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STABILITY AND APPROXIMATIONS OF EIGENVALUES AND EIGENFUNCTIONS FOR THE NEUMANN LAPLACIAN, PART I

RODRIGO BAÑUELOS, MICHAEL M. H. PANG

Abstract. We investigate stability and approximation properties of the lowest nonzero eigenvalue and corresponding eigenfunction of the Neumann Laplacian on domains satisfying a heat kernel bound condition. The results and proofs in this paper will be used and extended in a sequel paper to obtain stability results for domains in $\mathbb{R}^2$ with a snowflake type boundary.

1. Introduction

The goal of this paper and its sequel [13] is to prove stability results for the smallest positive Neumann eigenvalue and its associated eigenfunctions of domains in $\mathbb{R}^2$ with a snowflake type fractal boundary. In particular, our goal is that our results should apply to the Koch snowflake domain and its usual sequence of approximating polygons from inside.

Suppose the Neumann Laplacian $-\Delta_\Omega \geq 0$ defined on a domain Ω in $\mathbb{R}^d$ has discrete spectrum. The numerical computation of its eigenvalues and eigenfunctions often assumes that if Ω is replaced by an approximating domain with polygonal or piecewise smooth boundary, then the eigenvalues and eigenfunctions will not change too much. This continuous dependence of the Neumann eigenvalues and eigenfunctions on the domain, however, is not obvious. Moreover, it is known that even if Ω has smooth boundary, the spectrum of its Neumann Laplacian does not necessarily remain discrete under "small" perturbations (see [8, 9]). Therefore the approximating domain, apart from being "close" to Ω, must also satisfy some "regularity conditions". Burenkov and Davies [3] studied this problem when Ω and its approximating domain both have a boundary satisfying a uniform Holder condition, and they obtained explicit estimates for the change in the Neumann eigenvalues. More recently, Renka [14], Benjai [1], and Neuberger, Sieben and Swift [10] have numerically computed the Neumann eigenvalues and eigenfunctions of the Koch snowflake domain. However, the boundary of the Koch snowflake domain does not satisfy a uniform Holder condition. This motivates us to prove stability results for the Neumann eigenvalues and eigenfunctions starting from a different set of assumptions. In this paper we mainly consider the case when the lowest positive Neumann eigenvalue has multiplicity 1. In [13] we shall extend these results, and

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their ideas of proof, to the case when the lowest positive Neumann eigenvalue has multiplicity at least 2, and show that these results apply to the Koch snowflake domain and its usual sequence of approximating polygons from inside.

Let Ω be a bounded domain in $\mathbb{R}^d$ and, for all sufficiently small $\delta > 0$, say $0 < \delta < \delta_0$, let Ω_δ be a subdomain of Ω satisfying

$$\partial\Omega_\delta \subseteq \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \le \delta\}. \tag{1.1}$$

Let $P_t^{\Omega}(x,y)$ and $P_t^{\Omega_\delta}(x,y)$ be the heat kernels corresponding to the semigroups generated by $-\Delta_\Omega$ and $-\Delta_{\Omega_\delta}$, respectively.

Hypothesis 1.1. Our main assumption on Ω and Ω_δ is that there exist $c_0 \ge 1$ and $N > 0$ such that, for all $0 < t \le 1$ and all $x, y \in \Omega$,

$$P_t^{\Omega}(x,y) \le c_0 t^{-N/2} \tag{1.2}$$

and that, for all $0 < \delta < \delta_0$, all $0 < t \le 1$ and all $x, y \in \Omega_\delta$,

$$P_t^{\Omega_\delta}(x,y) \le c_0 t^{-N/2}. \tag{1.3}$$

Remark 1.2. (i) Let Ω is the Koch snowflake domain in $\mathbb{R}^2$ and Ω_δ be its usual approximating polygons from inside. Then (1.2) is true by [6, Theorem 5.2]. Since each of the approximating polygons Ω_δ is a Lipschitz domain, an upper heat kernel bound of the form (1.3) holds for each Ω_δ with $N = 2$ (see ([5, Section 2.4]). In [13] we shall show that there exists $c_0 \ge 1$ such that (1.3) holds for all the approximating polygons with $N = 2$.

(ii) We note that since $P_t^{\Omega}(x,y)$ is continuous on $(0,\infty) \times \Omega \times \Omega$, by the parabolic Harnack inequality (see, for example, Lemma 2.1 below), if Ω has the extension property, then (1.2) holds with $N = d$ and some $c_0 > 0$ (see [5, p.77]).

Under Hypothesis 1.1, $-\Delta_\Omega$ and $-\Delta_{\Omega_\delta}$ have compact resolvent (see [5, p. 61]). We let $0 < \mu_2 \le \mu_3 \le \dots$ be the eigenvalues of $-\Delta_\Omega$, counting multiplicity, and let $\varphi_2, \varphi_3, \varphi_4, \dots$ be the eigenfunction associated to $\mu_2, \mu_3, \mu_4, \dots$ respectively. We assume that $|\Omega|^{-1/2}, \varphi_2, \varphi_3, \dots$ form a complete orthonormal system on $L^2(\Omega)$. We let $0 < \mu_2^\delta \le \mu_3^\delta \le \dots$ and $\varphi_2^\delta, \varphi_3^\delta, \varphi_4^\delta, \dots$ be the corresponding quantities for the Neumann Laplacian $-\Delta_{\Omega_\delta}$ on Ω_δ.

Theorem 1.3. *Suppose Ω and Ω_δ satisfy Hypothesis 1.1. Then*

$$\lim_{\delta\downarrow 0} \mu_2^\delta = \mu_2. \tag{1.4}$$

Theorem 1.4. *Suppose Ω and Ω_δ satisfy Hypothesis 1.1. If μ_2 has multiplicity 1, then there exists $\delta_1 > 0$ such that*

$$\mu_3^\delta \ge \mu_2 + \delta_1 \tag{1.5}$$

for all $0 < \delta < \delta_1$. Hence, from (1.4) and (1.5), μ_2^δ has multiplicity 1 for all $0 < \delta < \delta_1$.

Theorem 1.5. *Suppose Ω and Ω_δ satisfy Hypothesis 1.1 and assumed that μ_2 has multiplicity 1. If Ω' is a subdomain of Ω such that $\overline{\Omega'} \subseteq \Omega$, then*

$$\lim_{\delta\downarrow 0} \sup_{z \in \overline{\Omega'}} |\varphi_2^\delta(z) - \varphi_2(z)| = 0. \tag{1.6}$$

Remark 1.6. (i) Using Theorem 1.4 in the following example, one can show that multiplicity 2 of μ_2 is not stable under small perturbations. Let $\Omega(t)$, $0 \le t \le 1$, be a continuous family of convex deformations from a long thin rectangle to a square. That is,

(a) $\Omega(t)$ is a convex domain for $0 \le t \le 1$,
(b) $\Omega(0)$ is a long thin rectangle and $\Omega(1)$ is a square.

We can assume that $\Omega(t)$ is symmetric with respect to the x and y axes for all $t \in [0,1]$. For each $t \in [0,1]$, let $\mu(t)$ be the smallest non-zero Neumann eigenvalue of $\Omega(t)$. Then $\mu(0)$ and $\mu(1)$ have multiplicity 1 and 2, respectively. So we can let

$$t_0 = \inf\{t \in [0,1] : \mu(t) \text{has multiplicity } 2\}.$$

Let $\{t_n\}_{n=1}^{\infty}$ be a decreasing sequence of numbers in $[0,1]$ such that $t_n \downarrow t_0$ as $n \to \infty$ and that $\mu(t_n)$ has multiplicity 2 for all $n = 1, 2, 3, \dots$. Since $\operatorname{dist}(\partial\Omega(t_0), \partial\Omega(t_n)) \to 0$ as $n \to \infty$ and since the domains $\Omega(t)$ are convex and symmetric with respect to the x and y axes, for each $n = 1, 2, 3, \dots$ we can let $D(t_n)$ be an dilation of $\Omega(t_n)$ such that

(c) $D(t_n) \subseteq \Omega(t_0)$,
(d) $\operatorname{dist}(\partial\Omega(t_0), \partial D(t_n)) \to 0$ as $n \to \infty$.

Let λ_n be the smallest nonzero Neumann eigenvalue of $D(t_n)$. Then, since $D(t_n)$ is a dilation of $\Omega(t_n)$, the multiplicity of λ_n is the same as that of $\mu(t_n)$; i.e., λ_n has multiplicity 2 for all $n = 1, 2, 3, \dots$. Then, by Theorem 1.4, $\mu(t_0)$ must have multiplicity 2. In particular, $t_0 > 0$. Let $\{s_n\}_{n=1}^{\infty}$ be an increasing sequence on $[0,1]$ such that $s_n \uparrow t_0$ as $n \to \infty$. Then, just as for $\{t_n\}_{n=1}^{\infty}$ before, we can let $D(s_n)$ be a dilation of $\Omega(s_n)$ satisfying

(e) $D(s_n) \subseteq \Omega(t_0)$,
(f) $\operatorname{dist}(\partial\Omega(t_0), \partial D(s_n)) \to 0$ as $n \to \infty$.

Let ζ_n be the smallest nonzero Neumann eigenvalue of $D(s_n)$. Then ζ_n has the same multiplicity as that of $\mu(s_n)$ since $D(s_n)$ is a dilation of $\Omega(s_n)$. But, by the definition of t_0, ζ_n has multiplicity 1 for all $n = 1, 2, 3, \dots$.

(ii) Theorems 1.3, 1.4, and 1.5 will be extended to the case when μ_2 has multiplicity at least 2 in [13]. With an additional inductive argument, it is possible to extend these results to all higher Neumann eigenvalues and eigenfunctions and to more general elliptic operators, including some non-uniformly elliptic operators. We plan to return to these issues in a later paper.

(iii) We mention that spectral stability results for the Dirichlet Laplacian are much more extensive than those for the Neumann Laplacian. Sharp rates for the convergence of Dirichlet eigenvalues and eigenfunctions can be found in [12] and [7]. We refer the readers to the excellent article [4] for a recent survey of spectral stability results for the Dirichlet and Neumann Laplacians and for more general elliptic operators.

2. Proofs of Theorems 1.3, 1.4 and 1.5

Lemma 2.1 (see [11, Lemma 4.10])**.** *Let Σ be a domain in $\mathbb{R}^d$, let u be a solution of the parabolic equation*

$$\frac{\partial u}{\partial t} - \omega^{-1} \sum_{i,j=1}^{d} \Big\{ \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big)\Big\} = 0$$

in $\sum \times(\tau_1, \tau_2)$*, where* ω *and* $\{a_{ij}\}$ *satisfy*

$$\begin{gathered} 0 < \lambda^{-1} \leq \{a_{ij}(x)\} \leq \lambda < \infty \\ 0 < \lambda^{-1} \leq \omega(x) \leq \lambda < \infty \end{gathered} \quad (x \in \Sigma)$$

for some $\lambda \geq 1$*. Let* Σ' *be a subdomain of* Σ *and suppose that*

$$\operatorname{dist}(\Sigma', \partial\Sigma) > \eta \quad \text{and} \quad t_1 - \tau_1 \geq \eta^2.$$

Then

$$|u(x,t) - u(y,s)| \leq A[|x-y| + (t-s)^{1/2}\}^\alpha$$

for all $x, y \in \Sigma'$ *and* $t, s \in [t_1, \tau_2]$*, where* α *depends only on* d *and* λ *and*

$$A = \big(\frac{4}{\eta}\big)^\alpha \theta$$

where θ *is the oscillation of* u *in* $\Sigma \times (\tau_1, \tau_2)$*.*

Lemma 2.2 ([2])**.** *Let* Ω *and* Ω_δ*,* $0 < \delta \leq \delta_0$*, be as described in Section 1. Let* T_t^Ω *and* $T_t^{\Omega_\delta}$ *be the semigroups generated by the Neumann Laplacians* $-\Delta_\Omega$ *and* $-\Delta_{\Omega_\delta}$ *on* Ω *and* Ω_δ*, respectively. Then, for all* $f \in L^\infty(\Omega)$ *and compact subset* $K \subseteq \Omega$*, we have*

$$\lim_{\delta \downarrow 0} T_t^{\Omega_\delta}(f 1_{\Omega_\delta})(x) = T_t^\Omega f(x) \quad (a.e.\ x \in K)$$

Proposition 2.3. *For all* $t_0 \in (0,1]$ *and all* $x_0, y_0 \in \Omega$*, we have*

$$\lim_{\delta \downarrow 0} P_{t_0}^{\Omega_\delta}(x_0, y_0) = P_{t_0}^\Omega(x_0, y_0).$$

Proof. Applying Lemma 2.1 with

$$\begin{gathered} \Sigma = \Omega, \quad \tau_1 = \frac{1}{4} t_0, \quad \tau_2 = 1, \\ u(x,t) = P_t^\Omega(x, y_0), \quad \lambda = 1, \quad \omega(x) \equiv 1, \\ \Sigma' = B\Big(x_0, \frac{1}{4}\operatorname{dist}(x_0, \partial\Omega)\Big), \quad t_1 = \frac{1}{2} t_0, \\ \eta = \min\big\{\frac{1}{4}\operatorname{dist}(x_0, \partial\Omega), \frac{1}{2} t_0^{1/2}\big\}, \end{gathered}$$

we obtain, for all $t \in (t_1, \tau_2)$ and $x \in B(x_0, \frac{1}{4}\operatorname{dist}(x_0, \partial\Omega))$,

$$|P_t^\Omega(x, y_0) - P_t^\Omega(x_0, y_0)| \leq A|x - x_0|^\alpha \tag{2.1}$$

where $\alpha \in (0,1]$ depends only on d, and $A > 0$ depends only on d, $\operatorname{dist}(x_0, \partial\Omega)$, t_0, N and c_0 in (1.2). Similarly, we deduce that

$$|P_t^{\Omega_\delta}(x, y_0) - P_t^{\Omega_\delta}(x_0, y_0)| \leq A|x - x_0|^\alpha, \tag{2.2}$$

where α and A in (2.2) have the same values as in (2.1) for all $\delta > 0$ satisfying

$$0 < \delta < \min\big\{\delta_0, \frac{1}{2}\operatorname{dist}(x_0, \partial\Omega)\big\}.$$

For all $0 < r < \frac{1}{4}\operatorname{dist}(x_0, \partial\Omega)$, we have

$$|B(x_0, r)|^{-1} \int_{B(x_0,r)} P_{t_0}^{\Omega_\delta}(x, y_0)\, dx = P_{t_0}^{\Omega_\delta}(x_0, y_0) + \beta_1(\delta, t_0, x_0, y_0, r) \tag{2.3}$$

where, by (2.2), we have

$$|\beta_1(t_0, \delta, x_0, y_0, r)| \leq Ar^\alpha. \tag{2.4}$$

Similarly, we have

$$|B(x_0,r)|^{-1}\int_{B(x_0,r)} P_{t_0}^{\Omega}(x,y_0)\,dx = P_{t_0}^{\Omega}(x_0,y_0)+\beta_2(t_0,x_0,y_0,r) \tag{2.5}$$

where

$$|\beta_2(t_0,x_0,y_0,r)| \le Ar^{\alpha}. \tag{2.6}$$

Applying Lemma 2.2 to the left side of (2.3) and (2.5), we see that as $\delta \downarrow 0$ we have

$$P_{t_0}^{\Omega_\delta}(x_0,y_0)+\beta_1(\delta,t_0,x_0,y_0,r) \to P_{t_0}^{\Omega}(x_0,y_0)+\beta_2(t_0,x_0,y_0,r) \tag{2.7}$$

Let $\epsilon > 0$ be given. Then we can first fix $r \in \big(0, \frac14 \operatorname{dist}(x_0,\partial\Omega)\big)$ such that

$$0 < r \le \Big(\frac{\epsilon}{3A}\Big)^{1/\alpha}. \tag{2.8}$$

By (2.7), there exists $\delta_2 > 0$ such that, for all $0 < r < \frac14 \operatorname{dist}(x_0,\partial\Omega)$ and $0<\delta<\delta_2$, we have

$$|P_{t_0}^{\Omega_\delta}(x_0,y_0) - P_{t_0}^{\Omega}(x_0,y_0) + \beta_1 - \beta_2| < \frac{\epsilon}{3}. \tag{2.9}$$

Thus, by (2.4), (2.6), (2.8) and (2.9), we see that for all $\delta \in (0,\delta_2)$ we have

$$|P_{t_0}^{\Omega_\delta}(x_0,y_0) - P_{t_0}^{\Omega}(x_0,y_0)| \le \epsilon.$$

□

Notation. (i) Let $f:\Omega\to\mathbb{R}$ be a function on Ω. Then we write $R_\delta f:\Omega_\delta\to\mathbb{R}$ for the restriction of f to Ω_δ; i.e., $R_\delta f = 1_{\Omega_\delta} f$.

(ii) Let $f:\Omega_\delta\to\mathbb{R}$ be a function on Ω_δ. Then we write $E_\delta f:\Omega\to\mathbb{R}$ for the extension of f to Ω defined by

$$E_\delta f(x) = \begin{cases} f(x) & x\in\Omega_\delta \\ 0 & x\in\Omega\backslash\Omega_\delta. \end{cases}$$

Proposition 2.4. *Let $M \ge 0$ be a fixed number. For all sufficiently small $\delta > 0$, let $f_\delta \in L^\infty(\Omega_\delta)$ such that*

$$\|f_\delta\|_\infty \le M.$$

Then, for all $0 < t \le 1$,

$$\|T_t^{\Omega}(E_\delta f_\delta) - E_\delta(T_t^{\Omega_\delta} f_\delta)\|_{L^2(\Omega)} \to 0$$

as $\delta \downarrow 0$.

Proof. Let $t\in(0,1]$ and $\epsilon\in(0,1)$ be fixed. Choose $\delta_3>0$ sufficiently small so that

$$2c_0 t^{-\frac{N}{2}} M|\Omega\backslash\Omega_{\delta_3}| \le \frac{\epsilon}{2} \tag{2.10}$$

and

$$2c_0 t^{-\frac{N}{2}} M|\Omega||\Omega\backslash\Omega_{\delta_3}|^{1/2} \le \frac{\epsilon}{2}. \tag{2.11}$$

Then, for all $\delta\in(0,\delta_3]$ and $x\in\Omega_{\delta_3}$, we have

$$\begin{aligned} &(T_t^{\Omega} E_\delta f_\delta - E_\delta T_t^{\Omega_\delta} f_\delta)(x) \\ &= \int_{\Omega_\delta} [P_t^{\Omega}(x,y) - P_t^{\Omega_\delta}(x,y)] f_\delta(y)dy \\ &= \Big(\int_{\Omega_\delta\backslash\Omega_{\delta_3}} + \int_{\Omega_{\delta_3}}\Big)\big[P_t^{\Omega}(x,y) - P_t^{\Omega_\delta}(x,y)\big] f_\delta(y)dy. \end{aligned} \tag{2.12}$$

By (1.2), (1.3) and (2.10), we have

$$\Big|\int_{\Omega_\delta\backslash\Omega_{\delta_3}}[P_t^{\Omega}(x,y)-P_t^{\Omega_\delta}(x,y)]f_\delta(y)dy\Big|\le 2c_0t^{-\frac{N}{2}}M|\Omega\backslash\Omega_{\delta_3}|\le\frac{\epsilon}{2}. \tag{2.13}$$

By (1.2), (1.3) and Proposition 2.3, there exists $\delta_4=\delta_4(\epsilon,x)>0$ such that

$$\Big|\int_{\Omega_{\delta_3}}[P_t^{\Omega}(x,y)-P_t^{\Omega_\delta}(x,y)]f_\delta(y)dy\Big|\le\frac{\epsilon}{2} \tag{2.14}$$

for all $\delta\in(0,\delta_4)$. Therefore (2.12), (2.13) and (2.14) imply that

$$(T_t^{\Omega}E_\delta f_\delta-E_\delta T_t^{\Omega_\delta}f_\delta)(x)\to 0\quad\text{as }\delta\downarrow 0 \tag{2.15}$$

for all $x\in\Omega_{\delta_3}$. Since, by (1.2) and (1.3),

$$\|T_t^{\Omega}E_\delta f_\delta-E_\delta T_t^{\Omega_\delta}f_\delta\|_\infty\le 2c_0t^{-N/2}M|\Omega|,$$

there exists $\delta_5=\delta_5(\epsilon)>0$ such that

$$\|R_{\delta_3}(T_t^{\Omega}E_\delta f_\delta-E_\delta T_t^{\Omega_\delta}f_\delta)\|^2_{L^2(\Omega_{\delta_3})}\le\frac{\epsilon^2}{4} \tag{2.16}$$

for all $\delta\in(0,\delta_5)$. Also, by (2.11), we have

$$\int_{\Omega\backslash\Omega_{\delta_3}}|(T_t^{\Omega}E_\delta f_\delta-E_\delta T_t^{\Omega_\delta}f_\delta)(x)|^2dx\le(2c_0t^{-\frac{N}{2}}M|\Omega|)^2|\Omega\backslash\Omega_{\delta_3}|\le\frac{\epsilon^2}{4} \tag{2.17}$$

for all $\delta\in(0,\delta_3)$. The proposition now follows from (2.16) and (2.17). □

Proof of Theorem 1.3. Let $\epsilon\in(0,1)$ be given. For all sufficiently small $\delta>0$, let

$$\beta_1(\delta)=|\Omega_\delta|^{-1}\int_{\Omega_\delta}\varphi_2(x)dx.$$

Taking inner products and norms in $L^2(\Omega_\delta)$, we get

$$\begin{aligned}e^{-\mu_2^\delta t}&\ge\|R_\delta(\varphi_2-\beta_1(\delta))\|_2^{-2}\langle T_t^{\Omega_\delta}R_\delta(\varphi_2-\beta_1(\delta)),R_\delta(\varphi_2-\beta_1(\delta))\rangle\\&=\|R_\delta(\varphi_2-\beta_1(\delta))\|_2^{-2}\{\langle T_t^{\Omega_\delta}R_\delta\varphi_2,R_\delta\varphi_2\rangle-2\langle R_\delta\varphi_2,\beta_1(\delta)1_{\Omega_\delta}\rangle+\beta_1(\delta)^2|\Omega_\delta|\}.\end{aligned}$$

So for $0<\delta<\delta_6$, we have

$$\begin{aligned}e^{-\mu_2^\delta t}&\ge\|R_\delta(\varphi_2-\beta_1(\delta))\|_2^{-2}\{\langle T_t^{\Omega_\delta}(R_\delta\varphi_2-(R_\delta\varphi_2)1_{\Omega_{\delta_6}})\\&\quad+T_t^{\Omega_\delta}((R_\delta\varphi_2)1_{\Omega_{\delta_6}}),(R_\delta\varphi_2-(R_\delta\varphi_2)1_{\Omega_{\delta_6}})\\&\quad+(R_\delta\varphi_2)1_{\Omega_{\delta_6}}\rangle-2\langle R_\delta\varphi_2,\ \beta_1(\delta)1_{\Omega_\delta}\rangle+\beta_1(\delta)^2|\Omega_\delta|\}\\&=\|R_\delta(\varphi_2-\beta_1(\delta))\|_2^{-2}\{\langle T_t^{\Omega_\delta}(R_\delta\varphi_2-(R_\delta\varphi_2)1_{\Omega_{\delta_6}}),\\&\quad(R_\delta\varphi_2-(R_\delta\varphi_2)1_{\Omega_{\delta_6}})\rangle+2\langle T_t^{\Omega_\delta}(R_\delta\varphi_2-(R_\delta\varphi_2)1_{\Omega_{\delta_6}}),\\&\quad(R_\delta\varphi_2)1_{\Omega_{\delta_6}}\rangle+\langle T_t^{\Omega_\delta}(R_\delta\varphi_2)1_{\Omega_{\delta_6}},(R_\delta\varphi_2)1_{\Omega_{\delta_6}}\rangle\\&\quad-2\langle R_\delta\varphi_2,\ \beta_1(\delta)1_{\Omega_{\delta_6}}\rangle+\beta_1(\delta)^2|\Omega_\delta|\}.\end{aligned} \tag{2.18}$$

But

$$\begin{aligned}
&\langle T_t^{\Omega_\delta}((R_\delta\varphi_2)1_{\Omega_{\delta_6}}),(R_\delta\varphi_2)1_{\Omega_{\delta_6}}\rangle\\
&=\langle T_t^{\Omega_\delta}((R_\delta\varphi_2)1_{\Omega_{\delta_6}})-R_\delta T_t^{\Omega}(\varphi_2 1_{\Omega_{\delta_6}}),(R_\delta\varphi_2)1_{\Omega_{\delta_6}}\rangle\\
&\quad+\langle R_\delta T_t^{\Omega}(\varphi_2 1_{\Omega_{\delta_6}}),(R_\delta\varphi_2)1_{\Omega_{\delta_6}}\rangle\\
&=\langle T_t^{\Omega_\delta}((R_\delta\varphi_2)1_{\Omega_{\delta_6}})-R_\delta T_t^{\Omega}(\varphi_2 1_{\Omega_{\delta_6}}),(R_\delta\varphi_2)1_{\Omega_{\delta_6}}\rangle\\
&\quad+\langle R_\delta T_t^{\Omega}(\varphi_2 1_{\Omega_{\delta_6}})-R_\delta T_t^{\Omega}\varphi_2+R_\delta T_t^{\Omega}\varphi_2,\ (R_\delta\varphi_2)1_{\Omega_{\delta_6}}-R_\delta\varphi_2+R_\delta\varphi_2\rangle\\
&=\langle T_t^{\Omega_\delta}((R_\delta\varphi_2)1_{\Omega_{\delta_6}})-R_\delta T_t^{\Omega}(\varphi_2 1_{\Omega_{\delta_6}}),(R_\delta\varphi_2)1_{\Omega_{\delta_6}}\rangle\\
&\quad+\langle R_\delta T_t^{\Omega}(\varphi_2 1_{\Omega\setminus\Omega_{\delta_6}}),R_\delta(\varphi_2 1_{\Omega\setminus\Omega_{\delta_6}})\rangle-\langle R_\delta T_t^{\Omega}(\varphi_2 1_{\Omega\setminus\Omega_{\delta_6}}),R_\delta\varphi_2\rangle\\
&\quad-\langle R_\delta T_t^{\Omega}\varphi_2,R_\delta(\varphi_2 1_{\Omega\setminus\Omega_\delta})\rangle+\langle R_\delta T_t^{\Omega}\varphi_2,R_\delta\varphi_2\rangle.
\end{aligned}\tag{2.19}$$

From (2.18) and (2.19) we obtain

$$\begin{aligned}
e^{-\mu_2^\delta t}&\ge\|R_\delta(\varphi_2-\beta_1(\delta))\|_2^{-2}\Big\{\langle T_t^{\Omega_\delta}R_\delta(\varphi_2 1_{\Omega\setminus\Omega_{\delta_6}}),R_\delta(\varphi_2 1_{\Omega\setminus\Omega_{\delta_6}})\rangle\\
&\quad+2\langle T_t^{\Omega_\delta}R_\delta(\varphi_2 1_{\Omega\setminus\Omega_{\delta_6}}),R_\delta(\varphi_2 1_{\Omega_{\delta_6}})\rangle\\
&\quad-2\langle R_\delta\varphi_2,\beta_1(\delta)1_{\Omega_{\delta_6}}\rangle+\beta_1(\delta)^2|\Omega_\delta|\\
&\quad+\langle T_t^{\Omega_\delta}((R_\delta\varphi_2)1_{\Omega_{\delta_6}})-R_\delta T_t^{\Omega}(\varphi_2 1_{\Omega_{\delta_6}}),(R_\delta\varphi_2)1_{\Omega_{\delta_6}}\rangle\\
&\quad+\langle R_\delta T_t^{\Omega}(\varphi_2 1_{\Omega\setminus\Omega_{\delta_6}}),R_\delta(\varphi_2 1_{\Omega\setminus\Omega_{\delta_6}})\rangle-\langle R_\delta T_t^{\Omega}(\varphi_2 1_{\Omega\setminus\Omega_{\delta_6}}),R_\delta\varphi_2\rangle\\
&\quad-e^{-\mu_2 t}\langle R_\delta\varphi_2,R_\delta(\varphi_2 1_{\Omega\setminus\Omega_{\delta_6}})\rangle+e^{-\mu_2 t}\langle R_\delta\varphi_2,R_\delta\varphi_2\rangle\Big\}\\
&=A\{B_1+B_2-B_3+B_4+B_5+B_6-B_7-B_8+B_9\}.
\end{aligned}\tag{2.20}$$

Since φ_2 is orthogonal to 1 in $L^2(\Omega)$, we have

$$\lim_{\delta\downarrow 0}\beta_1(\delta)=0.\tag{2.21}$$

Hence

$$\lim_{\delta\downarrow 0}A=1,\quad\lim_{\delta\downarrow 0}B_3=0,\quad\lim_{\delta\downarrow 0}B_4=0.\tag{2.22}$$

Since

$$\|\varphi_2\|_\infty=e^{\mu_2 t}\|T_t^{\Omega}\varphi_2\|_\infty\le e^{\mu_2 t}c_0^{1/2}t^{-\frac{N}{4}}\quad(0<t\le 1),$$

we can choose $\delta_6>0$ sufficiently small so that

$$\|\varphi_2 1_{\Omega\setminus\Omega_{\delta_6}}\|_{L^2(\Omega)}\le\frac{\epsilon}{12}.$$

Then we have, for $0<\delta<\delta_6$,

$$|B_1|\le\frac{\epsilon}{12},\quad|B_2|\le\frac{\epsilon}{6},\quad|B_6|\le\frac{\epsilon}{12},\quad|B_7|\le\frac{\epsilon}{12},\quad|B_8|\le\frac{\epsilon}{12},\tag{2.23}$$

and

$$B_9=e^{-\mu_2 t}\Big\{\int_\Omega\varphi_2(x)^2dx-\int_{\Omega\setminus\Omega_\delta}\varphi_2(x)^2dx\Big\}=e^{-\mu_2 t}-B_{10}\tag{2.24}$$

where

$$0\le B_{10}=e^{-\mu_2 t}\int_{\Omega\setminus\Omega_\delta}\varphi_2(x)^2dx\le\frac{\epsilon^2}{144}<\frac{\epsilon}{12}.\tag{2.25}$$

By Proposition 2.4 we have

$$\lim_{\delta\downarrow 0}B_5=0.\tag{2.26}$$

Thus, by (2.22) and (2.26), there exists $\delta_7 \in (0, \delta_6]$ such that, for all $0 < \delta < \delta_7$,

$$|B_3| \le \frac{\epsilon}{12}, \quad |B_4| \le \frac{\epsilon}{12}, \quad |B_5| \le \frac{\epsilon}{12}, \quad |A - 1| \le \frac{\epsilon}{12}. \tag{2.27}$$

Then, by (2.20), (2.23), (2.24), (2.25) and (2.27), we have

$$e^{\mu_2^\delta t} \ge e^{-\mu_2 t} - \epsilon \quad (0 < \delta < \delta_7). \tag{2.28}$$

We next prove the reverse inequality of (2.28). For all $0 < \delta < \delta_6$ we have

$$\begin{aligned} e^{-\mu_2 t} &\ge \langle T_t^\Omega E_\delta \varphi_2^\delta, E_\delta \varphi_2^\delta \rangle \\ &= \langle T_t^\Omega E_\delta [(\varphi_2^\delta - \varphi_2^\delta 1_{\Omega_{\delta_6}}) + \varphi_2^\delta 1_{\Omega_{\delta_6}}], E_\delta [(\varphi_2^\delta - \varphi_2^\delta 1_{\Omega_{\delta_6}}) + \varphi_2^\delta 1_{\Omega_{\delta_6}}] \rangle \\ &= \langle T_t^\Omega E_\delta [\varphi_2^\delta 1_{\Omega \setminus \Omega_{\delta_6}}], E_\delta [\varphi_2^\delta 1_{\Omega \setminus \Omega_{\delta_6}}] \rangle + 2 \langle T_t^\Omega E_\delta [\varphi_2^\delta 1_{\Omega \setminus \Omega_{\delta_6}}], E_\delta [\varphi_2^\delta 1_{\Omega_{\delta_6}}] \rangle \\ &\quad + \langle T_t^\Omega E_\delta [\varphi_2^\delta 1_{\Omega_{\delta_6}}], E_\delta [\varphi_2^\delta 1_{\Omega_{\delta_6}}] \rangle \\ &= \langle T_t^\Omega E_\delta [\varphi_2^\delta 1_{\Omega \setminus \Omega_{\delta_6}}], E_\delta [\varphi_2^\delta 1_{\Omega \setminus \Omega_{\delta_6}}] \rangle + 2 \langle T_t^\Omega E_\delta [\varphi_2^\delta 1_{\Omega \setminus \Omega_{\delta_6}}], E_\delta [\varphi_2^\delta 1_{\Omega_{\delta_6}}] \rangle \\ &\quad + \langle T_t^\Omega E_\delta [\varphi_2^\delta 1_{\Omega_{\delta_6}}] - E_\delta T_t^{\Omega_\delta} (\varphi_2^\delta 1_{\Omega_{\delta_6}}),\ E_\delta [\varphi_2^\delta 1_{\Omega_{\delta_6}}] \rangle \\ &\quad + \langle E_\delta T_t^{\Omega_\delta} (\varphi_2^\delta 1_{\Omega_{\delta_6}}), E_\delta [\varphi_2^\delta 1_{\Omega_{\delta_6}}] \rangle. \end{aligned} \tag{2.29}$$

But

$$\begin{aligned} &\langle E_\delta T_t^{\Omega_\delta} (\varphi_2^\delta 1_{\Omega_{\delta_6}}), E_\delta (\varphi_2^\delta 1_{\Omega_{\delta_6}}) \rangle \\ &= \langle T_t^{\Omega_\delta} (\varphi_2^\delta 1_{\Omega_{\delta_6}}), \varphi_2^\delta 1_{\Omega_{\delta_6}} \rangle_{L^2(\Omega_\delta)} \\ &= \langle T_t^{\Omega_\delta} \varphi_2^\delta - T_t^{\Omega_\delta} (\varphi_2^\delta 1_{\Omega_\delta \setminus \Omega_{\delta_6}}), \varphi_2^\delta - \varphi_2^\delta 1_{\Omega_\delta \setminus \Omega_{\delta_6}} \rangle_{L^2(\Omega_\delta)} \\ &= e^{-\mu_2^\delta t} - 2 e^{-\mu_2^\delta t} \langle \varphi_2^\delta 1_{\Omega_\delta \setminus \Omega_{\delta_6}}, \varphi_2^\delta \rangle_{L^2(\Omega_\delta)} \\ &\quad + \langle T_t^{\Omega_\delta} (\varphi_2^\delta 1_{\Omega_\delta \setminus \Omega_{\delta_6}}), \varphi_2^\delta 1_{\Omega_\delta \setminus \Omega_{\delta_6}} \rangle_{L^2(\Omega_\delta)}. \end{aligned} \tag{2.30}$$

From (2.29) and (2.30) we have, for $0 < \delta < \delta_6$,

$$\begin{aligned} e^{-\mu_2 t} &\ge \langle T_t^\Omega E_\delta [\varphi_2^\delta 1_{\Omega \setminus \Omega_{\delta_6}}], E_\delta [\varphi_2^\delta 1_{\Omega \setminus \Omega_{\delta_6}}] \rangle \\ &\quad + 2 \langle T_t^\Omega E_\delta [\varphi_2^\delta 1_{\Omega \setminus \Omega_{\delta_6}}], E_\delta [\varphi_2^\delta 1_{\Omega_{\delta_6}}] \rangle \\ &\quad + \langle T_t^\Omega E_\delta [\varphi_2^\delta 1_{\Omega_{\delta_6}}] - E_\delta T_t^{\Omega_\delta} (\varphi_2^\delta 1_{\Omega_{\delta_6}}), E_\delta [\varphi_2^\delta 1_{\Omega_{\delta_6}}] \rangle \\ &\quad - 2 e^{-\mu_2^\delta t} \langle \varphi_2^\delta 1_{\Omega_\delta \setminus \Omega_{\delta_6}}, \varphi_2^\delta \rangle_{L^2(\Omega_\delta)} \\ &\quad + \langle T_t^{\Omega_\delta} (\varphi_2^\delta 1_{\Omega_\delta \setminus \Omega_{\delta_6}}), \varphi_2^\delta 1_{\Omega_\delta \setminus \Omega_{\delta_6}} \rangle_{L^2(\Omega_\delta)} + e^{-\mu_2^\delta t} \\ &= C_1 + C_2 + C_3 - C_4 + C_5 + e^{-\mu_2^\delta t}. \end{aligned} \tag{2.31}$$

We now need the following estimate from [15]:

$$\mu_2^\delta \le p_{d/2,1}^2 \pi^{d/2} \Gamma(\frac{d}{2} + 1)^{-1} |\Omega_\delta|^{-\frac{2}{d}} \tag{2.32}$$

where $p_{\nu,k}$ denotes the kth positive zero of the derivative of $x^{1-\nu} J_\nu(x)$ and $J_\nu(x)$ is the standard Bessel function of the first kind of order ν. From (1.3) and (2.32) we obtain, for $0 < t \le 1$ and all sufficiently small $\delta > 0$,

$$\|\varphi_2^\delta\|_\infty = e^{\mu_2^\delta t} \|T_t^{\Omega_\delta} \varphi_2^\delta\|_\infty \le e^{\mu_2^\delta t} c_0^{1/2} t^{-\frac{N}{4}} = c_0^{1/2} c_1^t t^{-N/4} \quad (0 < t \le 1) \tag{2.33}$$

where $c_1 \ge 1$ depends only on d and the volume of Ω. Hence we may assume that δ_6 is sufficiently small so that

$$\|\varphi_2^\delta 1_{\Omega_\delta \setminus \Omega_{\delta_6}}\|_2 \le \frac{\epsilon}{7} \quad (0 < \delta < \delta_6).$$

Then, for all $0 < \delta < \delta_6$, we have

$$|C_1| \le \frac{\epsilon}{7}, \quad |C_2| \le \frac{2\epsilon}{7}, \quad |C_4| \le \frac{2\epsilon}{7}, \quad |C_5| \le \frac{\epsilon}{7}. \tag{2.34}$$

By Proposition 2.4 we have, for all $0 < t \le 1$,

$$\lim_{\delta\downarrow 0} \|T_t^{\Omega} E_\delta(\varphi_2^\delta 1_{\Omega_{\delta_6}}) - E_\delta T_t^{\Omega_\delta}(\varphi_2^\delta 1_{\Omega_{\delta_6}})\|_{L^2(\Omega)} = 0.$$

Hence there exists $\delta_8 \in (0, \delta_6)$ sufficiently small such that

$$|C_3| \le \frac{\epsilon}{7} \quad (0 < \delta < \delta_8). \tag{2.35}$$

From (2.31), (2.34) and (2.35) we have

$$e^{-\mu_2 t} \ge e^{-\mu_2^\delta t} - \epsilon \quad (0 < \delta < \delta_8). \tag{2.36}$$

The inequality (1.4) now follows from (2.28) and (2.36). □

Proposition 2.5. *Suppose that μ_2 has multiplicity 1. Then*

$$\lim_{\delta\downarrow 0} \langle \varphi_3, E_\delta \varphi_2^\delta\rangle = 0.$$

Proof. For all sufficiently small $\delta > 0$, we have

$$\begin{aligned} e^{-\mu_2^\delta t}\langle \varphi_3, E_\delta\varphi_2^\delta\rangle &= \langle \varphi_3, E_\delta T_t^{\Omega_\delta}\varphi_2^\delta\rangle \\ &= \langle \varphi_3, E_\delta T_t^{\Omega_\delta}\varphi_2^\delta - T_t^{\Omega} E_\delta\varphi_2^\delta\rangle + \langle \varphi_3, T_t^{\Omega} E_\delta \varphi_2^\delta\rangle \\ &= \langle \varphi_3, E_\delta T_t^{\Omega_\delta}\varphi_2^\delta - T^{\Omega} E_\delta\varphi_2^\delta\rangle + e^{-\mu_3 t}\langle \varphi_3, E_\delta\varphi_2^\delta\rangle. \end{aligned}$$

Thus

$$(e^{-\mu_2^\delta t} - e^{-\mu_3 t})\langle \varphi_3, E_\delta \varphi_2^\delta\rangle = \langle \varphi_3, E_\delta T_t^{\Omega_\delta}\varphi_2^\delta - T_t^{\Omega} E_\delta \varphi_2^\delta\rangle. \tag{2.37}$$

Let $\epsilon \in (0,1)$ be given. Then (1.4) and Proposition 2.4 imply that for any $t \in (0,1]$ there exists $\delta_9 > 0$ such that, for all $0 < \delta < \delta_9$, we have

$$|\langle \varphi_3, E_\delta T_t^{\Omega_\delta}\varphi_2^\delta - T_t^{\Omega} E_\delta\varphi_2^\delta\rangle| \le \frac{1}{2}(e^{-\mu_2 t} - e^{-\mu_3 t})\epsilon \tag{2.38}$$

and

$$0 < \frac{1}{2}(e^{-\mu_2 t} - e^{-\mu_3 t}) \le e^{-\mu_2^\delta t} - e^{-\mu_3 t}. \tag{2.39}$$

Therefore, by (2.37), (2.38) and (2.39), we have

$$|\langle \varphi_3, E_\delta \varphi_2^\delta\rangle| \le \epsilon \quad (0 < \delta < \delta_9).$$

This proves the proposition. □

Proof of Theorem 1.4. Suppose (1.5) is false. Let $\{\epsilon_k\}_{k=1}^\infty$ be a decreasing sequence of positive numbers such that $\lim_{k\to\infty} \epsilon_k = 0$ and, by (1.4),

$$\lim_{k\to\infty} \mu_3^{\epsilon_k} = \mu_2. \tag{2.40}$$

Let

$$E_{\epsilon_k}\varphi_3^{\epsilon_k} = a_1(k)|\Omega|^{-1/2} + a_2(k)\varphi_2 + \sum_{\ell=3}^\infty a_\ell(k)\varphi_\ell,$$

$$E_{\epsilon_k}\varphi_2^{\epsilon_k} = b_1(k)|\Omega|^{-1/2} + b_2(k)\varphi_2 + \sum_{\ell=3}^\infty b_\ell(k)\varphi_\ell.$$

Then

$$a_1(k) = \int_\Omega E_{\epsilon_k}\varphi_3^{\epsilon_k}|\Omega|^{-1/2}dx = \int_{\Omega_{\epsilon_k}} \varphi_3^{\epsilon_k}dx|\Omega|^{-1/2} = 0. \tag{2.41}$$

We next want to show that

$$\big\|\sum_{\ell=3}^\infty a_\ell(k)\varphi_\ell\big\|_2 \to 0 \quad \text{as } k\to\infty. \tag{2.42}$$

Let $t\in(0,1]$ and consider

$$\begin{aligned}
&T_t^\Omega E_{\epsilon_k}\varphi_3^{\epsilon_k} - E_{\epsilon_k}T_t^{\Omega_{\epsilon_k}}\varphi_3^{\epsilon_k}\\
&= T_t^\Omega E_{\epsilon_k}\varphi_3^{\epsilon_k} - e^{-\mu_3^{\epsilon_k}t}E_{\epsilon_k}\varphi_3^{\epsilon_k}\\
&= (e^{-\mu_2 t} - e^{-\mu_3^{\epsilon_k}t})a_2(k)\varphi_2 + \sum_{\ell=3}^\infty a_\ell(k)(e^{-\mu_\ell t} - e^{-\mu_3^{\epsilon_k}t})\varphi_\ell\\
&= (e^{-\mu_2 t} - e^{-\mu_3^{\epsilon_k}t})a_2(k)\varphi_2 + \sum_{\ell=3}^\infty a_\ell(k)(e^{-\mu_\ell t} - e^{-\mu_2 t})\varphi_\ell\\
&\quad + \sum_{\ell=3}^\infty a_\ell(k)(e^{-\mu_2 t} - e^{-\mu_3^{\epsilon_k}t})\varphi_\ell.
\end{aligned} \tag{2.43}$$

Now

$$\begin{aligned}
\big\|\sum_{\ell=3}^\infty a_k(\ell)(e^{-\mu_\ell t} - e^{-\mu_2 t})\varphi_\ell\big\|_2^2 &= \sum_{\ell=3}^\infty a_k(\ell)^2(e^{-\mu_\ell t} - e^{-\mu_2 t})^2\\
&\ge (e^{-\mu_2 t} - e^{-\mu_3 t})^2\sum_{\ell=3}^\infty a_\ell(k)^2\\
&= (e^{-\mu_2 t} - e^{-\mu_3 t})^2\big\|\sum_{\ell=3}^\infty a_\ell(k)\varphi_\ell\big\|_2^2.
\end{aligned} \tag{2.44}$$

By Proposition 2.4 we have

$$\|T_t^\Omega E_{\epsilon_k}\varphi_3^{\epsilon_k} - E_{\epsilon_k}T_t^{\Omega_{\epsilon_k}}\varphi_3^{\epsilon_k}\|_2 \to 0 \quad \text{as } k\to\infty. \tag{2.45}$$

Thus, by (2.40), (2.41), (2.43) and (2.45), we obtain

$$\lim_{k\to\infty}\big\|\sum_{\ell=3}^\infty a_\ell(k)(e^{-\mu_\ell t} - e^{-\mu_2 t})\varphi_\ell\big\|_2^2 = 0. \tag{2.46}$$

So (2.42) follows from (2.44) and (2.46). By a similar argument we can show that

$$b_1(k) = 0 \tag{2.47}$$

and

$$\lim_{k\to\infty}\big\|\sum_{\ell=3}^\infty b_\ell(k)\varphi_\ell\big\|_2 = 0. \tag{2.48}$$

Since (2.41), (2.42), (2.47) and (2.48) imply that

$$\lim_{k\to\infty} a_2(k) = \lim_{k\to\infty} b_2(k) = 1,$$

we have

$$\begin{aligned}
0 &= \langle \varphi_3^{\epsilon_k}, \varphi_2^{\epsilon_k}\rangle_{L^2(\Omega_{\epsilon_k})} \\
&= \langle E_{\epsilon_k}\varphi_3^{\epsilon_k}, E_{\epsilon_k}\varphi_2^{\epsilon_k}\rangle_{L^2(\Omega)} \\
&= \Big\langle a_2(k)\varphi_2 + \sum_{\ell=3}^{\infty} a_\ell(k)\varphi_\ell, b_2(k)\varphi_2 + \sum_{\ell=3}^{\infty} b_\ell(k)\varphi_\ell \Big\rangle_{L^2(\Omega)} \\
&= a_2(k)b_2(k) + \Big\langle \sum_{\ell=3}^{\infty} a_\ell(k)\varphi_\ell, \sum_{\ell=3}^{\infty} b_\ell(k)\varphi_\ell \Big\rangle_{L^2(\Omega)} \to 1 \quad \text{as } k \to \infty
\end{aligned}$$

which gives a contradiction. Thus (1.5) holds. □

Proof of Theorem 1.5. By (1.4) there exists $\delta_{10} \in (0, \frac{1}{2})$ such that

$$\mu_2^\delta < 2\mu_2 \quad (0 < \delta < \delta_{10})$$

and that

$$D = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta_{10}\} \supseteq \overline{\Omega'}.$$

Applying Lemma 2.1 with $\Sigma = \Omega$ or $\Sigma = \Omega_\delta$ for $0 < \delta < \frac{1}{2}\delta_{10}$, $\Sigma' = D$, $\omega = 1$, $a_{ij} = \delta_{ij}$, $\tau_1 = 1$, $\tau_2 = 2$, $t_1 = \frac{3}{2}$, $\eta = \frac{1}{2}\delta_{10}$ and

$$u(x,t) = e^{-\mu_2^\delta t}\varphi_2^\delta(x)$$

for $0 < \delta < \frac{1}{2}\delta_{10}$, or

$$u(x,t) = e^{-\mu_2 t}\varphi_2(x),$$

we see that there exists $\alpha > 0$ such that

$$|\varphi_2^\delta(x) - \varphi_2^\delta(y)| \le B|x-y|^\alpha, \tag{2.49}$$

$$|\varphi_2(x) - \varphi_2(y)| \le B|x-y|^\alpha \tag{2.50}$$

for all $x, y \in D$ and $0 < \delta < \frac{1}{2}\delta_{10}$, where, by (1.2), (1.3), (1.4) and (2.33), we can assume that $\delta_{10} \in (0, \frac{1}{2})$ is sufficiently small that

$$B = (8/\delta_{10})^\alpha 2c_0^{1/2}c_1e^{4\mu_2}.$$

Let

$$E_\delta\varphi_2^\delta = b_2(\delta)\varphi_2 + \sum_{\ell=3}^{\infty} b_\ell(\delta)\varphi_\ell \quad (0 < \delta < \delta_{10}).$$

Then, as in the proof of Theorem 1.4, we have $\lim_{\delta\downarrow 0} b_2(\delta) = 1$ and

$$\lim_{\delta\downarrow 0}\Big\|\sum_{\ell=3}^{\infty} b_\ell(\delta)\varphi_\ell\Big\|_2 = 0.$$

Thus

$$\|\varphi_2 - E_\delta\varphi_2^\delta\|_2 \to 0 \quad \text{as } \delta \downarrow 0. \tag{2.51}$$

Let

$$r = \operatorname{dist}(\overline{\Omega'}, \partial D).$$

Suppose that (1.6) is false. Then there exist $\epsilon > 0$, a decreasing sequence of positive numbers $\{\eta_k\}_{k=1}^\infty$ and a sequence of points $\{z_k\}_{k=1}^\infty$ in $\overline{\Omega'}$ such that $\lim_{k\to\infty}\eta_k = 0$ and

$$|\varphi_2^{\eta_k}(z_k) - \varphi_2(z_k)| \ge \epsilon \quad (k = 1, 2, 3, \dots). \tag{2.52}$$

Then for all $w \in D$ satisfying

$$|w - z_k| \le \min\left\{r, \left(\frac{\epsilon}{6}\right)^{1/\alpha} B^{-\frac{1}{\alpha}}\right\}$$

we have, by (2.49) and (2.50),

$$|\varphi_2^{\eta_k}(z_k) - \varphi_2^{\eta_k}(w)| \le \frac{\epsilon}{6}, \tag{2.53}$$

$$|\varphi_2(z_k) - \varphi_2(w)| \le \frac{\epsilon}{6}, \tag{2.54}$$

hence, from (2.52), (2.53) and (2.54), we have

$$|\varphi_2^{\eta_k}(w) - \varphi_2(w)| \ge \frac{2\epsilon}{3}.$$

Let $R = \min\left\{r, (\frac{\epsilon}{6})^{1/\alpha} B^{-\frac{1}{\alpha}}\right\}$. Then

$$\int_{B(z_k,R)} |\varphi_2^{\eta_k} - \varphi_2|^2 dx \ge \frac{4\epsilon^2}{9} c_2 R^d > 0 \tag{2.55}$$

for all $k = 1, 2, 3, \dots$, where $c_2 > 0$ depends only on d. But (2.55) contradicts (2.51), hence (1.6) holds. $\square$

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Rodrigo Bañuelos
Department of Mathematics, Purdue University, West Lafayette, IN 47906, USA
E-mail address: banuelos@math.purdue.edu

Michael M. H. Pang
Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
E-mail address: pangm@math.missouri.edu