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MULTIPLE POSITIVE SOLUTIONS FOR SINGULAR m-POINT BOUNDARY-VALUE PROBLEMS WITH NONLINEARITIES DEPENDING ON THE DERIVATIVE

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Abstract. Using the fixed point theorem in cones, this paper shows the existence of multiple positive solutions for the singular m-point boundary-value problem

$$x''(t) + a(t)f(t, x(t), x'(t)) = 0, \quad 0 < t < 1,$$
$$x'(0) = 0, \quad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i),$$

where $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $a_i \in [0, 1)$, $i = 1, 2, \ldots, m-2$, with $0 < \sum_{i=1}^{m-2} a_i < 1$ and f maybe singular at $x = 0$ and $x' = 0$.

1. Introduction

The study of multi-point boundary-value problems (BVP) for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [5, 6]. Since then, many authors have studied general nonlinear multi-point BVP; see for examples [4, 17], and references therein. Gupta, Ntouyas and Tsamatos [4] considered the existence of a solution in $C^1[0, 1]$ for the m-point boundary-value problem

$$\begin{gathered} x''(t) = f(t, x(t), x'(t)) + e(t), \quad 0 < t < 1, \\ x'(0) = 0, \quad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i), \end{gathered} \tag{1.1}$$

where $\xi_i \in (0, 1)$, $i = 1, 2, \ldots, m-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $a_i \in \mathbb{R}$, $i = 1, 2, \ldots, m-2$, have the same sign, $\sum_{i=1}^{m-2} a_i \neq 1$, $e \in L^1[0, 1]$, $f : [0, 1] \times R^2 \to \mathbb{R}$ is a function satisfying Carathèodory conditions and a growth condition of the form $|f(t, u, v)| \leq p_1(t)|u| + q_1(t)|v| + r_1(t)$, where p_1, q_1, $r_1 \in L^1[0, 1]$. Recently, using Leray-Schauder continuation theorem, Ma and O'Regan proved the existence of positive solutions of $C^1[0, 1)$ solutions for the above BVP, where $f : [0, 1] \times R^2 \to \mathbb{R}$ satisfies the Carathèodory conditions (see [17]). Khan and Webb [10] obtained

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a interesting result which presents the multiplicity of existence of at least three solutions of a second-order three-point boundary-value problem. However, up to now, there are a fewer results on the existence of multiple solutions to (1.1). In view of the importance of the research on the multiplicity of positive solutions for differential equations [1, 2, 5, 9, 10, 11, 12, 14, 15], the goal of this paper is to fill this gap in the literature.

There are main four sections in this paper. In section 2, we give a special cone and its properties. In section 3, using the theory of fixed point index on a cone, we present the existence of multiple positive solutions to (1.1) with f may be singular at $x' = 0$ but not at $x = 0$. In section 4, under the condition f is singular at $x' = 0$ and $x = 0$, we present the existence of multiple positive solutions to (1.1). In section 5, under the condition f is singular at $x = 0$ but not at $x' = 0$, we present the existence of multiple positive solutions to (1.1).

2. Preliminaries

Let $C^1[0,1] = \{x : [0,1] \to \mathbb{R}$ such that $x(t)$ be continuous on $[0,1]$ and $x'(t)$ continuous on $[0,1]$ $\}$ with norm $\|x\| = \max\{\gamma\|x\|_1, \gamma\delta\|x\|_2\}$, where

$$\|x\|_1 = \max_{t\in[0,1]} |x(t)|, \quad \|x\|_2 = \max_{t\in[0,1]} |x'(t)|,$$

$$\gamma = \frac{\sum_{i=1}^{m-2} a_i(1-\xi_i)}{1-\sum_{i=1}^{m-2} a_i\xi_i}, \quad \delta = \sum_{i=1}^{m-2} a_i(1-\xi_i).$$

Let

$$P = \{x \in C^1[0,1] : x(t) \geq \gamma\|x\|_1, \forall t \in [0,1], x(0) \geq \delta\|x\|_2\}.$$

Obviously, $C^1[0,1]$ is a Banach space and P is a cone in $C^1[0,1]$.

Lemma 2.1. *Let Ω be a bounded open set in real Banach space E, $\theta \in \Omega$, P be a cone in E and $A : \bar{\Omega} \cap P \to P$ be continuous and completely continuous. Suppose*

$$\lambda Ax \neq x, \quad \forall x \in \partial\Omega \cap P, \lambda \in (0,1]. \tag{2.1}$$

Then $i(A, \Omega \cap P, P) = 1$.

Lemma 2.2. *Let Ω be a bounded open set in real Banach space E, $\theta \in \Omega$, P be a cone in E and $A : \bar{\Omega} \cap P \to P$ be continuous and completely continuous. Suppose*

$$Ax \nleq x, \quad \forall x \in \partial\Omega \cap P. \tag{2.2}$$

Then $i(A, \Omega \cap P, P) = 0$.

Let $\mathbb{R}_+ = (0, +\infty)$, $\mathbb{R}_- = (-\infty, 0)$, $\mathbb{R} = (-\infty, +\infty)$. The following conditions will be used in this article.

$$a(t) \in C(0,1) \cap L^1[0,1], \quad a(t) > 0, \ t \in (0,1); \tag{2.3}$$

$$f \in C([0,1] \times \mathbb{R}_+ \times \mathbb{R}_-, [0, +\infty)); \tag{2.4}$$

There exists $g \in C([0,+\infty) \times (-\infty, 0], [0, +\infty))$ such that

$$f(t,x,y) \leq g(x,y), \forall (t,x,y) \in [0,1] \times \mathbb{R}_+ \times \mathbb{R}_-. \tag{2.5}$$

For $x \in P$ and $t \in [0,1]$, define operator

$$\begin{aligned}(Ax)(t) = &-\int_0^t (t-s)a(s)f(s,x(s)x'(s))ds \\ &+ \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Big(\int_0^1 (1-s)a(s)f(s,x(s),x'(s))ds \\ &- \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)a(s)f(s,x(s),x'(s))\Big)ds\,.\end{aligned} \tag{2.6}$$

Lemma 2.3 ([17]). *Assume* (2.1). *Then for* $y \in C[0,1]$ *the problem*

$$\begin{gathered} x'' + y(t) = 0, t \in (0,1) \\ x'(0) = \sum_{i=1}^{m-2} b_i x'(\xi_i), \quad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i)\end{gathered} \tag{2.7}$$

has a unique solution

$$x(t) = -\int_0^t (t-s)y(s)ds + Mt + N, \tag{2.8}$$

where,

$$M = \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(s)y(s)ds}{\sum_{i=1}^{m-2} b_i - 1},$$

$$\begin{aligned}N = \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Big(&\int_0^1 (1-s)a(s)y(s)ds - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)a(s)y(s)ds \\ &- \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(s)y(s)ds}{\sum_{i=1}^{m-2} b_i - 1}(1-\sum_{i=1}^{m-2} a_i\xi_i)\Big).\end{aligned}$$

Further, if $y \ge 0$*, for all* $t \in [0,1]$*,* x *satisfies*

$$\inf_{t\in[0,1]} x(t) \ge \gamma \|x\|_1, \tag{2.9}$$

where $\gamma = \big(\sum_{i=1}^{m-2} a_i(1-\xi_i)\big)/\big(1-\sum_{i=1}^{m-2} a_i\xi_i\big)$.

Lemma 2.4. *Suppose* (2.3)–(2.5)*hold. Then* $A : P \to P$ *is a completely continuous operator.*

Proof. For $x \in P$, from (2.6), one has

$$
\begin{aligned}
(Ax)(t) &\geq -\int_0^1 (1-s)a(s)f(s,x(s),x'(s))ds \\
&\quad + \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Big(\int_0^1 (1-s)a(s)f(s,x(s),x'(s))ds \\
&\quad - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)a(s)f(s,x(s),x'(s))ds\Big) \\
&\geq \frac{\sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i}\int_0^1 (1-s)a(s)f(s,x(s),x'(s))ds \\
&\quad - \sum_{i=1}^{m-2} a_i \int_0^1 (\xi_i - s)a(s)f(s,x(s),x'(s))ds \\
&\geq \frac{\sum_{i=1}^{m-2} a_i(1-\xi_i)}{1-\sum_{i=1}^{m-2} a_i}\int_0^1 a(s)f(s,x(s),x'(s))ds \geq 0, \quad t \in [0,1],
\end{aligned} \tag{2.10}
$$

$$
\begin{aligned}
|(Ax)(t)| &= -\int_0^1 (t-s)a(s)f(s,x(s)x'(s))ds \\
&\quad + \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Big(\int_0^1 (1-s)a(s)f(s,x(s),x'(s))ds \\
&\quad - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)a(s)f(s,x(s),x'(s))ds\Big) \\
&\leq \frac{1}{1-\sum_{i=1}^{m-2} a_i}\int_0^1 (1-s)a(s)f(s,x(s),x'(s))ds \\
&\leq \frac{1}{1-\sum_{i=1}^{m-2} a_i}\int_0^1 a(s)f(s,x(s),x'(s))ds \\
&\leq \frac{1}{1-\sum_{i=1}^{m-2} a_i}\int_0^1 a(s)ds \max_{0\leq c\leq\|x\|,-\|x\|\leq c'\leq 0} g(c,c') < +\infty, \quad t \in [0,1],
\end{aligned} \tag{2.11}
$$

$$
\begin{aligned}
|(Ax)'(t)| &= \big|-\int_0^t a(s)f(s,x(s)x'(s))ds\big| \\
&= \int_0^t a(s)f(s,x(s)x'(s))ds \\
&\leq \int_0^1 a(s)f(s,x(s)x'(s))ds \\
&\leq \int_0^1 a(s)ds \max_{0\leq c\leq\|x\|,-\|x\|\leq c'\leq 0} g(c,c') < +\infty,
\end{aligned} \tag{2.12}
$$

which implies that A is well defined.

$$
(Ax)(0) = \frac{1}{1-\sum_{i=1}^{m-2} a_i}(\int_0^1 (1-s)a(s)f(s,x(s),x'(s))ds
$$

$$
\begin{aligned}
&- \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)a(s)f(s, x(s), x'(s))ds) \\
&\geq \frac{1}{1 - \sum_{i=1}^{m-2} a_i} (\sum_{i=1}^{m-2} a_i \int_0^1 (1-s)a(s)f(s, x(s), x'(s))ds \\
&- \sum_{i=1}^{m-2} a_i \int_0^1 (\xi_i - s)a(s)f(s, x(s), x'(s))ds) \\
&= \sum_{i=1}^{m-2} a_i(1-\xi_i) \int_0^1 a(s)f(s, x(s), x'(s))ds.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\|Ax\|_2 &= \max_{t\in[0,1]} |(Ax)'(t)| \\
&= \max_{t\in[0,1]} | - \int_0^t a(s)f(s, x(s)x'(s))ds| \\
&= \int_0^1 a(s)f(s, x(s)x'(s))ds.
\end{aligned}
$$

Then

$$
(Ax)(0) \geq \delta \|Ax\|_2. \tag{2.13}
$$

By Lemma 2.3, we have $(Ax)(t) \geq \gamma\|Ax\|_1$. As a result $Ax \in P$, which implies $AP \subseteq P$. By a standard argument, we know that $A : P \to P$ is continuous and completely continuous. □

3. Singularities at $x' = 0$ but not at $x = 0$

In this section the nonlinearity f may be singular at $x' = 0$ but not at $x = 0$. We will assume that the following conditions hold.

- (H1) $a(t) \in C(0,1) \cap L^1[0,1]$, $a(t) > 0$, $t \in (0,1)$
- (H2) $f(t,u,z) \leq h(u)[g(z) + r(z)]$, where $f \in C([0,1] \times \mathbb{R}_+ \times \mathbb{R}_-, \mathbb{R}_+)$, $g(z) > 0$ continuous and nondecreasing on $\mathbb{R}_-$, $h(u) \geq 0$ continuous and nondecreasing on $\mathbb{R}_+$, $r(z) > 0$ continuous and non-increasing on $(-\infty, 0]$;
- (H3)
$$
\sup_{c\in\mathbb{R}_+} \frac{c}{-\frac{\sum_{i=1}^{m-2} a_i\xi_i + 1}{1 - \sum_{i=1}^{m-2} a_i} I^{-1}(h(c)\int_0^1 a(s)ds)} > 1,
$$
where $I(z) = \int_z^0 \frac{du}{g(u)+r(u)}$, $z \in \mathbb{R}_-$;
- (H4) There exists a function $g_1 \in C([0,+\infty) \times (-\infty,0], [0,+\infty))$, such that $f(t,u,z) \geq g_1(u,z), \forall (t,u,z) \in [0,1] \times \mathbb{R}_+ \times \mathbb{R}_-$, and $\lim_{u\to+\infty} \frac{g_1(u,z)}{u} = +\infty$, uniformly for $z \in \mathbb{R}_-$.
- (H5) There exists a function $\Psi_H \in C([0,1], \mathbb{R}_+)$ and a constant $0 \leq \delta < 1$ such that $f(t,u,z) \geq \Psi_H(t)u^\delta$, for all $(t,u,z) \in [0,1] \times [0,H] \times \mathbb{R}_-$.

For $n \in \{1, 2, \dots\}$ and $x \in P$, define operator

$$\begin{aligned} (A_n x)(t) = &-\int_0^t (t-s)a(s)f(s, x(s), -|x'(s)| - \frac{1}{n})ds \\ &+ \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Big(\int_0^1 (1-s)a(s)f(s, x(s), -|x'(s)| - \frac{1}{n})ds \\ &- \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)a(s)f(s, x(s), -|x'(s)| - \frac{1}{n})ds\Big), \quad t \in [0,1]. \end{aligned} \tag{3.1}$$

Theorem 3.1. *Suppose* (H1)–(H5) *hold. Then* (1.1) *has at least two positive solutions* $x_{0,1}, x_{0,2} \in C^1[0,1] \cap C^2(0,1)$ *with* $x_{0,1}(t), x_{0,2}(t) > 0$, $t \in [0,1]$.

Proof. Choose $R_1 > 0$ such that

$$\frac{R_1}{-\frac{\sum_{i=1}^{m-2} a_i\xi_i + 1}{1-\sum_{i=1}^{m-2} a_i} I^{-1}(h(R_1)\int_0^1 a(s)ds)} > 1. \tag{3.2}$$

From the continuity of I^{-1} and h, we can choose $\varepsilon > 0$ and $\varepsilon < R_1$ with

$$\frac{R_1}{-\frac{\sum_{i=1}^{m-2} a_i\xi_i + 1}{1-\sum_{i=1}^{m-2} a_i} I^{-1}(h(R_1)\int_0^1 a(s)ds + I(-\varepsilon))} > 1, \tag{3.3}$$

$n_0 \in \{1, 2, \dots\}$ with $\frac{1}{n_0} < \min\{\varepsilon, \delta/2\}$ and let $N_0 = \{n_0, n_0 + 1, \dots\}$.

Lemma 2.4 guarantees that for $n \in N_0$, $A_n : P \to P$ is a completely continuous operator. Let

$$\Omega_1 = \{x \in C^1[0,1] : \|x\| < R_1\}.$$

We show that

$$x \neq \mu A_n x, \quad \forall x \in P \cap \partial\Omega_1,\ \mu \in (0,1],\ n \in N_0. \tag{3.4}$$

In fact, if there exists an $x_0 \in P \cap \partial\Omega_1$ and $\mu_0 \in (0,1]$ such that $x_0 = \mu_0 A_n x_0$,

$$\begin{aligned} x_0(t) = &-\mu_0 \int_0^t (t-s)a(s)f(s, x_0(s), -|x_0'(s)| - \frac{1}{n})ds \\ &+ \frac{\mu_0}{1-\sum_{i=1}^{m-2} a_i}\Big(\int_0^1 (1-s)a(s)f(s, x_0(s), -|x_0'(s)| - \frac{1}{n})ds \\ &- \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)a(s)f(s, x_0(s), -|x_0'(s)| - \frac{1}{n})ds\Big), \quad t \in [0,1]. \end{aligned}$$

Then

$$x_0'(t) = -\mu_0 \int_0^t a(s)f(s, x_0(s), -|x_0'(s)| - \frac{1}{n})ds, \quad \forall t \in [0,1]. \tag{3.5}$$

Obviously, $x_0'(t) \le 0$, $t \in (0,1)$, and since $x_0(1) > 0$, $x_0(t) > 0, t \in [0,1]$. Differentiating (3.5), we have

$$\begin{gathered} x_0''(t) + \mu_0 a(t)f(t, x_0(t), x_0'(t) - \frac{1}{n}) = 0, \quad 0 < t < 1, \\ x_0'(0) = 0, \quad x_0(1) = \sum_{i=1}^{m-2} a_i x_0(\xi_i). \end{gathered} \tag{3.6}$$

Then

$$\begin{aligned}
-x_0''(t) &= \mu_0 a(t) f(t, x_0(t), x_0'(t) - \frac{1}{n}) \\
&\le a(t)h(x_0(t))[g(x_0'(t) - \frac{1}{n}) + r(x_0'(t) - \frac{1}{n})], \quad \forall t \in (0,1),
\end{aligned}$$

$$\frac{-x_0''(t)}{g(x_0'(t) - \frac{1}{n}) + r(x_0'(t) - \frac{1}{n})} \le a(t)h(x_0(t)), \quad \forall t \in (0,1).$$

Integrating from 0 to t, we have

$$I(x_0'(t) - \frac{1}{n}) - I(-\frac{1}{n}) \le \int_0^t a(s)h(x_0(s))ds \le h(R_1)\int_0^t a(s)ds,$$

and

$$I(x_0'(t) - \frac{1}{n}) \le h(R_1)\int_0^t a(s)ds + I(-\varepsilon).$$

Then

$$x_0'(t) \ge I^{-1}(h(R_1)\int_0^t a(s)ds + I(-\varepsilon));$$

that is,

$$-x_0'(t) \le -I^{-1}(h(R_1)\int_0^t a(s)ds + I(-\varepsilon)), \quad t \in (0,1). \tag{3.7}$$

Then

$$\begin{aligned}
x_0(0) &= \frac{\mu_0}{1 - \sum_{i=1}^{m-2} a_i}\int_0^1 -x_0'(s)ds - \frac{\mu_0 \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} -x_0'(s)ds}{1 - \sum_{i=1}^{m-2} a_i} \\
&\le \frac{1}{1 - \sum_{i=1}^{m-2} a_i}\int_0^1 -I^{-1}\Big(h(R_1)\int_0^s a(\tau)d\tau + I(-\varepsilon)\Big)ds \\
&\quad + \frac{\sum_{i=1}^{m-2} a_i}{1 - \sum_{i=1}^{m-2} a_i}\int_0^{\xi_i} -I^{-1}(h(R_1)\int_0^s a(\tau)d\tau + I(-\varepsilon))ds \\
&\le \frac{1}{1 - \sum_{i=1}^{m-2} a_i}\int_0^1 -I^{-1}\Big(h(R_1)\int_0^1 a(\tau)d\tau + I(-\varepsilon)\Big)ds \\
&\quad + \frac{\sum_{i=1}^{m-2} a_i}{1 - \sum_{i=1}^{m-2} a_i}\int_0^{\xi_i} -I^{-1}(h(R_1)\int_0^1 a(\tau)d\tau + I(-\varepsilon))ds \\
&= \frac{1 + \sum_{i=1}^{m-2} a_i\xi_i}{1 - \sum_{i=1}^{m-2} a_i}\Big(-I^{-1}\Big(h(R_1)\int_0^1 a(s)ds + I(-\varepsilon)\Big)\Big).
\end{aligned}$$

Since $x_0(0) \ge x_0(t) \ge \gamma\|x_0\|_1 \ge \gamma x_0(0) \ge \gamma\delta\|x_0\|_2, x_0(0) \ge \|x_0\| = R_1$,

$$\frac{R_1}{-\frac{1+\sum_{i=1}^{m-2} a_i\xi_i}{1-\sum_{i=1}^{m-2} a_i} I^{-1}(h(R_1)\int_0^1 a(s)ds + I(-\varepsilon))} \le 1, \tag{3.8}$$

which is a contradiction to (3.3). Then (3.4) holds.

From Lemma 2.1, for $n \in N_0$,

$$i(A_n, \Omega_1 \cap P, P) = 1. \tag{3.9}$$

Now we show that there exists a set Ω_2 such that

$$A_n x \not\le x, \quad \forall x \in \partial\Omega_2 \cap P. \tag{3.10}$$

Choose a^* with $0 < a^* < 1$. Let

$$N^* = \Big(\frac{1}{\gamma a^* \frac{\sum_{i=1}^{m-2} a_i(1-\xi_i)}{1-\sum_{i=1}^{m-2} a_i} \int_0^1 a(s)ds}\Big)^{-1} + 1.$$

From (H4), there exists $R_2 > R_1$ such that

$$g_1(x,y) \geq N^* x, \quad \forall x \geq R_2,\ y \in \mathbb{R}_-. \tag{3.11}$$

Let $\Omega_2 = \{x \in C^1[0,1] : \|x\| < \frac{R_2}{a^*}\}$. Then

$$Ax \not\leq x, \quad \forall x \in \partial\Omega_2 \cap P.$$

In fact, if there exists $x_0 \in \partial\Omega_2 \cap P$ with $x_0 \geq A_n x_0$. By the definition of the cone and Lemma 2.3, one has

$$x_0(t) \geq \gamma\|x_0\|_1 \geq \gamma x(0) \geq \gamma\delta\|x_0\|_2, \quad x_0(t) \geq \frac{R_2}{a^*} > R_2, \quad \forall t \in [0,1],$$

from (3.11),

$$\begin{aligned}
\gamma x_0(t) &\geq \gamma A_n x_0(t) \\
&= \gamma\Big(-\int_0^t (t-s)a(s)f(s,x_0(s),-|x_0'(s)| - \frac{1}{n})ds \\
&\quad + \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Big(\int_0^1 (1-s)a(s)f(s,x_0(s),-|x_0'(s)| - \frac{1}{n})ds \\
&\quad - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)a(s)f(s,x_0(s),-|x_0'(s)| - \frac{1}{n})ds\Big)\Big) \\
&\geq \gamma\Big(-\int_0^t (t-s)a(s)f(s,x_0(s),-|x_0'(s)| - \frac{1}{n})ds \\
&\quad + \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Big(\int_0^1 (1-s)a(s)f(s,x_0(s),-|x_0'(s)| - \frac{1}{n})ds \\
&\quad - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)a(s)f(s,x_0(s),-|x_0'(s)| - \frac{1}{n})ds\Big)\Big) \\
&\geq \gamma\Big(\frac{\sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i}\int_0^t (t-s)a(s)f(s,x_0(s),-|x_0'(s)| - \frac{1}{n})ds \\
&\quad - \frac{\sum_{i=1}^{m-2} a_i \int_0^1 (\xi_i - s)a(s)f(s,x_0(s),-|x_0'(s)| - \frac{1}{n})ds}{1-\sum_{i=1}^{m-2} a_i}\Big) \\
&= \frac{\gamma\sum_{i=1}^{m-2} a_i(1-\xi_i)}{1-\sum_{i=1}^{m-2} a_i}\int_0^1 a(s)f(s,x_0(s),-|x_0'(s)| - \frac{1}{n})ds \\
&\geq \frac{\gamma\sum_{i=1}^{m-2} a_i(1-\xi_i)}{1-\sum_{i=1}^{m-2} a_i}\int_0^1 a(s)g_1(x_0(s),-|x_0'(s)| - \frac{1}{n})ds \\
&\geq \frac{\gamma\sum_{i=1}^{m-2} a_i(1-\xi_i)}{1-\sum_{i=1}^{m-2} a_i}\int_0^1 a(s)dsN^* x_0(s) \\
&\geq a^* \frac{\gamma\sum_{i=1}^{m-2} a_i(1-\xi_i)}{1-\sum_{i=1}^{m-2} a_i}\int_0^1 a(s)dsN^* \frac{R_2}{a^*} > \frac{R_2}{a^*}.
\end{aligned}$$

Then $\|x_0\| \geq \gamma\|x_0\|_1 > \frac{R_2}{a^*}$, which is a contradiction to $x_0 \in \partial\Omega_2 \cap P$. Then (3.10) holds. From Lemma 2.2,

$$i(A_n, \Omega_2 \cap P, P) = 0. \tag{3.12}$$

which with (3.9) guarantee that

$$i(A_n, (\Omega_2 - \bar{\Omega}_1) \cap P, P) = -1. \tag{3.13}$$

From this equality and (3.9), A_n has two fixed points with $x_{n,1} \in \Omega_1 \cap P, x_{n,2} \in (\Omega_2 - \bar{\Omega}_1) \cap P$.

For each $n \in N_0$, there exists $x_{n,1} \in \Omega_1 \cap P$ such that $x_{n,1} = A_n x_{n,1}$; that is,

$$\begin{aligned} x_{n,1}(t) = &-\int_0^t (t-s)a(s)f(s, x_{n,1}(s), -|x'_{n,1}(s)| - \frac{1}{n})ds \\ &+ \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Big(\int_0^1 (1-s)a(s)f(s, x_{n,1}(s), -|x'_{n,1}(s)| - \frac{1}{n})ds \\ &- \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)a(s)f(s, x_{n,1}(s), -|x'_{n,1}(s)| - \frac{1}{n})ds\Big). \end{aligned} \tag{3.14}$$

As in the proof of (3.5), we have $x'_{n,1}(t) \leq 0$, $t \in (0,1)$ and

$$x'_{n,1}(t) = -\int_0^t a(s)f(s, x_{n,1}(s), x'_{n,1}(s) - \frac{1}{n})ds, \quad n \in N_0,\ t \in (0,1).$$

Now we consider $\{x_{n,1}(t)\}_{n\in N_0}$ and $\{x'_{n,1}(t)\}_{n\in N_0}$. Since $\|x_{n,1}\| \leq R_1$, it follows that

$$\{x_{n,1}(t)\} \text{ is uniformly bounded on } [0,1], \tag{3.15}$$

$$\{x'_{n,1}(t)\} \text{ is uniformly bounded on } [0,1]. \tag{3.16}$$

Then

$$\{x_{n,1}(t)\} \text{ is equicontinuous on } [0,1]. \tag{3.17}$$

As in the proof as (3.6),

$$\begin{gathered} x''_{n,1}(t) + a(t)f(t, x_{n,1}(t), x'_{n,1}(t) - \frac{1}{n}) = 0, \quad 0 < t < 1, \\ x'_{n,1}(0) = 0, x_{n,1}(1) = \sum_{i=1}^{m-2} a_i x_{n,1}(\xi_i). \end{gathered} \tag{3.18}$$

Now we show that for all $t_1, t_2 \in [0,1]$,

$$|I(x'_{n,1}(t_2) - \frac{1}{n}) - I(x'_{n,1}(t_1) - \frac{1}{n})| \leq h(R_1)|\int_{t_1}^{t_2} a(t)dt|. \tag{3.19}$$

From (3.18),

$$\begin{aligned} -x''_{n,1}(t) &= a(t)f(t, x_{n,1}(t), x'_{n,1}(t) - \frac{1}{n}) \\ &\leq a(t)h(x_{n,1}(t))\big[g\big(x'_{n,1}(t) - \frac{1}{n}\big) + r(x'_{n,1}(t) - \frac{1}{n})\big], \quad \forall t \in (0,1), \end{aligned}$$

and

$$\begin{aligned} x''_{n,1}(t) &= -a(t)f\big(t, x_{n,1}(t), x'_{n,1}(t) - \frac{1}{n}\big) \\ &\geq -a(t)h(x_{n,1}(t))\big[g\big(x'_{n,1}(t) - \frac{1}{n}\big) + r(x'_{n,1}(t) - \frac{1}{n})\big], \quad \forall t \in (0,1), \end{aligned}$$

so

$$\frac{-x''_{n,1}(t)}{g(x'_{n,1}(t)-\frac{1}{n})+r(x'_{n,1}(t)-\frac{1}{n})} \le a(t)h(x_{n,1}(t)),\ \forall t\in(0,1), \tag{3.20}$$

and

$$\frac{x''_{n,1}(t)}{g(x'_{n,1}(t)-\frac{1}{n})+r(x'_{n,1}(t)-\frac{1}{n})} \ge -a(t)h(x_{n,1}(t)),\quad \forall t\in(0,1). \tag{3.21}$$

Then, for all t_1, t_2 in $[0,1]$ and $t_1<t_2$,

$$\begin{aligned}\Big|-\int_{t_1}^{t_2}\frac{1}{g(x'_{n,1}(s)-\frac{1}{n})+r(x'_{n,1}(s)-\frac{1}{n})}d(x'_{n,1}(s)-\frac{1}{n})\Big| &\le h(R_1)\int_{t_1}^{t_2}a(t)dt\\ &= h(R_1)\Big|\int_{t_1}^{t_2}a(t)dt\Big|,\end{aligned}$$

Inequality (3.19) holds.

Since I^{-1} is uniformly continuous on $[0, I(-R_1-\epsilon)]$, for all $\bar\epsilon>0$, there exists $\epsilon'>0$ such that

$$|I^{-1}(s_1)-I^{-1}(s_2)|<\bar\epsilon,\quad \forall |s_1-s_2|<\epsilon',\ s_1,s_2\in[0,I(-R_1-\epsilon)]. \tag{3.22}$$

And (3.19) guarantees that for $\epsilon'>0$, there exists $\delta'>0$ such that

$$|I(x'_{n,1}(t_2)-\frac{1}{n})-I(x'_{n,1}(t_1)-\frac{1}{n})|<\epsilon',\quad \forall\ |t_1-t_2|<\delta',\ t_1,t_2\in[0,1]. \tag{3.23}$$

From this inequality and (3.22),

$$\begin{aligned}|x'_{n,1}(t_2)-x'_{n,1}(t_1)| &= |x'_{n,1}(t_2)-\frac{1}{n}-(x'_{n,1}(t_1)-\frac{1}{n})|\\ &= |I^{-1}(I(x'_{n,1}(t_2)-\frac{1}{n}))-I^{-1}(I(x'_{n,1}(t_1)-\frac{1}{n}))|\\ &< \bar\epsilon,\quad \forall|t_1-t_2|<\delta',\ t_1,t_2\in[0,1];\end{aligned} \tag{3.24}$$

that is,

$$\{x'_{n,1}(t)\}\text{ is equicontinuous on }[0,1]. \tag{3.25}$$

From (3.15)–(3.17), (3.25) and the Arzela-Ascoli Theorem, $\{x_{n,1}(t)\}$ and $\{x'_{n,1}(t)\}$ are relatively compact on $C[0,1]$, which implies there exists a subsequence $\{x_{n_j,1}\}$ of $\{x_{n,1}\}$ and function $x_{0,1}(t)\in C[0,1]$ such that

$$\lim_{j\to+\infty}\max_{t\in[0,1]}|x_{n_j,1}(t)-x_{0,1}(t)|=0,\quad \lim_{j\to+\infty}\max_{t\in[0,1]}|x'_{n_j,1}(t)-x'_{0,1}(t)|=0.$$

Since $x'_{n_j,1}(0)=0$, $x_{n_j,1}(1)=\sum_{i=1}^{m-2}a_ix_{n_j,1}(\xi_i)$, $x'_{n_j,1}(t)<0$, $x_{n_j,1}(t)>0$, $t\in(0,1)$, $j\in\{1,2,\dots\}$,

$$x'_{0,1}(0)=0,\quad x_{0,1}(1)=\sum_{i=1}^{m-2}a_ix_{0,1}(\xi_i),\quad x'_{0,1}(t)\le 0,\quad x_{0,1}(t)\ge 0,\quad t\in(0,1). \tag{3.26}$$

For $(t,x_{n_j,1}(t),x'_{n_j,1}(t)-\frac{1}{n})\in[0,1]\times[0,R_1+\epsilon]\times(-\infty,0)$, from (H5) there exists a function $\Psi_{R_1}\in C([0,1],\mathbb{R}_+)$ such that

$$f(t,x_{n_j,1}(t),x'_{n_j,1}(t)-\frac{1}{n_j})ds\ge\Psi_{R_1}(t)(x_{n_j,1}(t))^\delta,\quad 0\le\delta<1.$$

Then, for $n \in N_0$,

$$
\begin{aligned}
x_{n_j,1}(t) &= -\int_0^t (t-s)a(s)f(s, x_{n_j,1}(s), x'_{n_j,1}(s) - \frac{1}{n_j})ds \\
&\quad + \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Big(\int_0^1 (1-s)a(s)f(s, x_{n_j,1}(s), x'_{n_j,1}(s) - \frac{1}{n_j})ds \\
&\quad - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)a(s)f(s, x_{n_j,1}(s), x'_{n_j,1}(s) - \frac{1}{n_j})ds\Big) \\
&\geq -\int_0^1 (1-s)a(s)f(s, x_{n_j,1}(s), x'_{n_j,1}(s) - \frac{1}{n_j})ds \\
&\quad + \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Big(\int_0^1 (1-s)a(s)f(s, x_{n_j,1}(s), x'_{n_j,1}(s) - \frac{1}{n_j})ds \\
&\quad - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)a(s)f(s, x_{n_j,1}(s), x'_{n_j,1}(s) - \frac{1}{n_j})ds\Big) \\
&\geq \frac{\sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i}\int_0^1 (1-s)a(s)f(s, x_{n_j,1}(s), x'_{n_j,1}(s) - \frac{1}{n_j})ds \\
&\quad - \frac{\sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i}\int_0^1 (\xi_i - s)a(s)f(s, x_{n_j,1}(s), x'_{n_j,1}(s) - \frac{1}{n_j})ds \\
&= \frac{\sum_{i=1}^{m-2} a_i(1-\xi_i)}{1-\sum_{i=1}^{m-2} a_i}\int_0^1 a(s)f(s, x_{n_j,1}(s), x'_{n_j,1}(s) - \frac{1}{n_j})ds \\
&\geq \frac{\sum_{i=1}^{m-2} a_i(1-\xi_i)}{1-\sum_{i=1}^{m-2} a_i}\int_0^1 a(s)\Psi_{R_1}(s)(x_{n_j,1}(s))^\delta ds \\
&\geq \frac{\sum_{i=1}^{m-2} a_i(1-\xi_i)}{1-\sum_{i=1}^{m-2} a_i}\int_0^1 a(s)\Psi_{R_1}(s)\gamma^\delta ds \|x_{n_j,1}\|_1^\delta,
\end{aligned}
$$

which implies

$$
\|x_{n_j,1}\|_1 \geq \Big(\frac{\sum_{i=1}^{m-2} a_i(1-\xi_i)}{1-\sum_{i=1}^{m-2} a_i}\int_0^1 a(s)\Psi_{R_1}(s)\gamma^\delta ds\Big)^{\frac{1}{1-\delta}},
$$

and

$$
x_{n_j,1}(t) \geq \Big(\frac{\sum_{i=1}^{m-2} a_i(1-\xi_i)}{1-\sum_{i=1}^{m-2} a_i}\int_0^1 a(s)\Psi_{R_1}(s)\gamma^\delta ds\Big)^{\frac{1}{1-\delta}} = a_0 > 0.
$$

Thus

$$
\begin{aligned}
x'_{n_j,1}(t) &= -\int_0^t a(s)f(s, x_{n_j,1}(s), x'_{n_j,1}(s) - \frac{1}{n_j})ds \\
&\leq -\int_0^t a(s)\Psi_{R_1}(s)(x_{n_j,1}(s))^\delta ds \\
&\leq -\int_0^t a(s)\Psi_{R_1}(s)ds a_0^\delta, \quad t \in [0,1], n \in N_0.
\end{aligned}
$$

Consequently,

$$\inf_{j\geq 1}\min_{s\in[\frac{1}{2},t]}|x'_{n_j,1}(s)|>0,\quad t\in[\frac{1}{2},1),$$
$$\inf_{j\geq 1}\min_{s\in[t,\frac{1}{2}]}|x'_{n_j,1}(s)|>0,\quad t\in(0,\frac{1}{2}].$$

Since

$$x'_{n_j,1}(t)-x'_{n_j,1}(\frac{1}{2})=-\int_{1/2}^{t}a(s)f(s,x_{n_j,1}(s),x'_{n_j,1}(s)-\frac{1}{n_j})ds,\quad t\in(0,1),$$

and

$$\begin{aligned}f(t,x_{n_j,1}(t),x'_{n_j,1}(t)-\frac{1}{n_j})&\leq h(x_{n_j,1}(t))[g(x'_{n_j,1}(t)-\frac{1}{n_j})+r(x'_{n_j,1}(t)-\frac{1}{n_j})]\\&\leq h(\frac{R_1}{\gamma})[g(-\int_0^t a(s)\Psi_{R_1}(s)dsa_0^{\delta})+r(-\frac{R_1}{\gamma\delta}-\epsilon)],\end{aligned}$$

letting $j\to+\infty$, the Lebesgue Dominated Convergence Theorem guarantees that

$$x'_{0,1}(t)-x'_{0,1}(\frac{1}{2})=-\int_{1/2}^{t}a(s)f(s,x_{0,1}(s),x'_{0,1}(s))ds,\quad t\in(0,1). \tag{3.27}$$

Differentiating, we have

$$x''_{0,1}(t)+a(t)f(t,x_{0,1}(t),x'_{0,1}(t))=0,\quad 0<t<1,$$

and from (3.26) $x_{0,1}(t)$ is a positive solution of (1.1) with $x_{0,1}\in C^1[0,1]\cap C^2(0,1)$.

For the set $\{x_{n,2}\}_{n\in N_0}\subseteq(\Omega_2-\overline{\Omega}_1)\cap P$, the proof is as that for the set $\{x_{n,1}\}_{n\in N_0}$. We can obtain a convergent subsequence $\{x_{n_i,2}\}_{n\in N_0}$ of $\{x_{n,2}\}_{n\in N_0}$ with $\lim_{i\to+\infty}x_{n_i,2}=x_{0,2}\in C^1[0,1]\cap C^2(0,1)$. Moreover, $x_{0,2}$ is a positive solution to (1.1). □

Example 3.1 In (1.1), let $f(t,u,z)=\mu[1+(-z)^{-a}][1+u^b+u^d]$ and $a(t)\equiv 1$ with $0<a<1, b>1, 0<d<1$ and $\mu>0$. If

$$\mu<\sup_{c\in\mathbb{R}_+}\frac{I(-\frac{c(1-\sum_{i=1}^{m-2}a_i)}{1+\sum_{i=1}^{m-2}a_i\xi_i})}{1+c^b+c^d}. \tag{3.28}$$

Then (1.1) has at least two positive solutions $x_{0,1},x_{0,2}\in C^1[0,1]\cap C^2(0,1)$.

We apply Theorem 3.1 with $g(z)=(-z)^{-a}, r(z)=1, h(u)=\mu(1+u^b+u^d), \Psi(t)=\mu, g_1(u,z)=\mu u^b$. (H1), (H2), (H4), (H5) hold. Also

$$\begin{aligned}&\sup_{c\in\mathbb{R}_+}\frac{c}{-\frac{\sum_{i=1}^{m-2}a_i\xi_i+1}{1-\sum_{i=1}^{m-2}a_i}I^{-1}(h(c)\int_0^1 a(s)ds)}\\&=\sup_{c\in\mathbb{R}_+}\frac{c}{-\frac{\sum_{i=1}^{m-2}a_i\xi_i+1}{1-\sum_{i=1}^{m-2}a_i}I^{-1}(\mu(1+c^b+c^d))},\end{aligned}$$

and (3.28) guarantees that (H3) holds.

4. Singularities at $x' = 0$ and $x = 0$

In this section the nonlinearity f may be singular at $x' = 0$ and $x = 0$. We assume that the following conditions hold.

(P1) $a(t) \in C[0,1]$, $a(t) > 0$, $t \in (0,1)$;

(P2) $f(t,u,z) \le [h(u)+\omega(u)][g(z)+r(z)]$, where $f \in C([0,1]\times\mathbb{R}_+\times\mathbb{R}_-,\mathbb{R}_+)$, $g(z) > 0$ continuous and non-increasing on $(-\infty,0]$, $\omega(u) > 0$ continuous and non-increasing on $[0,+\infty)$, $h(u) \ge 0$ continuous and nondecreasing on $\mathbb{R}_+$, $r(z) > 0$ continuous and nondecreasing on $\mathbb{R}_-$;

(P3)
$$\sup_{c\in\mathbb{R}_+} \frac{c}{-\frac{\sum_{i=1}^{m-2} a_i\xi_i+1}{1-\sum_{i=1}^{m-2} a_i}(I^{-1}[-\max_{t\in[0,1]} a(t)(ch(c)+\int_0^c \omega(s)ds)])} > 1,$$
where $I(z) = \int_z^0 \frac{udu}{g(u)+r(u)}$, $z \in \mathbb{R}_-$, $\int_0^a \omega(s)ds < +\infty$;

(P4) There exists a function $g_1 \in C([0,+\infty)\times(-\infty,0],[0,+\infty))$, such that $f(t,u,z) \ge g_1(u,z), \forall (t,u,z) \in [0,1]\times\mathbb{R}_+\times\mathbb{R}_-$, and $\lim_{u\to+\infty} \frac{g_1(u,z)}{u} = +\infty$, uniformly for $z \in \mathbb{R}_-$.

(P5) There exists a function $\Psi_H \in C([0,1],\mathbb{R}_+)$ with $f(t,u,z) \ge \Psi_H(t)$, for all $(t,u,z) \in [0,1]\times[0,H]\times[-H,0)$.

For $n \in \{1,2,\dots\}$, $x \in P$, $t \in [0,1]$, define operator
$$\begin{aligned}(A_nx)(t) &= -\int_0^t (t-s)a(s)f(s,x(s)+\frac{1}{n},-|x'(s)|-\frac{1}{n})ds \\ &\quad+\frac{1}{1-\sum_{i=1}^{m-2} a_i}\Big(\int_0^1 (1-s)a(s)f(s,x(s)+\frac{1}{n},-|x'(s)|-\frac{1}{n})ds \\ &\quad-\sum_{i=1}^{m-2} a_i\int_0^{\xi_i} (\xi_i-s)a(s)f(s,x(s)+\frac{1}{n},-|x'(s)|-\frac{1}{n})ds\Big).\end{aligned} \tag{4.1}$$

Theorem 4.1. *Suppose* (P1)–(P5) *hold. Then* (1.1) *has at least two positive solutions $x_{0,1}, x_{0,2} \in C^1[0,1]\cap C^2(0,1)$ and $x_{0,1}(t), x_{0,2}(t) > 0$, $t \in [0,1]$.*

Proof. Choose $R_1 > 0$ such that
$$\frac{R_1}{-\frac{\sum_{i=1}^{m-2} a_i\xi_i+1}{1-\sum_{i=1}^{m-2} a_i}(I^{-1}[-\max_{t\in[0,1]} a(t)(R_1h(R_1)+\int_0^{R_1} \omega(s)ds)])} > 1. \tag{4.2}$$
From the continuity of I^{-1} and h, we can choose $\epsilon > 0$ and $\epsilon < R_1$ such that
$$\frac{R_1}{-\frac{\sum_{i=1}^{m-2} a_i\xi_i+1}{1-\sum_{i=1}^{m-2} a_i}(I^{-1}[I(-\epsilon)-\max_{t\in[0,1]} a(t)((R_1+\epsilon)h(R_1+\epsilon)+\int_0^{R_1+\epsilon} \omega(s)ds)])} \tag{4.3}$$
is greater than 1, $n_0 \in \{1,2,\dots\}$ with $\frac{1}{n_0} < \min\{\epsilon,\delta/2\}$ and let $N_0 = \{n_0, n_0+1,\dots\}$. Then Lemma 2.4 guarantees that for $n \in N_0$, $A_n : P \to P$ is a completely continuous operator. Let
$$\Omega_1 = \{x \in C^1[0,1] : \|x\| < R_1\}.$$
We show that
$$x \ne \mu A_nx, \quad \forall x \in P\cap\partial\Omega_1,\ \mu \in (0,1],\ n \in N_0. \tag{4.4}$$

In fact, if there exists an $x_0 \in P \cap \partial\Omega_1$ and $\mu_0 \in (0,1]$ with $x_0 = \mu_0 A_n x_0$,

$$
\begin{aligned}
x_0(t) = &-\mu_0 \int_0^t (t-s)a(s)f(s, x_0(s)+\frac{1}{n}, -|x_0'(s)|-\frac{1}{n})ds \\
&+ \frac{\mu_0}{1-\sum_{i=1}^{m-2} a_i}\Big(\int_0^1 (1-s)a(s)f(s, x_0(s)+\frac{1}{n}, -|x_0'(s)|-\frac{1}{n})ds \\
&- \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)a(s)f(s, x_0(s)+\frac{1}{n}, -|x_0'(s)|-\frac{1}{n})ds\Big), \quad t\in[0,1].
\end{aligned}
$$

Then

$$
x_0'(t) = -\mu_0 \int_0^t a(s)f(s, x_0(s)+\frac{1}{n}, -|x_0'(s)|-\frac{1}{n})ds, \quad \forall t\in[0,1]. \tag{4.5}
$$

Obviously, $x_0'(t) \le 0$, $t\in(0,1)$, and since $x_0(1) > 0$, $x_0(t) > 0, t \in [0,1]$. Differentiating (4.5), we have

$$
\begin{gathered}
x_0''(t) + \mu_0 a(t) f(t, x_0(t)+\frac{1}{n}, x_0'(t)-\frac{1}{n}) = 0, \quad 0<t<1, \\
x_0'(0) = 0, \quad x_0(1) = \sum_{i=1}^{m-2} a_i x_0(\xi_i).
\end{gathered} \tag{4.6}
$$

Then, for $t\in(0,1)$,

$$
\begin{aligned}
-x_0''(t) &= \mu_0 a(t) f(t, x_0(t)+\frac{1}{n}, x_0'(t)-\frac{1}{n}) \\
&\le a(t)\big[h\big(x_0(t)+\frac{1}{n}\big) + \omega(x_0(t)+\frac{1}{n})\big]\big[g\big(x_0'(t)-\frac{1}{n}\big) + r(x_0'(t)-\frac{1}{n})\big],
\end{aligned}
$$

and

$$
\frac{-x_0''(t)}{g(x_0'(t)-\frac{1}{n}) + r(x_0'(t)-\frac{1}{n})} \le a(t)[h(x_0(t)+\frac{1}{n}) + \omega(x_0(t)+\frac{1}{n})], \quad \forall t\in(0,1),
$$

and

$$
\begin{aligned}
&\frac{-x_0''(t)(x_0'(t)-\frac{1}{n})}{g(x_0'(t)-\frac{1}{n}) + r(x_0'(t)-\frac{1}{n})} \\
&\ge a(t)[h(x_0(t)+\frac{1}{n}) + \omega(x_0(t)+\frac{1}{n})](x_0'(t)-\frac{1}{n}) \\
&\ge a(t)[h(R_1+\epsilon) + \omega(x_0(t)+\frac{1}{n}(1-t))](x_0'(t)-\frac{1}{n}) \\
&= a(t)[h(R_1+\epsilon)(x_0'(t)-\frac{1}{n}) + \omega(x_0(t)+\frac{1}{n}(1-t))(x_0'(t)-\frac{1}{n})]
\end{aligned} \tag{4.7}
$$

Integrating from 0 to t, we have

$$
\begin{aligned}
&I(x_0'(t)-\frac{1}{n}) - I(-\frac{1}{n}) \\
&\ge \int_0^t a(s)\big[h(R_1+\epsilon)(x_0'(s)-\frac{1}{n}) + \omega(x_0(s)+\frac{1}{n}(1-s))(x_0'(s)-\frac{1}{n})\big]ds \\
&\ge \max_{t\in[0,1]} a(t)h(R_1+\epsilon)\Big(\int_0^t x_0'(s)ds - \int_0^t \frac{1}{n}ds\Big) + \int_0^t \omega(x_0(s)+\frac{1}{n}(1-s))d(x_0(s) \\
&\quad + \frac{1}{n}(1-s))
\end{aligned}
$$

$$\geq -\max_{t\in[0,1]} a(t)(h(R_1+\epsilon)(R_1+\epsilon)+\int_0^{R_1+\epsilon}\omega(s)ds),$$

$$I(x_0'(t)-\frac{1}{n})\geq I(-\epsilon)-\max_{t\in[0,1]} a(t)(h(R_1+\epsilon)(R_1+\epsilon)+\int_0^{R_1+\epsilon}\omega(s)ds).$$

Then

$$x_0'(t)\geq I^{-1}\Big(I(-\epsilon)-\max_{t\in[0,1]}\{a(t)\}\big(h(R_1+\epsilon)(R_1+\epsilon)+\int_0^{R_1+\epsilon}\omega(s)ds\big)\Big);$$

that is,

$$-x_0'(t)\leq -I^{-1}\Big(I(-\epsilon)-\max_{t\in[0,1]} a(t)(h(R_1+\epsilon)(R_1+\epsilon)+\int_0^{R_1+\epsilon}\omega(s)ds)\Big),\quad t\in(0,1). \tag{4.8}$$

Since

$$\begin{aligned}
x_0(0)&=\frac{\mu_0}{1-\sum_{i=1}^{m-2}a_i}\int_0^1 -x_0'(s)ds-\frac{\mu_0\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}-x_0'(s)ds}{1-\sum_{i=1}^{m-2}a_i}\\
&\leq \frac{1}{1-\sum_{i=1}^{m-2}a_i}\int_0^1 -I^{-1}\Big(I(-\epsilon)-\max_{t\in[0,1]}\{a(t)\}(h(R_1+\epsilon)(R_1+\epsilon)\\
&\quad+\int_0^{R_1+\epsilon}\omega(s)ds)\Big)ds+\frac{\sum_{i=1}^{m-2}a_i}{1-\sum_{i=1}^{m-2}a_i}\int_0^{\xi_i}-I^{-1}\Big(I(-\epsilon)\\
&\quad-\max_{t\in[0,1]} a(t)(h(R_1+\epsilon)(R_1+\epsilon)+\int_0^{R_1+\epsilon}\omega(s)ds)\Big)ds\\
&=\frac{1+\sum_{i=1}^{m-2}a_i\xi_i}{1-\sum_{i=1}^{m-2}a_i}(-I^{-1}\Big(I(-\epsilon)-\max_{t\in[0,1]} a(t)(h(R_1+\epsilon)(R_1+\epsilon)\\
&\quad+\int_0^{R_1+\epsilon}\omega(s)ds)\Big).
\end{aligned}$$

Since $x_0(0)\geq x_0(t)\geq\gamma\|x_0\|_1\geq\gamma x_0(0)\geq\gamma\delta\|x_0\|_2$, $x_0(0)\geq\|x_0\|=R_1$. So

$$\frac{R_1}{-\frac{\sum_{i=1}^{m-2}a_i\xi_i+1}{1-\sum_{i=1}^{m-2}a_i}(I^{-1}[I(-\epsilon)-\max\{a(t)\}((R_1+\epsilon)h(R_1+\epsilon)+\int_0^{R_1+\epsilon}\omega(s)ds)])}\leq 1, \tag{4.9}$$

which is a contradiction to (4.3). Then (4.4) holds.

From Lemma 2.1, for $n\in N_0$,

$$i(A_n,\Omega_1\cap P,P)=1. \tag{4.10}$$

Now we show that there exists a set Ω_2 such that

$$A_nx\not\leq x,\quad \forall x\in\partial\Omega_2\cap P. \tag{4.11}$$

Choose a^*,N^* as in section 3. Let

$$\Omega_2=\{x\in C^1[0,1]:\|x\|<\frac{R_2}{a^*}\}.$$

Then

$$A_nx\not\leq x,\quad \forall x\in\partial\Omega_2\cap P.$$

In fact, if there exists $x_0 \in \partial\Omega_2 \cap P$ with $x_0 \geq A_n x_0$. By the definition of the cone and Lemma 2.3, one has

$$x_0(t) \geq \gamma \|x_0\|_1 \geq \gamma x(0) \geq \gamma\delta \|x_0\|_2,$$

and so $x_0(t) \geq \frac{R_2}{a^*} > R_2$, for all $t \in [0,1]$, $x_0(t) + \frac{1}{n} > R_2$, from (3.11),

$$\begin{aligned}
\gamma x_0(t) &\geq \gamma A_n x_0(t)\\
&= \gamma\Big(-\int_0^t (t-s)a(s)f(s, x_0(s) + \frac{1}{n}, -|x_0'(s)| - \frac{1}{n})ds\\
&\quad + \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Big(\int_0^1 (1-s)a(s)f(s, x_0(s) + \frac{1}{n}, -|x_0'(s)| - \frac{1}{n})ds\\
&\quad - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)a(s)f(s, x_0(s) + \frac{1}{n}, -|x_0'(s)| - \frac{1}{n})ds\Big)\Big)\\
&\geq \gamma\Big(-\int_0^t (t-s)a(s)f(s, x_0(s) + \frac{1}{n}, -|x_0'(s)| - \frac{1}{n})ds\\
&\quad + \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Big(\int_0^1 (1-s)a(s)f(s, x_0(s) + \frac{1}{n}, -|x_0'(s)| - \frac{1}{n})ds\\
&\quad - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)a(s)f(s, x_0(s) + \frac{1}{n}, -|x_0'(s)| - \frac{1}{n})ds\Big)\Big)\\
&\geq \gamma\Big(\frac{\sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i} \int_0^t (t-s)a(s)f(s, x_0(s) + \frac{1}{n}, -|x_0'(s)| - \frac{1}{n})ds\\
&\quad - \frac{\sum_{i=1}^{m-2} a_i \int_0^1 (\xi_i - s)a(s)f\big(s, x_0(s) + \frac{1}{n}, -|x_0'(s)| - \frac{1}{n}\big)ds}{1-\sum_{i=1}^{m-2} a_i}\Big)\\
&= \frac{\gamma\sum_{i=1}^{m-2} a_i(1-\xi_i)}{1-\sum_{i=1}^{m-2} a_i} \int_0^1 a(s)f(s, x_0(s) + \frac{1}{n}, -|x_0'(s)| - \frac{1}{n})ds\\
&\geq \frac{\gamma\sum_{i=1}^{m-2} a_i(1-\xi_i)}{1-\sum_{i=1}^{m-2} a_i} \int_0^1 a(s)g_1(x_0(s) + \frac{1}{n}, -|x_0'(s)| - \frac{1}{n})ds\\
&\geq \frac{\gamma\sum_{i=1}^{m-2} a_i(1-\xi_i)}{1-\sum_{i=1}^{m-2} a_i} \int_0^1 a(s)dsN^* x_0(s)\\
&\geq a^* \frac{\gamma\sum_{i=1}^{m-2} a_i(1-\xi_i)}{1-\sum_{i=1}^{m-2} a_i} \int_0^1 a(s)dsN^* \frac{R_2}{a^*} > \frac{R_2}{a^*}.
\end{aligned}$$

Then $\|x_0\| \geq \gamma\|x_0\|_1 > \frac{R_2}{a^*}$, which is a contradiction to $x_0 \in \partial\Omega_2 \cap P$. Then (4.11) holds.

From Lemma 2.2,

$$i(A_n, \Omega_2 \cap P, P) = 0. \tag{4.12}$$

This equality and (4.10) guarantee,

$$i(A_n, (\Omega_2 - \bar{\Omega}_1) \cap P, P) = -1. \tag{4.13}$$

From this equality and (4.10), A_n has two fixed points with $x_{n,1} \in \Omega_1 \cap P, x_{n,2} \in (\Omega_2 - \overline{\Omega}_1) \cap P$. For each $n \in N_0$, there exists $x_{n,1} \in \Omega_1 \cap P$ with $x_{n,1} = A_n x_{n,1}$;

that is,

$$\begin{aligned} x_{n,1}(t) = &-\int_0^t (t-s)a(s)f(s,x_{n,1}(s)+\frac{1}{n},-|x'_{n,1}(s)|-\frac{1}{n})ds \\ &+\frac{1}{1-\sum_{i=1}^{m-2}a_i}\Big(\int_0^1(1-s)a(s)f(s,x_{n,1}(s)+\frac{1}{n},-|x'_{n,1}(s)|-\frac{1}{n})ds \\ &-\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}(\xi_i-s)a(s)f(s,x_{n,1}(s)+\frac{1}{n},-|x'_{n,1}(s)|-\frac{1}{n})ds\Big). \end{aligned} \tag{4.14}$$

As in the same proof of (3.5), $x'_n(t)\le 0$, $t\in(0,1)$ and

$$x'_{n,1}(t) = -\int_0^t a(s)f(s,x_{n,1}(s)+\frac{1}{n},x'_{n,1}(s)-\frac{1}{n})ds, \quad n\in N_0,\ t\in(0,1).$$

Now we consider $\{x_{n,1}(t)\}_{n\in N_0}$ and $\{x'_{n,1}(t)\}_{n\in N_0}$, since $\|x_{n,1}\|\le R_1$, it follows that

$$\{x_{n,1}(t)\} \text{ is uniformly bounded on } [0,1], \tag{4.15}$$

$$\{x'_{n,1}(t)\} \text{ is uniformly bounded on } [0,1]. \tag{4.16}$$

Then

$$\{x_{n,1}(t)\} \text{ is equicontinuous on } [0,1]. \tag{4.17}$$

As in the same proof of (3.6),

$$\begin{gathered} x''_{n,1}(t)+a(t)f(t,x_{n,1}(t)+\frac{1}{n},x'_{n,1}(t)-\frac{1}{n})=0, \quad 0<t<1, \\ x'_{n,1}(0)=0, \quad x_{n,1}(1)=\sum_{i=1}^{m-2}a_ix_{n,1}(\xi_i). \end{gathered} \tag{4.18}$$

Now we show for all t_1, t_2 in $[0,1]$,

$$\begin{aligned} &|I(x'_{n,1}(t_2)-\frac{1}{n})-I(x'_{n,1}(t_1)-\frac{1}{n})| \\ &\le \max_{t\in[0,1]}a(t)\big[h(R_1+\epsilon)(|x_{n,1}(t_2)-x_{n,1}(t_1)|+|t_2-t_1|) \\ &\quad +|\int_{x_{n,1}(t_1)+\frac{1}{n}(1-t_1)}^{x_{n,1}(t_2)+\frac{1}{n}(1-t_2)}\omega(s)dt|\big]. \end{aligned} \tag{4.19}$$

From (4.18), it follows that for $t\in(0,1)$,

$$\begin{aligned} -x''_{n,1}(t) &= a(t)f(t,x_{n,1}(t)+\frac{1}{n},x'_{n,1}(t)-\frac{1}{n}) \\ &\le a(t)[h(x_{n,1}(t)+\frac{1}{n})+\omega(x_{n,1}(t)+\frac{1}{n})][g(x'_{n,1}(t)-\frac{1}{n})+r(x'_{n,1}(t)-\frac{1}{n})], \end{aligned}$$

$$\begin{aligned} x''_{n,1}(t) &= -a(t)f(t,x_{n,1}(t)+\frac{1}{n},x'_{n,1}(t)-\frac{1}{n}) \\ &\ge -a(t)[h(x_{n,1}(t)+\frac{1}{n})+\omega(x_{n,1}(t)+\frac{1}{n})][g(x'_{n,1}(t)-\frac{1}{n})+r(x'_{n,1}(t)-\frac{1}{n})], \end{aligned}$$

and so for $t \in (0,1)$,

$$\begin{aligned}
&\frac{-x''_{n,1}(t)(x'_{n,1}(t)-\frac{1}{n})}{g(x'_{n,1}(t)-\frac{1}{n})+r(x'_{n,1}(t)-\frac{1}{n})}\\
&\geq a(t)[h(x_{n,1}(t)+\frac{1}{n})+\omega(x_{n,1}(t)+\frac{1}{n})](x'_{n,1}(t)-\frac{1}{n})\\
&\geq a(t)[h(R_1+\epsilon)(x'_{n,1}(t)-\frac{1}{n})+\omega(x_{n,1}(t)+\frac{1}{n}(1-t))(x_{n,1}(t)+\frac{1}{n}(1-t))'],
\end{aligned} \tag{4.20}$$

$$\begin{aligned}
&\frac{x''_{n,1}(t)(x'_{n,1}(t)-\frac{1}{n})}{g(x'_{n,1}(t)-\frac{1}{n})+r(x'_{n,1}(t)-\frac{1}{n})}\\
&\leq -a(t)[h(x_{n,1}(t)+\frac{1}{n})+\omega(x_{n,1}(t)+\frac{1}{n})](x'_{n,1}(t)-\frac{1}{n})\\
&\leq -a(t)[h(R_1+\epsilon)(x'_{n,1}(t)-\frac{1}{n})+\omega(x_{n,1}(t)+\frac{1}{n}(1-t))(x_{n,1}(t)+\frac{1}{n}(1-t))'].
\end{aligned} \tag{4.21}$$

Then, for all t_1, $t_2 \in [0,1]$ and $t_1 < t_2$,

$$\begin{aligned}
&I(x'_{n,1}(t_2)-\frac{1}{n})-I(x'_{n,1}(t_1)-\frac{1}{n})\\
&\geq \max_{t\in[0,1]} a(t)[h(R_1+\epsilon)(\int_{t_1}^{t_2} x'_{n,1}(s)dt-\frac{1}{n}(t_2-t_1)+\int_{x_{n,1}(t_1)+\frac{1}{n}(1-t_1)}^{x_{n,1}(t_2)+\frac{1}{n}(1-t_2)}\omega(s)dt]\\
&\geq -\max_{t\in[0,1]} a(t)\big[h(R_1+\epsilon)(|x_{n,1}(t_2)-x_{n,1}(t_1)|+|t_2-t_1|)\\
&\quad +|\int_{x_{n,1}(t_1)+\frac{1}{n}(1-t_1)}^{x_{n,1}(t_2)+\frac{1}{n}(1-t_2)}\omega(s)dt|\big],
\end{aligned}$$

$$\begin{aligned}
&I(x'_{n,1}(t_1)-\frac{1}{n})-I(x'_{n,1}(t_2)-\frac{1}{n})\\
&\leq \max_{t\in[0,1]} a(t)\big[h(R_1+\epsilon)(|x_{n,1}(t_2)-x_{n,1}(t_1)|+|t_2-t_1|)\\
&\quad +|\int_{x_{n,1}(t_1)+\frac{1}{n}(1-t_1)}^{x_{n,1}(t_2)+\frac{1}{n}(1-t_2)}\omega(s)dt|\big].
\end{aligned}$$

Therefore, (4.19) holds. Since I^{-1} is uniformly continuous on $[0, I(-R_1-\epsilon)]$, for all $\bar\epsilon > 0$, there exists $\epsilon' > 0$ such that

$$|I^{-1}(s_1)-I^{-1}(s_2)|<\bar\epsilon, \forall\ |s_1-s_2|<\epsilon', s_1,s_2\in[0,I(-R_1-\epsilon)]. \tag{4.22}$$

Then (4.19) guarantees that for $\epsilon' > 0$, there exists $\delta' > 0$ such that

$$|I(x'_{n,1}(t_2)-\frac{1}{n})-I(x'_{n,1}(t_1)-\frac{1}{n})|<\epsilon', \quad \forall|t_1-t_2|<\delta', t_1,t_2\in[0,1]. \tag{4.23}$$

From this inequality and (4.22),

$$\begin{aligned}
|x'_{n,1}(t_2)-x'_{n,1}(t_1)| &= |x'_{n,1}(t_2)-\frac{1}{n}-(x'_{n,1}(t_1)-\frac{1}{n})|\\
&= |I^{-1}(I(x'_{n,1}(t_2)-\frac{1}{n}))-I^{-1}(I(x'_{n,1}(t_1)-\frac{1}{n}))|\\
&< \bar\epsilon, \quad \forall|t_1-t_2|<\delta', t_1,t_2\in[0,1];
\end{aligned} \tag{4.24}$$

that is,

$$\{x'_{n,1}(t)\} \text{ is equicontinuous on } [0,1]. \tag{4.25}$$

From (4.15)–(4.17), (4.25) and the Arzela-Ascoli Theorem, $\{x_{n,1}(t)\}$ and $\{x'_{n,1}(t)\}$ are relatively compact on $C[0,1]$, which implies, there exists a subsequence $\{x_{n_j,1}\}$ of $\{x_{n,1}\}$ and function $x_{0,1}(t) \in C[0,1]$ such that

$$\lim_{j\to+\infty} \max_{t\in[0,1]} |x_{n_j,1}(t) - x_{0,1}(t)| = 0, \quad \lim_{j\to+\infty} \max_{t\in[0,1]} |x'_{n_j,1}(t) - x'_{0,1}(t)| = 0.$$

Since $x'_{n_j,1}(0) = 0$, $x_{n_j,1}(1) = \sum_{i=1}^{m-2} a_i x_{n_j,1}(\xi_i)$, $x'_{n_j,1}(t) < 0$, $x_{n_j,1}(t) > 0$, $t \in (0,1)$, $j \in \{1,2,\dots\}$,

$$x'_{0,1}(0) = 0, x_{0,1}(1) = \sum_{i=1}^{m-2} a_i x_{0,1}(\xi_i), x'_{0,1}(t) \le 0, x_{0,1}(t) \ge 0, \quad t \in (0,1). \tag{4.26}$$

For $(t, x_{n_j,1}(t) + \frac{1}{n_j}, x'_{n_j,1}(t) - \frac{1}{n_j}) \in [0,1] \times [0, R_1+\epsilon] \times (-\infty, 0)$, from (P5) there exists a function $\Psi_{R_1} \in C([0,1], \mathbb{R}_+)$ such that

$$f(t, x_{n_j,1}(t) + \frac{1}{n_j}, x'_{n_j,1}(t) - \frac{1}{n_j})ds \ge \Psi_{R_1}(t).$$

Then, for $n \in N_0$,

$$\begin{aligned}
x_{n_j,1}(t) &= -\int_0^t (t-s)a(s)f(s, x_{n_j,1}(s) + \frac{1}{n_j}, x'_{n_j,1}(s) - \frac{1}{n_j})ds \\
&\quad + \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Big(\int_0^1 (1-s)a(s)f(s, x_{n_j,1}(s) + \frac{1}{n_j}, x'_{n_j,1}(s) - \frac{1}{n_j})ds \\
&\quad - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)a(s)f(s, x_{n_j,1}(s) + \frac{1}{n_j}, x'_{n_j,1}(s) - \frac{1}{n_j})ds\Big) \\
&\ge -\int_0^1 (1-s)a(s)f(s, x_{n_j,1}(s) + \frac{1}{n_j}, x'_{n_j,1}(s) - \frac{1}{n_j})ds \\
&\quad + \frac{1}{1-\sum_{i=1}^{m-2} a_i}(\int_0^1 (1-s)a(s)f(s, x_{n_j,1}(s) + \frac{1}{n_j}, x'_{n_j,1}(s) - \frac{1}{n_j})ds \\
&\quad - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)a(s)f(s, x_{n_j,1}(s) + \frac{1}{n_j}, x'_{n_j,1}(s) - \frac{1}{n_j})ds) \\
&\ge \frac{\sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i} \int_0^1 (1-s)a(s)f(s, x_{n_j,1}(s) + \frac{1}{n_j}, x'_{n_j,1}(s) - \frac{1}{n_j})ds \\
&\quad - \frac{\sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i} \int_0^1 (\xi_i - s)a(s)f(s, x_{n_j,1}(s) + \frac{1}{n_j}, x'_{n_j,1}(s) - \frac{1}{n_j})ds \\
&= \frac{\sum_{i=1}^{m-2} a_i(1-\xi_i)}{1-\sum_{i=1}^{m-2} a_i} \int_0^1 a(s)f(s, x_{n_j,1}(s) + \frac{1}{n_j}, x'_{n_j,1}(s) - \frac{1}{n_j})ds \\
&\ge \frac{\sum_{i=1}^{m-2} a_i(1-\xi_i)}{1-\sum_{i=1}^{m-2} a_i} \int_0^1 a(s)\Psi_{R_1}(s)ds = a_0,
\end{aligned}$$

and

$$x'_{n_j,1}(t) = -\int_0^t a(s)f(s, x_{n_j,1}(s) + \frac{1}{n_j}, x'_{n_j,1}(s) - \frac{1}{n_j})ds$$

$$\leq -\int_0^t a(s)\Psi_{R_1+\epsilon}(s)ds, \quad t\in[0,1],\ n\in N_0.$$

Thus,

$$\inf_{j\geq 1}\min\{\min_{s\in[\frac12,t]} x_{n_j,1}(s),\ \min_{s\in[\frac12,t]}|x'_{n_j,1}(s)|\}>0, \quad t\in[\frac12,1),$$
$$\inf_{j\geq 1}\min\{\min_{s\in[t,\frac12]} x_{n_j,1}(s),\ \min_{s\in[t,\frac12]}|x'_{n_j,1}(s)|\}>0, \quad t\in(0,\frac12].$$

From (P2), we have

$$\begin{aligned}
&f\big(t,x_{n_j,1}(t)+\frac{1}{n_j},x'_{n_j,1}(t)-\frac{1}{n_j}\big)\\
&\leq [h(x_{n_j,1}(t)+\frac{1}{n_j})+\omega(x_{n_j,1}(t)+\frac{1}{n_j})][g(x'_{n_j,1}(t)-\frac{1}{n_j})+r(x'_{n_j,1}(t)-\frac{1}{n_j})]\\
&\leq [h(\frac{R_1}{\gamma}+\epsilon)+\omega(a_0)][r(-\int_0^t a(s)\Psi_{R_1+\epsilon}(s)ds)+g(-\frac{R_1}{\gamma\delta}-\epsilon)].
\end{aligned}$$

and since

$$x'_{n_j,1}(t)-x'_{n_j,1}(\frac12)=-\int_{1/2}^t a(s)f(s,x_{n_j,1}(s)+\frac{1}{n_j},x'_{n_j,1}(s)-\frac{1}{n_j})ds, \quad t\in(0,1),$$

letting $j\to+\infty$, the Lebesgue Dominated Convergence Theorem guarantees that

$$x'_{0,1}(t)-x'_{0,1}(\frac12)=-\int_{1/2}^t a(s)f(s,x_{0,1}(s),x'_{0,1}(s))ds, t\in(0,1). \tag{4.27}$$

Differentiating, we have

$$x''_{0,1}(t)+a(t)f(t,x_{0,1}(t),x'_{0,1}(t))=0, \quad 0<t<1,$$

and from (4.26), $x_{0,1}(t)$ is a positive solution of (1.1) with $x_{0,1}\in C^1[0,1]\cap C^2(0,1)$.

For the set $\{x_{n,2}\}_{n\in N_0}\subseteq(\Omega_2-\overline{\Omega}_1)\cap P$, as in proof for the set $\{x_{n,1}\}_{n\in N_0}$, we obtain a convergent subsequence $\{x_{n_i,2}\}_{n\in N_0}$ of $\{x_{n,2}\}_{n\in N_0}$ with $\lim_{i\to+\infty}x_{n_i,2}=x_{0,2}\in C^1[0,1]\cap C^2(0,1)$. Moreover, $x_{0,2}$ is a positive solution to (1.1). □

Example 4.1 In (1.1), let $f(t,u,z)=\mu[1+(-z)^{-a}][1+u^b+u^{-d}]$ and $a(t)\equiv 1$ with $0<a<1$, $b>1$, $0<d<1$ and $\mu>0$. If

$$\mu<\sup_{c\in\mathbb{R}_+}\frac{I(-\frac{c(1-\sum_{i=1}^{m-2}a_i)}{\sum_{i=1}^{m-2}a_i\xi_i+1})}{\max_{t\in[0,1]}a(t)(c+c^{1-d}+\frac{c^{1+b}}{1+b})}. \tag{4.28}$$

Then equation (1.1) has at least two positive solutions $x_{0,1},x_{0,2}\in C^1[0,1]\cap C^2(0,1)$. We apply Theorem 4.1 with $g(z)=1$, $r(z)=(-z)^{-a}$, $h(u)=\mu u^{-d}$, $\omega(u)=\mu(1+u^b)$, $\Psi(t)=\mu$, $g_1(u,z)=\mu u^b$. Note that (P1), (P2), (P4) and (P5) hold, and that

$$\begin{aligned}
&\sup_{c\in\mathbb{R}_+}\frac{c}{-\frac{\sum_{i=1}^{m-2}a_i\xi_i+1}{1-\sum_{i=1}^{m-2}a_i}I^{-1}[\max_{t\in[0,1]}a(t)(ch(c)+\int_0^c\omega(s)ds)]}\\
&=\sup_{c\in\mathbb{R}_+}\frac{c}{-\frac{\sum_{i=1}^{m-2}a_i\xi_i+1}{1-\sum_{i=1}^{m-2}a_i}I^{-1}[\mu\max_{t\in[0,1]}a(t)(c+c^{1-d}+\frac{c^{1+b}}{1+b})]}
\end{aligned}$$

Then (4.28) guarantees that (P3) holds.

5. Singularities at $x = 0$ but not $x' = 0$

In this section the nonlinearity f may be singular at $x = 0$, but not at $x' = 0$. We assume that the following conditions hold.

(S1) $a(t) \in C[0,1]$, $a(t) > 0$, $t \in (0,1)$;

(S2) $f(t,u,z) \le [h(u) + \omega(u)]r(z)$, where $f \in C([0,1] \times \mathbb{R}_+ \times \mathbb{R}_-, \mathbb{R}_+)$, $\omega(u) > 0$ continuous and non-increasing on $[0,+\infty)$, $h(u) \ge 0$ continuous and nondecreasing on $\mathbb{R}_+$, $r(z) > 0$ continuous and nondecreasing on $\mathbb{R}_-$;

(S3)
$$\sup_{c\in\mathbb{R}_+} \frac{c}{-\frac{\sum_{i=1}^{m-2} a_i\xi_i+1}{1-\sum_{i=1}^{m-2} a_i}(I^{-1}[-\max_{t\in[0,1]} a(t)(ch(c) + \int_0^c \omega(s)ds)])} > 1,$$
where $I(z) = \int_z^0 \frac{udu}{r(u)}$, $z \in \mathbb{R}_-$, $\int_0^a \omega(s)ds < +\infty$, $a \in \mathbb{R}_+$;

(S4) There exists a function $g_1 \in C([0,+\infty) \times (-\infty,0],[0,+\infty))$, such that $f(t,u,z) \ge g_1(u,z), \forall (t,u,z) \in [0,1] \times \mathbb{R}_+ \times \mathbb{R}_-$, and $\lim_{u\to+\infty} \frac{g_1(u,z)}{u} = +\infty$, uniformly for $z \in \mathbb{R}_-$.

(S5) There exists a function $\Psi_H \in C([0,1],\mathbb{R}_+)$ and a constant $0 \le \delta$ such that $f(t,u,z) \ge \Psi_H(t)(-z)^\delta$, for all $(t,u,z) \in [0,1] \times [0,H] \times [-H,0)$.

For $n \in \{1,2,\dots\}$, $x \in P$, define operator
$$\begin{aligned}(A_nx)(t) = &-\int_0^t (t-s)a(s)f(s,x(s)+\frac{1}{n},-|x'(s)|)ds \\ &+\frac{1}{1-\sum_{i=1}^{m-2} a_i}\Big(\int_0^1 (1-s)a(s)f(s,x(s)+\frac{1}{n},-|x'(s)|)ds \\ &-\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i-s)a(s)f(s,x(s)+\frac{1}{n},-|x'(s)|)ds\Big), \quad t\in[0,1].\end{aligned} \tag{5.1}$$

Theorem 5.1. *Suppose* (S1)–(S5) *hold. Then* (1.1) *has at least two positive solutions* $x_{0,1}$, $x_{0,2}$ *in* $C^1[0,1] \cap C^2(0,1)$ *with* $x_{0,1}(t), x_{0,2}(t) > 0$, $t \in [0,1]$.

Proof. Choose $R_1 > 0$ such that
$$\frac{R_1}{-\frac{\sum_{i=1}^{m-2} a_i\xi_i+1}{1-\sum_{i=1}^{m-2} a_i}(I^{-1}[-\max_{t\in[0,1]} a(t)(R_1h(R_1) + \int_0^{R_1} \omega(s)ds)])} > 1. \tag{5.2}$$
From the continuity of I^{-1} and h, we can choose $\epsilon > 0$ and $\epsilon < R_1$ such that
$$\frac{R_1}{-\frac{\sum_{i=1}^{m-2} a_i\xi_i+1}{1-\sum_{i=1}^{m-2} a_i}(I^{-1}[-\max_{t\in[0,1]} a(t)(R_1h(R_1+\epsilon) + \int_0^{R_1+\epsilon} \omega(s)ds)])} > 1. \tag{5.3}$$
Let $n_0 \in \{1,2,\dots\}$ with $\frac{1}{n_0} < \min\{\epsilon, \delta/2\}$ and let $N_0 = \{n_0, n_0+1, \dots\}$. Then Lemma 2.4 guarantees that for $n \in N_0$, $A_n : P \to P$ is a completely continuous operator. Let
$$\Omega_1 = \{x \in C^1[0,1] : \|x\| < R_1\}.$$
We show that
$$x \ne \mu A_nx, \quad \forall x \in P \cap \partial\Omega_1,\ \mu \in (0,1],\ n \in N_0. \tag{5.4}$$

In fact, if there exists an $x_0 \in P \cap \partial\Omega_1$ and $\mu_0 \in (0,1]$ with $x_0 = \mu_0 A_n x_0$,

$$\begin{aligned} x_0(t) = &-\mu_0 \int_0^t (t-s)a(s)f(s, x_0(s)+\frac{1}{n}, -|x_0'(s)|)ds \\ &+ \frac{\mu_0}{1-\sum_{i=1}^{m-2} a_i}\Big(\int_0^1 (1-s)a(s)f(s, x_0(s)+\frac{1}{n}, -|x_0'(s)|)ds \\ &- \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)a(s)f(s, x_0(s)+\frac{1}{n}, -|x_0'(s)|)ds\Big). \end{aligned}$$

Then

$$x_0'(t) = -\mu_0 \int_0^t a(s)f(s, x_0(s)+\frac{1}{n}, -|x_0'(s)|)ds, \quad \forall t \in [0,1]. \tag{5.5}$$

Obviously, $x_0'(t) \le 0$, $t \in (0,1)$, and since $x_0(1) > 0$, $x_0(t) > 0, t \in [0,1]$. Differentiating (5.5), we have

$$\begin{gathered} x_0''(t) + \mu_0 a(t) f(t, x_0(t)+\frac{1}{n}, \quad x_0'(t)) = 0,\ 0 < t < 1, \\ x_0'(0) = 0, \quad x_0(1) = \sum_{i=1}^{m-2} a_i x_0(\xi_i). \end{gathered} \tag{5.6}$$

and

$$\begin{aligned} -x_0''(t) &= \mu_0 a(t) f(t, x_0(t)+\frac{1}{n}, x_0'(t)) \\ &\le a(t)[h(x_0(t)+\frac{1}{n}) + \omega(x_0(t)+\frac{1}{n})]r(x_0'(t)), \quad \forall t \in (0,1). \end{aligned}$$

Then

$$\frac{-x_0''(t)}{r(x_0'(t))} \le a(t)[h(x_0(t)+\frac{1}{n}) + \omega(x_0(t)+\frac{1}{n})], \quad \forall t \in (0,1),$$

and

$$\begin{aligned} \frac{-x_0''(t)x_0'(t)}{r(x_0'(t))} &\ge a(t)[h(x_0(t)+\frac{1}{n}) + \omega(x_0(t)+\frac{1}{n})]x_0'(t) \\ &\ge a(t)[h(R_1+\epsilon) + \omega(x_0(t)+\frac{1}{n})]x_0'(t). \end{aligned}$$

Integrating from 0 to t, we have

$$\begin{aligned} I(x_0'(t)) &\ge \int_0^t a(s)[h(R_1+\epsilon)x_0'(s)) + \omega(x_0(s)+\frac{1}{n})x_0'(s)]ds \\ &\ge \max_{t\in[0,1]} a(t) \int_0^t [h(R_1+\epsilon)x_0'(s)\omega(x_0(s)+\frac{1}{n})x_0'(s)]ds \\ &\ge -\max_{t\in[0,1]} a(t)(h(R_1+\epsilon)R_1 + \int_0^t \omega(x_0(s)+\frac{1}{n})d(x_0(s)+\frac{1}{n})) \\ &= -\max_{t\in[0,1]} a(t)(h(R_1+\epsilon)R_1 + \int_{x_0(0)+\frac{1}{n}}^{x_0(t)+\frac{1}{n}} \omega(s)ds) \\ &\ge -\max_{t\in[0,1]} a(t)(h(R_1+\epsilon)R_1 + \int_0^{R_1+\epsilon} \omega(s)ds) \end{aligned}$$

and

$$I(x_0'(t)) \geq -\max_{t\in[0,1]} a(t)(h(R_1+\epsilon)R_1 + \int_0^{R_1+\epsilon} \omega(s)ds).$$

Then

$$x_0'(t) \geq I^{-1}(-\max_{t\in[0,1]} a(t)(h(R_1+\epsilon)R_1 + \int_0^{R_1+\epsilon} \omega(s)ds));$$

that is,

$$-x_0'(t) \leq -I^{-1}(-\max_{t\in[0,1]} a(t)(h(R_1+\epsilon)R_1 + \int_0^{R_1+\epsilon} \omega(s)ds)),\ t\in(0,1). \tag{5.7}$$

Since

$$\begin{aligned}
&x_0(0)\\
&= \frac{\mu_0}{1-\sum_{i=1}^{m-2} a_i}\int_0^1 -x_0'(s)ds - \frac{\mu_0\sum_{i=1}^{m-2} a_i\int_0^{\xi_i} -x_0'(s)ds}{1-\sum_{i=1}^{m-2} a_i}\\
&\leq \frac{1}{1-\sum_{i=1}^{m-2} a_i}\int_0^1 -I^{-1}\Big(-\max_{t\in[0,1]} a(t)(h(R_1+\epsilon)R_1 + \int_0^{R_1+\epsilon}\omega(s)ds)\Big)ds\\
&\quad + \frac{\sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i}\int_0^{\xi_i} -I^{-1}\Big(I(-\max_{t\in[0,1]} a(t)(h(R_1+\epsilon)(R_1) + \int_0^{R_1+\epsilon}\omega(s)ds))\Big)ds\\
&= \frac{1+\sum_{i=1}^{m-2} a_i\xi_i}{1-\sum_{i=1}^{m-2} a_i}(-I^{-1}\Big(-\max_{t\in[0,1]} a(t)(h(R_1+\epsilon)R_1 + \int_0^{R_1+\epsilon}\omega(s)ds)\Big),
\end{aligned} \tag{5.8}$$

$x_0(0) \geq x_0(t) \geq \gamma\|x_0\|_1 \geq \gamma x_0(0) \geq \gamma\delta\|x_0\|_2, x_0(0) \geq \|x_0\| = R_1$. So

$$\frac{R_1}{-\frac{\sum_{i=1}^{m-2} a_i\xi_i+1}{1-\sum_{i=1}^{m-2} a_i}(I^{-1}[-\max_{t\in[0,1]} a(t)(R_1h(R_1+\epsilon) + \int_0^{R_1+\epsilon}\omega(s)ds)])} \leq 1, \tag{5.9}$$

which is a contradiction to (5.3). Then (5.4) holds. From Lemma 2.1, for $n \in N_0$,

$$i(A_n, \Omega_1 \cap P, P) = 1. \tag{5.10}$$

Now we show that there exists a set Ω_2 such that

$$A_nx \not\leq x, \quad \forall\, x \in \partial\Omega_2 \cap P. \tag{5.11}$$

Choose a^*, N^* as in section 3. Let

$$\Omega_2 = \{x \in C^1[0,1] : \|x\| < \frac{R_2}{a^*}\}.$$

Then

$$A_nx \not\leq x, \quad \forall x \in \partial\Omega_2 \cap P.$$

In fact, if there exists $x_0 \in \partial\Omega_2 \cap P$ with $x_0 \geq A_nx_0$, by the definition of the cone and Lemma 2.3, one has

$$x_0(t) \geq \gamma\|x_0\|_1 \geq \gamma x(0) \geq \gamma\delta\|x_0\|_2,$$

$x_0(t) \geq \frac{R_2}{a^*} > R_2$ for all $t \in [0,1]$. Then $x_0(t) + \frac{1}{n} > R_2$. From (3.11),

$$\begin{aligned}
\gamma x_0(t) &\geq \gamma A_nx_0(t)\\
&= \gamma\Big(-\int_0^t (t-s)a(s)f(s, x_0(s)+\frac{1}{n}, x_0'(s))ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{1-\sum_{i=1}^{m-2}a_i}\Big(\int_0^1(1-s)a(s)f(s,x_0(s)+\frac{1}{n},x_0'(s))ds\\
&-\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}(\xi_i-s)a(s)f(s,x_0(s)+\frac{1}{n},x_0'(s))ds\Big)\Big)\\
&\geq\gamma\Big(-\int_0^t(t-s)a(s)f(s,x_0(s)+\frac{1}{n},x_0'(s))ds\\
&+\frac{1}{1-\sum_{i=1}^{m-2}a_i}\Big(\int_0^1(1-s)a(s)f(s,x_0(s)+\frac{1}{n},x_0'(s))ds\\
&-\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}(\xi_i-s)a(s)f(s,x_0(s)+\frac{1}{n},x_0'(s))ds\Big)\Big)\\
&\geq\gamma\Big(\frac{\sum_{i=1}^{m-2}a_i}{1-\sum_{i=1}^{m-2}a_i}\int_0^t(t-s)a(s)f(s,x_0(s)+\frac{1}{n},x_0'(s))ds\\
&-\frac{\sum_{i=1}^{m-2}a_i\int_0^1(\xi_i-s)a(s)f(s,x_0(s)+\frac{1}{n},x_0'(s)}{1-\sum_{i=1}^{m-2}a_i})\Big)\\
&=\frac{\gamma\sum_{i=1}^{m-2}a_i(1-\xi_i)}{1-\sum_{i=1}^{m-2}a_i}\int_0^1a(s)f(s,x_0(s)+\frac{1}{n},x_0'(s))ds\\
&\geq\frac{\gamma\sum_{i=1}^{m-2}a_i(1-\xi_i)}{1-\sum_{i=1}^{m-2}a_i}\int_0^1a(s)g_1(x_0(s)+\frac{1}{n},x_0'(s))ds\\
&\geq\frac{\gamma\sum_{i=1}^{m-2}a_i(1-\xi_i)}{1-\sum_{i=1}^{m-2}a_i}\int_0^1a(s)dsN^*x_0(s)\\
&\geq a^*\frac{\gamma\sum_{i=1}^{m-2}a_i(1-\xi_i)}{1-\sum_{i=1}^{m-2}a_i}\int_0^1a(s)dsN^*\frac{R_2}{a^*}>\frac{R_2}{a^*};
\end{aligned}
$$

that is, $\|x_0\|\geq\gamma\|x_0\|_1>\frac{R_2}{a^*}$, which is a contradiction to $x_0\in\partial\Omega_2\cap P$. Then (5.11) holds. From Lemma 2.2,

$$
i(A_n,\Omega_2\cap P,P)=0. \tag{5.12}
$$

This equality and (5.10) guarantee

$$
i(A_n,(\Omega_2-\bar{\Omega}_1\cap P,P)=-1. \tag{5.13}
$$

From this equality and (5.10), A_n has two fixed points with $x_{n,1}\in\Omega_1\cap P, x_{n,2}\in(\Omega_2-\bar{\Omega}_1)\cap P$.

For each $n\in N_0$, there exists $x_{n,1}\in\Omega_1\cap P$ such that $x_{n,1}=A_nx_{n,1}$; that is, for $t\in[0,1]$,

$$
\begin{aligned}
x_{n,1}(t)=&-\int_0^t(t-s)a(s)f(s,x_{n,1}(s)+\frac{1}{n},-|x_{n,1}'(s)|)ds\\
&+\frac{1}{1-\sum_{i=1}^{m-2}a_i}\Big(\int_0^1(1-s)a(s)f(s,x_{n,1}(s)+\frac{1}{n},-|x_{n,1}'(s)|)ds\\
&-\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}(\xi_i-s)a(s)f(s,x_{n,1}(s)+\frac{1}{n},-|x_{n,1}'(s)|)ds\Big).
\end{aligned} \tag{5.14}
$$

As in the proof of (3.5), $x_n'(t) \le 0$, $t \in (0,1)$ and

$$x_{n,1}'(t) = -\int_0^t a(s)f(s, x_{n,1}(s) + \frac{1}{n}, x_{n,1}'(s))ds, \quad n \in N_0,\ t \in (0,1).$$

Now we consider $\{x_{n,1}(t)\}_{n\in N_0}$ and $\{x_{n,1}'(t)\}_{n\in N_0}$, since $\|x_{n,1}\| \le R_1$,

$$\{x_{n,1}(t)\} \text{ is uniformly bounded on } [0,1], \tag{5.15}$$

$$\{x_{n,1}'(t)\} \text{ is uniformly bounded on } [0,1]. \tag{5.16}$$

Then

$$\{x_{n,1}(t)\} \text{ is equicontinuous on } [0,1]. \tag{5.17}$$

As in the proof of (3.6),

$$\begin{gathered} x_{n,1}''(t) + a(t)f(t, x_{n,1}(t) + \frac{1}{n}, x_{n,1}'(t)) = 0, 0 < t < 1, \\ x_{n,1}'(0) = 0, x_{n,1}(1) = \sum_{i=1}^{m-2} a_i x_{n,1}(\xi_i). \end{gathered} \tag{5.18}$$

Now we show that for all $t_1, t_2 \in [0,1]$,

$$\begin{aligned} &|I(x_{n,1}'(t_2)) - I(x_{n,1}'(t_1))| \\ &\le \max_{t\in[0,1]} a(t)\big[h(R_1+\epsilon)(|x_{n,1}(t_2) - x_{n,1}(t_1)| + |t_2 - t_1|) \\ &\quad + |\int_{x_{n,1}(t_1)+\frac{1}{n}(1-t_1)}^{x_{n,1}(t_2)+\frac{1}{n}(1-t_2)} \omega(s)dt|\big]. \end{aligned} \tag{5.19}$$

From (5.18),

$$\begin{aligned} -x_{n,1}''(t) &= a(t)f(t, x_{n,1}(t) + \frac{1}{n}, x_{n,1}'(t)) \\ &\le a(t)[h(x_{n,1}(t) + \frac{1}{n}) + \omega(x_{n,1}(t) + \frac{1}{n})]r(x_{n,1}'(t)), \quad \forall t \in (0,1), \end{aligned}$$

$$\begin{aligned} x_{n,1}''(t) &= -a(t)f(t, x_{n,1}(t) + \frac{1}{n}, x_{n,1}'(t)) \\ &\ge -a(t)[h(x_{n,1}(t) + \frac{1}{n}) + \omega(x_{n,1}(t) + \frac{1}{n})]r(x_{n,1}'(t)), \quad \forall t \in (0,1), \end{aligned}$$

$$\begin{aligned} &\frac{-x_{n,1}''(t)x_{n,1}'(t)}{r(x_{n,1}'(t))} \\ &\ge a(t)[h(x_{n,1}(t) + \frac{1}{n}) + \omega(x_{n,1}(t) + \frac{1}{n})]x_{n,1}'(t) \\ &\ge a(t)[h(R_1+\epsilon)x_{n,1}'(t) + \omega(x_{n,1}(t) + \frac{1}{n}(1-t))x_{n,1}'(t)], \quad \forall t \in (0,1), \end{aligned} \tag{5.20}$$

$$\begin{aligned} &\frac{x_{n,1}''(t)x_{n,1}'(t)}{r(x_{n,1}'(t))} \\ &\le -a(t)[h(x_{n,1}(t) + \frac{1}{n}) + \omega(x_{n,1}(t) + \frac{1}{n})]x_{n,1}'(t) \\ &\le -a(t)[h(R_1+\epsilon)x_{n,1}'(t) + \omega(x_{n,1}(t) + \frac{1}{n}(1-t))x_{n,1}'(t)], \quad \forall t \in (0,1). \end{aligned} \tag{5.21}$$

Since the right-hand sides of (5.20) and (5.21) are positive, for all t_1, $t_2 \in [0,1]$ and $t_1 < t_2$,

$$\begin{aligned}
&I(x'_{n,2}(t_2)) - I(x'_{n,1}(t_1))\\
&\geq \max_{t\in[0,1]} a(t)[h(R_1+\epsilon)\Big(\int_{t_1}^{t_2} x'_{n,1}(s)dt + \int_{t_1}^{t_2} \omega(x_{n,1}(s)+\frac{1}{n})x'_{n,1}(s)ds\Big)]\\
&\geq -\max_{t\in[0,1]} a(t)[h(R_1+\epsilon)(|x_{n,1}(t_2)-x_{n,1}(t_1)|) + |\int_{x_{n,1}(t_1)+\frac{1}{n}(1-t_1)}^{x_{n,1}(t_2)+\frac{1}{n}(1-t_2)} \omega(s)dt|],\\
&I(x'_{n,1}(t_1)) - I(x'_{n,1}(t_2))\\
&\leq \max_{t\in[0,1]} a(t)[h(R_1+\epsilon)(|x_{n,1}(t_2)-x_{n,1}(t_1)|) + |\int_{x_{n,1}(t_1)+\frac{1}{n}(1-t_1)}^{x_{n,1}(t_2)+\frac{1}{n}(1-t_2)} \omega(s)dt|]\,,
\end{aligned}$$

(5.19) holds.

Since I^{-1} is uniformly continuous on $[0, I(-R_1-\epsilon)]$, for all $\bar{\epsilon} > 0$, there exists $\epsilon' > 0$ such that

$$|I^{-1}(s_1) - I^{-1}(s_2)| < \bar{\epsilon}, \quad \forall |s_1 - s_2| < \epsilon',\ s_1, s_2 \in [0, I(-R_1-\epsilon)]. \tag{5.22}$$

Also (5.19) guarantees that for $\epsilon' > 0$, there exists $\delta' > 0$ such that

$$|I(x'_{n,1}(t_2)) - I(x'_{n,1}(t_1))| < \epsilon', \quad \forall\ |t_1 - t_2| < \delta',\ t_1, t_2 \in [0,1]. \tag{5.23}$$

From this inequality and (5.22),

$$\begin{gathered}
|x'_{n,1}(t_2) - x'_{n,1}(t_1)| = |I^{-1}(I(x'_{n,1}(t_2))) - I^{-1}(I(x'_{n,1}(t_1)))| < \bar{\epsilon},\\
\forall |t_1 - t_2| < \delta',\ t_1, t_2 \in [0,1];
\end{gathered} \tag{5.24}$$

that is,

$$\{x'_{n,1}(t)\} \text{ is equi-continuous on } [0,1]. \tag{5.25}$$

From (5.15)–(5.17), (5.25) and the Arzela-Ascoli Theorem, $\{x_{n,1}(t)\}$ and $\{x'_{n,1}(t)\}$ are relatively compact on $C^1[0,1]$. This implies, there exists a subsequence $\{x_{n_j,1}\}$ of $\{x_{n,1}\}$ and function $x_{0,1}(t) \in C^1[0,1]$ such that

$$\lim_{j\to+\infty} \max_{t\in[0,1]} |x_{n_j,1}(t) - x_{0,1}(t)| = 0, \quad \lim_{j\to+\infty} \max_{t\in[0,1]} |x'_{n_j,1}(t) - x'_{0,1}(t)| = 0.$$

Since $x'_{n_j,1}(0) = 0$, $x_{n_j,1}(1) = \sum_{i=1}^{m-2} a_i x_{n_j,1}(\xi_i)$, $x'_{n_j,1}(t) < 0$, $x_{n_j,1}(t) > 0$, $t \in (0,1)$, $j \in \{1,2,\dots\}$,

$$x'_{0,1}(0) = 0, x_{0,1}(1) = \sum_{i=1}^{m-2} a_i x_{0,1}(\xi_i), x'_{0,1}(t) \leq 0, x_{0,1}(t) \geq 0,\ t \in (0,1). \tag{5.26}$$

For $(t, x_{n_j,1}(t) + \frac{1}{n_j}, x'_{n_j,1}(t)) \in [0,1] \times [0, R_1+\epsilon] \times (-\infty, 0)$, from (S5) there exists a function $\Psi_{R_1} \in C([0,1], \mathbb{R}_+)$ such that

$$f(t, x_{n_j,1}(t) + \frac{1}{n_j}, x'_{n_j,1}(t))ds \geq \Psi_{R_1}(t)(-x'_{n_j,1}(t))^{\delta}, \quad 0 \leq \delta < 1.$$

Then, for $n \in N_0$,

$$\begin{aligned}
x'_{n_j,1}(t) &= -\int_0^t a(s) f(s, x_{n_j,1}(s) + \frac{1}{n_j}, x'_{n_j,1}(s))ds\\
&\leq -\int_0^t a(s)\Psi_{R_1+\epsilon}(s)(-x'_{n_j,1}(s))^{\delta} ds, \quad t \in [0,1],\ n \in N_0,
\end{aligned}$$

which implies

$$x'_{n_j,1}(t) \le -(1-\delta)\Big(\int_0^t a(s)\Psi_{R_1+\epsilon}(s)ds\Big)^{\frac{1}{1-\delta}},$$

and

$$\begin{aligned}
x_{n_j,1}(t) &= -\int_0^t (t-s)a(s)f(s, x_{n_j,1}(s)+\frac{1}{n_j}, x'_{n_j,1}(s))ds \\
&\quad + \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Big(\int_0^1 (1-s)a(s)f(s, x_{n_j,1}(s)+\frac{1}{n_j}, x'_{n_j,1}(s))ds \\
&\quad - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)a(s)f(s, x_{n_j,1}(s)+\frac{1}{n_j}, x'_{n_j,1}(s))ds\Big) \\
&\ge -\int_0^1 (1-s)a(s)f(s, x_{n_j,1}(s)+\frac{1}{n_j}, x'_{n_j,1}(s))ds \\
&\quad + \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Big(\int_0^1 (1-s)a(s)f(s, x_{n_j,1}(s)+\frac{1}{n_j}, x'_{n_j,1}(s))ds \\
&\quad - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)a(s)f(s, x_{n_j,1}(s)+\frac{1}{n_j}, x'_{n_j,1}(s))ds\Big) \\
&\ge \frac{\sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i}\int_0^1 (1-s)a(s)f(s, x_{n_j,1}(s)+\frac{1}{n_j}, x'_{n_j,1}(s))ds \\
&\quad - \frac{\sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i}\int_0^1 (\xi_i - s)a(s)f(s, x_{n_j,1}(s)+\frac{1}{n_j}, x'_{n_j,1}(s))ds \\
&= \frac{\sum_{i=1}^{m-2} a_i(1-\xi_i)}{1-\sum_{i=1}^{m-2} a_i}\int_0^1 a(s)f(s, x_{n_j,1}(s)+\frac{1}{n_j}, x'_{n_j,1}(s))ds \\
&\ge \frac{\sum_{i=1}^{m-2} a_i(1-\xi_i)}{1-\sum_{i=1}^{m-2} a_i}\int_0^1 a(s)\Psi_{R_1+\epsilon}(s)(-x'_{n_j,1}(s))^{\delta}ds \\
&\ge \frac{\sum_{i=1}^{m-2} a_i(1-\xi_i)}{1-\sum_{i=1}^{m-2} a_i}\int_0^1 a(s)\Psi_{R_1+\epsilon}(s)\Big((1-\delta)\Big(\int_0^s a(\tau)\Psi_{R_1+\epsilon}(\tau)d\tau\Big)^{\frac{1}{1-\delta}}\Big)^{\delta}ds \\
&=: F, \quad t \in [0,1].
\end{aligned}$$

Since

$$\begin{aligned}
&f(t, x_{n_j,1}(t)+\frac{1}{n_j}, x'_{n_j,1}(t)) \\
&\le [h(x_{n_j,1}(t)+\frac{1}{n_j}) + \omega(x_{n_j,1}(t)+\frac{1}{n_j})]r(x'_{n_j,1}(t)) \\
&\le [h(\frac{R_1}{\gamma}+\epsilon) + \omega(F)]r\Big((\delta-1)\Big(\int_0^t a(s)\Psi_{R_1+\epsilon}(s)ds\Big)^{\frac{1}{1-\delta}}\Big),
\end{aligned}$$

and

$$x'_{n_j,1}(t) - x'_{n_j,1}(\frac{1}{2}) = -\int_{1/2}^t a(s)f(s, x_{n_j,1}(s)+\frac{1}{n_j}, x'_{n_j,1}(s))ds, \qquad t \in (0,1),$$

letting $j \to +\infty$, the Lebesgue Dominated Convergence Theorem guarantees that

$$x'_{0,1}(t) - x'_{0,1}(\frac{1}{2}) = -\int_{1/2}^{t} a(s)f(s, x_{0,1}(s), x'_{0,1}(s))ds, \ t \in (0,1). \tag{5.27}$$

Differentiating, we have

$$x''_{0,1}(t) + a(t)f(t, x_{0,1}(t), x'_{0,1}(t)) = 0, \quad 0 < t < 1\,.$$

From this equality and from (5.26), $x_{0,1}(t)$ is a positive solution of (1.1) with $x_{0,1} \in C^1[0,1] \cap C^2(0,1)$.

For the set $\{x_{n,2}\}_{n\in N_0} \subseteq (\Omega_2 - \overline{\Omega}_1) \cap P$, as in the proof for the set $\{x_{n,1}\}_{n\in N_0}$, we obtain a convergent subsequence $\{x_{n_i,2}\}_{n\in N_0}$ of $\{x_{n,2}\}_{n\in N_0}$ with $\lim_{i\to+\infty} x_{n_i,2} = x_{0,2} \in C^1[0,1] \cap C^2(0,1)$. Moreover, $x_{0,2}$ is a positive solution to (1.1). □

Example 5.1 In (1.1), let $f(t,u,z) = \mu[1 + (-z)^a][1 + u^b + u^{-d}]$ and $a(t) \equiv 1$ with $0 \le a < 1, b > 1, 0 < d < 1$ and $\mu > 0$. If

$$\mu < \sup_{c\in\mathbb{R}_+} \frac{I(\frac{c(1-\sum_{i=1}^{m-2} a_i)}{\sum_{i=1}^{m-2} a_i\xi_i + 1})}{\max_{t\in[0,1]} a(t)(c + c^{1-d} + \frac{c^{1+b}}{1+b})}, \tag{5.28}$$

Then (1.1) has at least two positive solutions $x_{0,1}, x_{0,2} \in C^1[0,1] \cap C^2(0,1)$.

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