sigma, 0]$, we can say that x_m converges uniformly to x on $[-\sigma, T]$ where $x = \varphi_0$ on $[-\sigma, 0]$.

Now, for each $t \in [0,T]$ and each $m \ge 1$, let $\delta_m(t) = t_p^m$, $\theta_m(t) = t_{p+1}^m$, if $t \in]t_p^m, t_{p+1}^m]$ and $\delta_m(0) = \theta_m(0) = 0$. For $t \in]t_p^m, t_{p+1}^m]$ we get

$$\begin{aligned} x''_m(t) = z_p^m &\in F(\tau(t_p^m)x_m, y_p^m) + \frac{1}{m} B(0,1). \\ &= F(\tau(\delta_m(t))x_m, x'_m(t_p^m)) + \frac{1}{m} B(0,1). \end{aligned}$$

Thus for all $m \ge 1$ and a.e. on $[0,T]$,

$$x''_m(t) \in F(\tau(\delta_m(t))x_m, x'_m(\delta_m(t))) + \frac{1}{m} B(0,1) \tag{3.14}$$

Also, for all $m \ge 1$ and all $t \in [0,T]$,

$$\tau(\theta_m(t))x_m \in B_\sigma(\varphi_0, r) \cap K_0 \tag{3.15}$$

$$x_m(t) \in B(\varphi_0(0), r) \tag{3.16}$$

$$x_m(\theta_m(t)) \in P(x_m(\delta_m(t))) \subseteq K \tag{3.17}$$

Claim: For each $t \in [0,T], \lim_{m\to\infty} \tau(\theta_m(t))x_m = \tau(t)x$ in $C([-\sigma,0],\mathbb{R}^n)$. Let $t \in [0,T]$. then

$$
\begin{aligned}
&\|\tau(\theta_m(t))x_m - \tau(t)x\|_\sigma \\
&\le \|\tau(\theta_m(t))x_m - \tau(t)x_m\|_\sigma + \|\tau(t)x_m - \tau(t)x\|_\sigma \\
&\le \sup_{-\sigma\le s\le 0} \|x_m(\theta_m(t)+s) - x_m(t+s)\| + \|\tau(t)x_m - \tau(t)x\|_\sigma \\
&\le \sup_{-\sigma\le s_1\le s_2\le T\,, |s_2-s_1|\le \frac{1}{m}} \|x_m(s_2) - x_m(s_1)\| + \|\tau(t)x_m - \tau(t)x\|_\sigma \\
&\le \sup_{-\sigma\le s_1\le s_2\le 0,\, |s_2-s_1|\le \frac{1}{m}} \|x_m(s_2) - x_m(s_1)\| \\
&\quad + \sup_{-\sigma\le s_1\le 0\le s_2\le T,\, |s_2-s_1|\le \frac{1}{m}} \|x_m(s_2) - x_m(s_1)\| \\
&\quad + \sup_{0\le s_1\le s_2\le T,\, |s_2-s_1|\le \frac{1}{m}} \|x_m(s_2) - x_m(s_1)\| + \|\tau(t)x_m - \tau(t)x\|_\sigma \\
&\le \sup_{-\sigma\le s_1\le s_2\le 0\,, |s_2-s_1|\le \frac{1}{m}} \|\varphi_0(s_2) - \varphi_0(s_1)\| \\
&\quad + \sup_{-\sigma\le s_1\le 0,\, |s_1|\le \frac{1}{m}} \|x_m(0) - x_m(s_1)\| + \sup_{0\le s_2\le T,\, |s_2|\le \frac{1}{m}} \|x_m(s_2) - x_m(0)\| \\
&\quad + \sup_{0\le s_1\le s_2\le T,\, |s_2-s_1|\le \frac{1}{m}} \|x_m(s_2) - x_m(s_1)\| + \|\tau(t)x_m - \tau(t)x\|_\sigma \\
&\le 2 \sup_{-\sigma\le s_1\le s_2\le 0,\, |s_2-s_1|\le \frac{1}{m}} \|\varphi_0(s_2) - \varphi_0(s_1)\| \\
&\quad + 2 \sup_{0\le s_1\le s_2\le T,\, |s_2-s_1|\le \frac{1}{m}} \|x_m(s_2) - x_m(s_1)\| + \|\tau(t)x_m - \tau(t)x\|_\sigma \,.
\end{aligned}
$$

Using the continuity of φ_0, the fact that (x'_m) is uniformly bounded, the uniform convergence of (x_m) towards x and the preceding estimate, we get

$$\lim_{m\to\infty} \|\tau(\theta_m(t))x_m - \tau(t)x\|_\sigma = 0.$$

Similarly, for each $t \in [0,T]$, $\lim_{m\to\infty} \tau(\delta_m(t))x_m = \tau(t)x$ in $C([-\sigma,0],\mathbb{R}^n)$. Also, since $\lim_{m\to\infty} \delta_m(t) = t$ and (x''_m) is uniformly bounded, then

$$\lim_{m\to\infty} x'_m(\delta_m(t)) = x'(t) \quad \forall t \in [0,T]. \tag{3.18}$$

Thus by the upper semicontinuity of F, and by (3.12), we obtain

$$x''(t) \in \overline{Co} F(\tau(t)x, x'(t)) \subseteq \partial V(x'(t)) \text{a.e. on } [0,T]. \tag{3.19}$$

Our aim now is proving the relation

$$x''(t) \in F(\tau(t)x, x'(t)) \quad \text{a.e. on } [0,T]. \tag{3.20}$$

Since F is upper semicontinuous with closed values, then by [1, prop. 1.1.2], the graph of F is closed in $[0,T]\times\mathbb{R}^n\times\mathbb{R}^n$. So, if we prove that the sequence (x''_m) has a subsequence converges strongly point wise to x'' then the relation (3.14) assures that the relation (3.20) is true.

In order to show that (x''_m) has a subsequence converges strongly point wise to x'', we note that the condition (H2) and property (ii) of Lemma 3.1 give

$$z_p^m - w_p^m \in F(\tau(t_p^m)x_m, y_p^m) \subseteq \partial V(y_p^m) = \partial V(x'_m(t_p^m)), \tag{3.21}$$

for $p = 0,1,2,\dots,\nu_m - 2$.

From the definition of the subdifferential of V, for $p = 0, 1, 2, \dots, \nu_m - 2$, we have

$$\begin{aligned}
V(x'_m(t^m_{p+1})) - V(x'_m(t^m_p)) &\geq \langle z^m_p - w^m_p, x'_m(t^m_{p+1}) - x'_m(t^m_p)\rangle \\
&= \langle z^m_p - w^m_p, \int_{t^m_p}^{t^m_{p+1}} x''_m(s)\,ds\rangle \\
&= \langle z^m_p, z^m_p(t^m_{p+1} - t^m_p)\rangle - \langle w^m_p, \int_{t^m_p}^{t^m_{p+1}} x''_m(s)\,ds\rangle \\
&= h^m_{p+1}\|z^m_p\|^2 - \langle w^m_p, \int_{t^m_p}^{t^m_{p+1}} x''_m(s)ds\rangle \\
&= \int_{t^m_p}^{t^m_{p+1}} \|x''_m(s)\|^2 ds - \langle w^m_p, \int_{t^m_p}^{t^m_{p+1}} x''_m(s)ds\rangle
\end{aligned} \tag{3.22}$$

Analogously,

$$\begin{aligned}
V(x'_m(T)) - V(x'_m(t^m_{\nu_m-1})) &\geq \langle z^m_{\nu_m-1} - w^m_{\nu_m-1}, \int_{t^m_{\nu_m-1}}^{T} x''_m(s)\,ds\rangle \\
&= \int_{t^m_{\nu_m-1}}^{T} \|x''_m(s)\|^2\,ds - \langle w^m_{\nu_m-1}, \int_{t_{\nu_m-1}}^{T} x''_m(s)\,ds\rangle
\end{aligned} \tag{3.23}$$

By adding the $\nu_m - 1$ inequalities from (3.22) and the inequality (3.23), we get

$$\begin{aligned}
&V(x'_m(T)) - V(x'_m(0)) \\
&= V(x'_m(T)) - V(x'_m(t^m_{\nu_m-1})) + V(x'_m(t^m_{\nu_m-1})) - V(x'_m(t^m_{\nu_m-2})) + \dots \\
&\quad + V(x'_m(t^m_1)) - V(x'_m(0)) \\
&\geq \int_0^T \|x''_m(s)\|^2\,ds - \sum_{p=0}^{\nu_m-2} \langle w^m_p, \int_{t^m_p}^{t^m_{p+1}} x''_m(s)ds\rangle - \langle w^m_{\nu_m-1}, \int_{t^m_{\nu_m-1}}^{T} x''_m(s)\,ds\rangle
\end{aligned} \tag{3.24}$$

Now,

$$\begin{aligned}
&\sum_{p=0}^{\nu_m-2} |\langle w^m_p, \int_{t^m_p}^{t^m_{p+1}} x''_m(s)\,ds\rangle| + |\langle w^m_{\nu_m-1}, \int_{t^m_{\nu_m-1}}^{T} x''_m(s)\,ds\rangle| \\
&\leq \sum_{p=0}^{\nu_m-2} \|w^m_p\|(M+1)(t^m_{p+1} - t^m_p) + \|w^m_{\nu_m-1}\|(M+1)(T - t^m_{\nu_m-1}) \\
&\leq \frac{T(M+1)}{m}
\end{aligned}$$

Hence, by passing to the limit as $m \to \infty$ in (3.24) we obtain

$$V(x'_m(T)) - V(y_0) \geq \lim_{m\to\infty} \sup \int_0^T \|x''_m(s)\|^2 ds. \tag{3.25}$$

On the other hand from relation (3.19) and [5, Lemma 3.3], we obtain

$$\frac{d}{dt} V(x'(t)) = \|x''(t)\|^2, \quad \text{a.e. on } [0, T]\,.$$

Thus, $V(x'(T)) - V(x'(0)) = \int_0^T \|x''(s)\|^2 ds$, which yields directly that

$$V(x'(T)) - V(y_0) = \int_0^T \|x''(s)\|^2 ds \tag{3.26}$$

Therefore, by (3.25) and (3.26), we get

$$\int_0^T \|x''(s)\|^2 ds \geq \limsup_{m\to\infty} \int_0^T \|x_m''(s)\|^2\, ds \tag{3.27}$$

Since (x_m'') converges weakly to x'' in $L^2([0,T],\mathbb{R}^n)$. hence

$$\int_0^T \|x''(s)\|^2\, ds \leq \liminf_{m\to\infty} \int_0^T \|x_m''(s)\|^2\, ds \tag{3.28}$$

By (3.27) and (3.28), we obtain

$$\lim_{m\to\infty} \int_0^T \|x_m''(s)\|^2\, ds = \int_0^T \|x''(s)\|^2\, ds,$$

this means that the sequences (x_m'') converges strongly to x'' in $L^2([0,T],\mathbb{R}^n)$. Consequently there is a subsequence of (x_m''), denoted again by (x_m''), converges point wise to x''. From the facts that the graph of F is closed, $\tau(t)x_m$ converges uniformly to $\tau(t)x$, (x_m') converges uniformly to x' and (x_m'') converges point wise to $x''$the relation(18) is proved.

It remains to prove the following two properties:

(1) $(x(t), x'(t)) \in K \times \Omega$, for all $t \in [0,T]$.
(2) $x(s) \in P(x(t))$ for all $t, s \in [0,T]$ and $t \leq s$.

To prove the first property we note that the property (iii) of l Lemma 3.1 implies that $x_m(\delta_m(t)) \in \overline{B(\varphi_0(0), r)} \cap K$ and $x_m'(\delta_m(t)) \in \overline{B(y_0, r)} \cap \Omega$. Since $\lim_{m\to\infty} x_m(\delta_m(t)) = x(t)$ and $\lim_{m\to\infty} x_m'(\delta_m(t)) = x'(t)$ then $x(t) \in \overline{B(\varphi_0(0), r)} \cap K$ and $x'(t) \in \overline{B(y_0, r)} \cap \Omega$.

To prove the second property, let $t, s \in [0,T]$ be such that $t \leq s$. Then for m large enough, we can find p, $q \in \{0, 1, 2, \dots, \nu_m - 2\}$ such that $p > q$, $t \in [t_q^m, t_{q+1}^m]$ and $s \in [t_p^m, t_{p+1}^m]$. Assume that $j = p - q$. Using property (v) of Lemma 3.1 and condition (H4) we get

$$P(x_m(t_p^m)) \subseteq P(x_m(t_{p-1}^m)) \subseteq P(x_m(t_{p-2}^m)) \subseteq \cdots \subseteq P(x_m(t_q^m)).$$

This implies $P(x_m(\delta_m(s))) \subseteq P(x_m(\delta_m(t)))$. Since $x_m(\delta_m(s)) \in P(x_m(\delta_m(s)))$, it follows that $P(x_m(\delta_m(t)))$ and hence the second property is proved. $\square$

AS an example, let $K = \mathbb{R}$ and $P(x) = [x, \infty)$. Then $x \preceq y$ if and only if $y \in P(x)$; i.e., if and only if $x \leq y$. Then the solution obtained above is monotone in the usual sense.

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