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ASYMPTOTIC BEHAVIOR FOR A QUADRATIC NONLINEAR SCHRÖDINGER EQUATION

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Abstract. We study the initial-value problem for the quadratic nonlinear Schrödinger equation

$$\begin{gathered} iu_t + \frac{1}{2}u_{xx} = \partial_x \overline{u}^2, \quad x \in \mathbb{R},\ t > 1, \\ u(1,x) = u_1(x), \quad x \in \mathbb{R}. \end{gathered}$$

For small initial data $u_1 \in \mathbf{H}^{2,2}$ we prove that there exists a unique global solution $u \in \mathbf{C}([1,\infty); \mathbf{H}^{2,2})$ of this Cauchy problem. Moreover we show that the large time asymptotic behavior of the solution is defined in the region $|x| \le C\sqrt{t}$ by the self-similar solution $\frac{1}{\sqrt{t}} MS(\frac{x}{\sqrt{t}})$ such that the total mass

$$\frac{1}{\sqrt{t}} \int_{\mathbb{R}} MS(\frac{x}{\sqrt{t}}) dx = \int_{\mathbb{R}} u_1(x) dx,$$

and in the far region $|x| > \sqrt{t}$ the asymptotic behavior of solutions has rapidly oscillating structure similar to that of the cubic nonlinear Schrödinger equations.

1. Introduction

We consider the quadratic nonlinear Schrödinger equation

$$\begin{gathered} iu_t + \frac{1}{2}u_{xx} = \partial_x \overline{u}^2, \quad x \in \mathbb{R},\ t > 1, \\ u(1,x) = u_1(x), \quad x \in \mathbb{R}. \end{gathered} \tag{1.1}$$

In general, the quadratic type nonlinearities in the one dimensional case are considered to be subcritical with respect to the large time asymptotic behavior of solutions. Different types of the quadratic nonlinearities, including derivatives of the unknown function were considered previously (see [6, 7, 12, 15, 20] and references cited therein). We choose the initial time value $t = 1$ for the convenience of the forthcoming calculations (note that by the change $t' = t - 1$ it can be transformed to the usual case of the initial time value $t' = 0$).

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If we replace the nonlinear term of (1.1) by $\partial_x u^2$, then we can represent the solution u by the formula (see [17])

$$u(t,x) = s(t,x) - \frac{1}{2}\frac{(2s(t,x)\psi(t,x) + \partial_x\psi(t,x))\exp\big(2\int_0^x s(t,y)dy\big)}{1+\psi(t,x)\exp\big(2\int_0^x s(t,y)dy\big)}$$

through the Hope-Cole transformation, where

$$s(t,x) = \frac{1}{\sqrt{t}}e^{\frac{ix^2}{2t}}\frac{1}{C - 2\sqrt{2i}\int_0^{\frac{x}{\sqrt{t}}\sqrt{-\frac{i}{2}}} e^{-y^2}dy},$$

C is a constant determined by

$$\int u_1(x)dx = \int s(t,x)dx = \int e^{\frac{i\xi^2}{2}}\frac{1}{C - 2\sqrt{2i}\int_0^{\xi\sqrt{-\frac{i}{2}}} e^{-y^2}dy}d\xi$$

and ψ is the solution of the linear Schrödinger equation $i\psi_t + \frac{1}{2}\psi_{xx} = 0$ with the initial data

$$\psi(1,x) = \Big(\exp\big(-2\int_0^x (u_1(y) - s(1,y))dy\big) - 1\Big)\exp\big(-2\int_0^x s(1,y)dy\big).$$

See Appendix 8 for details. However the Hope-Cole transformation can not be applied to our problem. Recently in [6] we considered the nonlinear Schrödinger equation

$$iu_t + \frac{1}{2}u_{xx} = \lambda(\overline{u}_x)^2 + \mu u_x^2,$$

with $\lambda,\mu \in \mathbf{C}$. We applied a method similar to the normal forms of Shatah [18], making a transformation of the original equation with quadratic nonlinearity to a nonlinear Schrödinger equation with critical cubic nonlinearity

$$\begin{aligned}\mathcal{L}(u - \frac{\mu}{2}u^2 - \lambda\mathcal{G}(\overline{u},\overline{u})) &= -\mu u(\mathcal{L}u) + 2\lambda\mathcal{G}(\overline{\mathcal{L}u},\overline{u})\\ &= -\lambda\mu u(\overline{u}_x)^2 - \lambda\mu u u_x^2 + 2|\lambda|^2\mathcal{G}(\overline{u},u_x^2) + 2\lambda\overline{\mu}\mathcal{G}(\overline{u},\overline{u}_x^2),\end{aligned}$$

where $\mathcal{L} = i\partial_t + \frac{1}{2}\partial_x^2$, $\mathcal{G}$ is a symmetric bilinear operator. Then for small initial data $u_0 \in \mathbf{H}^{3,1}$ we obtained the large time asymptotic behavior of small solutions which has an additional logarithmic oscillation. We note that if $\int u(t,x)dx = 0$ in (1.1), then by introducing a new variable $v = \int_{-\infty}^x udx$, we have for v,

$$iv_t + \frac{1}{2}v_{xx} = (\overline{v}_x)^2.$$

Therefore in the case of $\int u_1(x)dx = 0$, asymptotic behavior of solutions has been shown in [6].

In [7] we studied the one dimensional quadratic nonlinear Schr ödinger equation $iu_t + \frac{1}{2}u_{xx} = t^{-\alpha}|u_x|^2$ with $\alpha \in (0,1)$. Heuristically the solution should have a quasilinear character if $\alpha \in (\frac{1}{2},1)$. However we showed that the asymptotic behavior of solutions does not have a quasilinear character for all range $\alpha \in (0,1)$ due to the special structure of the nonlinear term. For the case $\alpha \in [\frac{1}{2},1)$ we proved that if the initial data $u_0 \in \mathbf{H}^{3,0} \cap \mathbf{H}^{2,2}$ are small then the solution has a slow time-decay as $t^{-\frac{\alpha}{2}}$. And the derivative u_x of the solution has a quasilinear behavior $t^{-1/2}$ as $t \to \infty$. When $\alpha \in (0,\frac{1}{2})$, if we assume that the initial data u_0 are analytic and small, then the same result as for the case $\alpha \in [\frac{1}{2},1)$ holds.

The aim of the present paper is to prove existence of global solutions and large time behavior of solutions to the Cauchy problem (1.1). Here we use the method similar to the normal forms of Shatah and the transformation in [11]. Also we use the following factorization formulas for the free Schrödinger evolution group

$$\mathcal{U}(t)\mathcal{F}^{-1} = M(t)\mathcal{D}_t\mathcal{V}(t),$$
$$\mathcal{F}\mathcal{U}(-t) = i\mathcal{V}(-t)\overline{E}(t)\mathcal{D}_{1/t}.$$

Here we denote

$$M(t) = e^{\frac{i}{2t}x^2}, \quad E(t) = e^{\frac{it}{2}\xi^2},$$

the dilation operator $(\mathcal{D}_a\phi)(x) = \frac{1}{\sqrt{ia}}\phi(\frac{x}{a})$ and $\mathcal{V}(t) = \mathcal{F}M(t)\mathcal{F}^{-1}$. Note that $\mathcal{D}_{1/t}M(t) = E(t)\mathcal{D}_{1/t}$. So we represent the solution

$$u(t) = \mathcal{U}(t)\mathcal{F}^{-1}w(t) = \mathcal{D}_tE(t)v(t),$$

where $w = \mathcal{V}(-t)v(t)$. The direct Fourier transform $\hat{\phi}(\xi)$ of the function $\phi(x)$ is defined by

$$\mathcal{F}\phi = \hat{\phi} = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ix\xi}\phi(x)dx,$$

then the inverse Fourier transformation is given by

$$\mathcal{F}^{-1}\phi = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ix\xi}\phi(\xi)d\xi.$$

Denote the usual Lebesgue space $\mathbf{L}^p = \{\phi \in \mathbf{S}'; \|\phi\|_{\mathbf{L}^p} < \infty\}$, where the norm $\|\phi\|_{\mathbf{L}^p} = (\int_{\mathbb{R}} |\phi(x)|^p dx)^{1/p}$ if $1 \le p < \infty$ and $\|\phi\|_{\mathbf{L}^\infty} = \operatorname{ess.sup}_{x\in\mathbb{R}} |\phi(x)|$ if $p = \infty$. The weighted Lebesgue norm is $\|\phi\|_{\mathbf{L}^{p,a}} = \|\langle\cdot\rangle^a\phi\|_{\mathbf{L}^p}$. Weighted Sobolev space is

$$\mathbf{H}^{m,a} = \{\phi \in \mathbf{S}' : \|\phi\|_{\mathbf{H}^{m,a}} \equiv \|\langle i\partial\rangle^m\phi\|_{\mathbf{L}^{2,a}} < \infty\},$$

where $m, a \in \mathbb{R}$, $\langle x\rangle = \sqrt{1+x^2}$. The usual Sobolev space is $\mathbf{H}^m = \mathbf{H}^{m,0}$, so the index 0 we usually omit if it does not cause a confusion. Different positive constants we denote by the same letter C.

Denote $\mathbf{Y} = \{\phi \in \mathbf{L}^\infty, \phi' \in \mathbf{H}^{1,1}\}$.

Theorem 1.1. *Let the initial data $M(1)u_1 = e^{\frac{i}{2}x^2}u_1 \in \mathbf{Y}$ with a norm $\|u_1\|_{\mathbf{Y}} \le \varepsilon$, where $\varepsilon > 0$ is sufficiently small. Then there exists a unique solution $u \in \mathbf{C}([1,\infty);\mathbf{Y})$ of the Cauchy problem* (1.1).

We denote by $\frac{1}{\sqrt{t}}M\Psi(\frac{x}{\sqrt{t}})$ a self-similar solution of the quadratic nonlinear Schrödinger equation (1.1) such that the total mass

$$\frac{1}{\sqrt{t}}\int_{\mathbb{R}} M\Psi(\frac{x}{\sqrt{t}})dx = \int_{\mathbb{R}} u_1(x)dx.$$

The following theorem states that the large time asymptotic behavior of solution of (1.1) is defined by this self-similar solution in the region $|x| \le C\sqrt{t}$ and in the far region $|x| > \sqrt{t}$ it has rapidly oscillating structure similar to that of the cubic Schrödinger equations.

Theorem 1.2. *Let the initial data $M(1)u_1 = e^{\frac{i}{2}x^2}u_1 \in \mathbf{Y}$ with a norm $\|u_1\|_{\mathbf{Y}} \le \varepsilon$, where $\varepsilon > 0$ is sufficiently small. Then there exist unique functions H_j and $B_j \in \mathbf{L}^\infty$ (B_j are real-valued), $j = 1, 2$, such that the following asymptotic formula is valid*

$$u(t,x) = \frac{1}{\sqrt{t}}M(t)\Psi(\frac{x}{\sqrt{t}})$$

$$+\frac{1}{\sqrt{t}}M(t)\sum_{j=1}^{2}H_j(\frac{x}{t})\exp\Big(iB_j(\frac{x}{t})\log\frac{1+\frac{|x|}{\sqrt{t}}}{1+\frac{|x|}{t}}\Big)$$
$$+O(t^{-\frac{1}{2}-\varkappa})$$

for $t\to\infty$ uniformly with respect to $x\in\mathbb{R}$, where $\varkappa>0$.

We organize the rest of our paper as follows. In Sections 2–6 we prove some preliminary estimates. Section 7 is devoted to the proof of Theorem 1.1. We prove Theorem 1.2 in Section 8 . Section 10 is devoted to the proof of existence of the self-similar solution.

2. Transformation of equation

We represent the solution $u(t)=\mathcal{U}(t)\mathcal{F}^{-1}w(t)$, where the free Schrödinger evolution group $\mathcal{U}(t)=\mathcal{F}^{-1}e^{\frac{it}{2}\xi^2}\mathcal{F}$. Applying the Fourier transformation to equation (1.1), changing the dependent variable $w(t,\xi)=e^{\frac{it}{2}\xi^2}\widehat{u}(t,\xi)$, we get

$$w_t(t,\xi)=\frac{\xi}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{itS}\overline{w(t,\eta-\xi)w(t,-\eta)}d\eta, \tag{2.1}$$

where $S=\frac{1}{2}(\xi^2+(\xi-\eta)^2+\eta^2)$. Using the identity

$$e^{itS}=\frac{d}{dt}\Big(\frac{2+itS}{(1+itS)^2}te^{itS}\Big)+\frac{itS-1}{(1+itS)^3}e^{itS}$$

in view of the symmetry $\eta\leftrightarrow\xi-\eta$ we rewrite equation (2.1) as

$$\begin{aligned}
&\partial_t(w(t,\xi)-\int_{\mathbb{R}}e^{itS}\overline{w(t,\eta-\xi)w(t,-\eta)}Ad\eta)\\
&=t^{-1}\int_{\mathbb{R}}e^{itS}\overline{w(t,\eta-\xi)w(t,-\eta)}\widetilde{A}d\eta\\
&\quad+\sqrt{\frac{2}{\pi}}\int_{\mathbb{R}^2}e^{itQ}\overline{w(t,\eta-\xi)}w(t,\eta-\zeta)w(t,\zeta)A\eta d\eta d\zeta,
\end{aligned}\tag{2.2}$$

where

$$\begin{gathered}
Q=\frac{1}{2}(\xi^2+(\xi-\eta)^2-(\eta-\zeta)^2-\zeta^2),\\
A=\frac{\xi t(2+itS)}{\sqrt{2\pi}(1+itS)^2}=-\frac{i\xi}{\sqrt{2\pi}}(1+t\partial_t)(\frac{1}{it}+S)^{-1},\\
\widetilde{A}=\frac{\xi t(itS-1)}{\sqrt{2\pi}(1+itS)^3}=-\frac{i\xi}{\sqrt{2\pi}}(2+t\partial_t)t\partial_t(\frac{1}{it}+S)^{-1}.
\end{gathered}$$

Now we return to the function $u=\mathcal{U}(t)\mathcal{F}^{-1}w(t)$ to get, from (2.2),

$$(i\partial_t+\frac{1}{2}\partial_x^2)(u-\mathcal{Q}(\overline{u},\overline{u}))=t^{-1}\widetilde{\mathcal{Q}}(\overline{u},\overline{u})+\mathcal{Q}(\overline{u},\partial_xu^2), \tag{2.3}$$

where

$$\begin{aligned}
\mathcal{Q}(\overline{u},\overline{u})&=\mathcal{U}(t)\mathcal{F}^{-1}\int_{\mathbb{R}}e^{itS}\overline{w(t,\eta-\xi)w(t,-\eta)}Ad\eta\\
&=-\frac{1}{\sqrt{2\pi}}\partial_x\mathcal{F}^{-1}\int_{\mathbb{R}}\overline{\widehat{u}(\eta-\xi)\widehat{u}(-\eta)}(1+t\partial_t)(\frac{1}{it}+S)^{-1}d\eta,
\end{aligned}$$

$$\begin{aligned}\widetilde{\mathcal{Q}}(\overline{u},\overline{u}) &= \mathcal{U}(t)\mathcal{F}^{-1}\int_{\mathbb{R}} e^{itS}\overline{w(t,\eta-\xi)w(t,-\eta)}\widetilde{A}d\eta \\ &= -\frac{1}{\sqrt{2\pi}}\partial_x\mathcal{F}^{-1}\int_{\mathbb{R}}\overline{\widehat{u}(\eta-\xi)\widehat{u}(-\eta)}(3+t\partial_t)\partial_t(\frac{1}{it}+S)^{-1}d\eta.\end{aligned}$$

In the next lemma we give an x-representation of the operator

$$\mathcal{I}(\phi,\psi) = \mathcal{F}^{-1}\int_{\mathbb{R}}\widehat{\phi}(\xi-\eta)\widehat{\psi}(\eta)(\frac{1}{it}+S)^{-1}d\eta.$$

Denote the convolution with a kernel g

$$(\psi*\phi)_g \equiv \int_{\mathbb{R}^2} g(t,y,z)\psi(x-y)\phi(x-z)dydz.$$

Lemma 2.1. *The representation $\mathcal{I}(\phi,\psi) = (\psi*\phi)_g$ is true, where*

$$g(t,y,z) \equiv \sqrt{\frac{2}{3\pi}}K_0\Big(\sqrt{\frac{4}{3it}(y^2-yz+z^2)}\Big)$$

and K_0 is the Macdonalds function.

Proof. We substitute the Fourier transformation

$$\begin{aligned}\mathcal{I}(\phi,\psi) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} d\eta\widehat{\psi}(\frac{\xi}{2}-\eta)\int_{\mathbb{R}} d\xi e^{ix\xi}\widehat{\phi}(\frac{\xi}{2}+\eta)\big(\frac{1}{it}+\frac{3}{4}\xi^2+\eta^2\big)^{-1} \\ &= \frac{8}{3(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}} d\eta\int_{\mathbb{R}} dye^{-iy\eta}\psi(x-y)\int_{\mathbb{R}} dze^{iz\eta}\phi(x-z) \\ &\quad\times\int_{\mathbb{R}} d\xi\cos(\frac{y+z}{2}\xi)(\xi^2+\frac{4}{3}\eta^2-\frac{4i}{3t})^{-1}.\end{aligned}$$

By [2] we find

$$\int_0^\infty \frac{\cos(z\xi)}{\xi^2+a^2}d\xi = \frac{\pi}{2a}e^{-a|z|}$$

for $\operatorname{Re} a>0$. Hence

$$\begin{aligned}\mathcal{I}(\phi,\psi) &= \frac{2}{\sqrt{6\pi}}\int_{\mathbf{R}^2} dydz\psi(x-y)\phi(x-z) \\ &\quad\times\int_0^\infty \frac{\cos((z-y)\eta)}{\sqrt{\eta^2-\frac{i}{t}}}e^{-\frac{1}{\sqrt{3}}|y+z|\sqrt{\eta^2-\frac{i}{t}}}d\eta.\end{aligned}$$

We compute by [2] for $\operatorname{Re} a>0$, $\operatorname{Re}\beta>0$

$$\int_0^\infty \frac{\cos((z-y)\eta)e^{-\beta\sqrt{\eta^2+a^2}}d\eta}{\sqrt{\eta^2+a^2}} = K_0(a\sqrt{\beta^2+(z-y)^2}). \tag{2.4}$$

Then taking $a=\frac{1}{\sqrt{it}}$, $\beta=\frac{1}{\sqrt{3}}|y+z|$ we find the representation of the lemma. The proof is complete. □

By Lemma 2.1 we can rewrite the operators $\mathcal{Q}$ and $\widetilde{\mathcal{Q}}$ in equation (2.3) as $\mathcal{Q}(\phi,\psi) = (\psi*\phi)_q$ and $\widetilde{\mathcal{Q}}(\phi,\psi) = (\psi*\phi)_{\widetilde{q}}$ with

$$\begin{aligned}q &= -\frac{1}{\sqrt{2\pi}}(1+t\partial_t)(\partial_y+\partial_z)g(t,y,z), \\ \widetilde{q} &= -\frac{1}{\sqrt{2\pi}}(2+t\partial_t)t\partial_t(\partial_y+\partial_z)g(t,y,z).\end{aligned}$$

In the same way as in paper [11] we change the variables $u(t,x) = t^{-1/2} e^{\frac{it}{2}\xi^2} v(t,\xi)$ and $\xi = \frac{x}{t}$ in equation (2.3) to obtain

$$\mathcal{L}(v - \overline{E}(\overline{Ev} * \overline{Ev})_h) = \mathcal{P}_1 - \mathcal{P}_2, \tag{2.5}$$

where

$$\mathcal{L} = i\partial_t + \frac{1}{2t^2}\partial_\xi^2, \mathcal{P}_1 = t^{-1}\overline{E}(\overline{Ev} * \overline{Ev})_{\widetilde{h}}, \mathcal{P}_2 = t^{-1}\overline{E}(\overline{Ev} * \partial_\zeta(E^2 v^2))_h,$$

and by changing $x = \xi t$, $y = \eta t$, $z = \zeta t$. Denote the operators

$$\begin{aligned}\mathcal{G}_{a,b}(\phi,\psi) &= \overline{E}^{a+b}(E^a\phi * E^b\psi)_h \\ &= \overline{E}^{a+b}\int_{\mathbb{R}^2} h(t,\eta,\zeta) e^{\frac{a}{2}it(\xi-\eta)^2}\phi(\xi-\eta) e^{\frac{b}{2}it(\xi-\zeta)^2}\psi(\xi-\zeta)d\eta d\zeta,\end{aligned}$$

$$\begin{aligned}\widetilde{\mathcal{G}}_{a,b}(\phi,\psi) &= \overline{E}^{a+b}(E^a\phi * E^b\psi)_{\widetilde{h}} \\ &= \overline{E}^{a+b}\int_{\mathbb{R}^2} \widetilde{h}(t,\eta,\zeta) e^{\frac{a}{2}it(\xi-\eta)^2}\phi(\xi-\eta) e^{\frac{b}{2}it(\xi-\zeta)^2}\psi(\xi-\zeta)d\eta d\zeta,\end{aligned}$$

$$\begin{aligned}\mathcal{H}_{a,b}(\phi,\psi) &= \overline{E}^{a+b}(E^a\phi * \partial_\zeta(E^b\psi))_h \\ &= \overline{E}^{a+b}\int_{\mathbb{R}^2} h(t,\eta,\zeta) e^{\frac{a}{2}it(\xi-\eta)^2}\phi(\xi-\eta)\partial_\zeta e^{\frac{b}{2}it(\xi-\zeta)^2}\psi(\xi-\zeta)d\eta d\zeta\end{aligned}$$

with

$$\begin{aligned} h(t,\eta,\zeta) &= -\frac{\sqrt{t}}{\pi\sqrt{3}}(1+t\partial_t)(\partial_\eta+\partial_\zeta)K_0\Big(\sqrt{\frac{4t}{3i}(\eta^2-\eta\zeta+\zeta^2)}\Big), \\ \widetilde{h}(t,\eta,\zeta) &= -\frac{\sqrt{t}}{\pi\sqrt{3}}(2+t\partial_t)t\partial_t(\partial_\eta+\partial_\zeta)K_0\Big(\sqrt{\frac{4t}{3i}(\eta^2-\eta\zeta+\zeta^2)}\Big). \end{aligned}$$

3. Preliminary estimates

Define the weighted Lebesgue norm $\|\phi\|_{\mathbf{L}^{p,a}} = \|\langle\cdot\rangle^a\phi\|_{\mathbf{L}^p}$ and define the linear operator

$$\mathbb{K}\phi = \int_{\mathbb{R}} K(\xi,\eta)(\phi(\xi-\eta)-\phi(\xi))d\eta.$$

Lemma 3.1. *Suppose that*

$$K(\xi,\eta) = O(e^{-\langle\eta\rangle}\langle\xi\eta\rangle^{-\alpha})$$

for all $\xi \in \mathbb{R}$, $\eta \in \mathbb{R}\backslash\{0\}$, where $\alpha > 1$. Then the estimates are true

$$\|\mathbb{K}\phi\|_{\mathbf{L}^{p,\theta}} \le C\|\partial_\xi\phi\|_{\mathbf{L}^{2,\lambda}}$$

if $\alpha > \frac{3}{2} + \frac{1}{p}$, where $\theta = \frac{3}{2} + \frac{1}{p} + \lambda$, $\lambda \ge 0$, $p = 2,\infty$, and

$$\|\mathbb{K}\phi\|_{\mathbf{L}^{2,1+\lambda}} \le C\|\phi\|_{\mathbf{L}^{2,\lambda}},$$

if $\alpha > 1$, where $\lambda \ge 0$.

Proof. Since

$$\|\phi(\cdot-\eta)-\phi(\cdot)\|_{\mathbf{L}^{p,\lambda}} = \Big\|\int_0^\eta \partial_z\phi(\cdot-z)dz\Big\|_{\mathbf{L}^{p,\lambda}} \le C|\eta|^{\frac{1}{2}+\frac{1}{p}}\langle\eta\rangle^\lambda\|\partial_\xi\phi\|_{\mathbf{L}^{2,\lambda}},$$

for $p=2,\infty$, we find

$$\|\mathbb{K}\phi\|_{\mathbf{L}^{p,\theta}} \le C\|\partial_\xi\phi\|_{\mathbf{L}^{2,\lambda}}\|\int_{\mathbb{R}} e^{-\langle\eta\rangle}\langle\eta\rangle^{\lambda}\langle\xi\eta\rangle^{-\alpha}|\eta|^{\frac{1}{2}+\frac{1}{p}}d\eta\|_{\mathbf{L}^{\infty,\theta-\lambda}} \le C\|\partial_\xi\phi\|_{\mathbf{L}^{2,\lambda}}$$

if $\alpha>\frac{3}{2}+\frac{1}{p}$. We now prove the second estimate. For $|\xi|<1$ we have

$$|\mathbb{K}\phi| \le C\|\phi\|_{\mathbf{L}^2}\Big(\int_{\mathbb{R}} e^{-\langle\eta\rangle}d\eta\Big)^{1/2} \le C\|\phi\|_{\mathbf{L}^2}.$$

For $|\xi|\ge 1$ we write

$$\begin{aligned}
||\xi|^{1+\lambda}\mathbb{K}\phi| &\le C|\xi|^{\lambda}|\phi(\xi)||\xi|\int_{\mathbb{R}}\langle\xi\eta\rangle^{-\alpha}e^{-\langle\eta\rangle}d\eta\\
&\quad + C|\xi|\int_{|\eta|>\frac{|\xi|}{2}}\langle\xi\eta\rangle^{-\alpha}\langle\xi-\eta\rangle^{\lambda}|\phi(\xi-\eta)|d\eta\\
&\quad + C|\xi|\int_{|\eta|<\frac{|\xi|}{2}}\langle\xi\eta\rangle^{-\alpha}\langle\xi-\eta\rangle^{\lambda}|\phi(\xi-\eta)|d\eta.
\end{aligned}$$

By the Cauchy-Schwarz inequality we find for the second summand

$$\begin{aligned}
|\xi|\int_{|\eta|>\frac{|\xi|}{2}}\langle\xi\eta\rangle^{-\alpha}\langle\xi-\eta\rangle^{\lambda}|\phi(\xi-\eta)|d\eta &\le |\xi|\|\phi\|_{\mathbf{L}^{2,\lambda}}\Big(\int_{|\eta|>\frac{|\xi|}{2}}\langle\xi\eta\rangle^{-2\alpha}d\eta\Big)^{1/2}\\
&\le C|\xi|^{\frac{3}{2}-2\alpha}\|\phi\|_{\mathbf{L}^{2,\lambda}}
\end{aligned}$$

and for the third summand we change $\eta=\frac{x}{|\xi|}$,

$$\begin{aligned}
&|\xi|\int_{|\eta|<\xi|/2}\langle\xi\eta\rangle^{-\alpha}\langle\xi-\eta\rangle^{\lambda}|\phi(\xi-\eta)|d\eta\\
&= C\int_{|x|<\xi^2/2}\langle x\rangle^{-\alpha}\langle\xi-\frac{x}{|\xi|}\rangle^{\lambda}|\phi(\xi-\frac{x}{|\xi|})|dx.
\end{aligned}$$

Then taking the $\mathbf{L}^2$ - norm we have

$$\begin{aligned}
\|\xi\mathbb{K}\phi\|_{\mathbf{L}^2(|\xi|\ge 1)} &\le C\big\|\phi\|_{\mathbf{L}^{2,\lambda}} + C\|\phi\|_{\mathbf{L}^{2,\lambda}}\||\xi|^{\frac{3}{2}-2\alpha}\big\|_{\mathbf{L}^2}\\
&\quad + C\|\int_{|x|<\frac{\xi^2}{2}}\langle x\rangle^{-\alpha}\langle\xi-\frac{x}{|\xi|}\rangle^{\lambda}|\phi(\xi-\frac{x}{|\xi|})|dx\|_{\mathbf{L}^2(|\xi|\ge 1)}
\end{aligned}$$

and

$$\begin{aligned}
&\|\int_{|x|<\frac{\xi^2}{2}}\langle x\rangle^{-\alpha}\langle\xi-\frac{x}{|\xi|}\rangle^{\lambda}|\phi(\xi-\frac{x}{|\xi|})|dx\|^2_{\mathbf{L}^2(|\xi|\ge 1)}\\
&= \int_{|\xi|\ge 1}d\xi\int_{|x|<\frac{\xi^2}{2}}\langle x\rangle^{-\alpha}\langle\xi-\frac{x}{|\xi|}\rangle^{\lambda}|\phi(\xi-\frac{x}{|\xi|})|dx\\
&\quad\times\int_{|y|<\frac{\xi^2}{2}}\langle y\rangle^{-\alpha}\langle\xi-\frac{y}{|\xi|}\rangle^{\lambda}|\phi(\xi-\frac{y}{|\xi|})|dy\\
&= \int_{\mathbb{R}}dx\langle x\rangle^{-\alpha}\int_{\mathbb{R}}dy\langle y\rangle^{-\alpha}\int_{|\xi|\ge 1,|\xi|\ge 2|x|,|\xi|\ge 2|y|}\langle\xi-\frac{x}{|\xi|}\rangle^{\lambda}|\phi(\xi-\frac{x}{|\xi|})|\\
&\quad\times\langle\xi-\frac{y}{|\xi|}\rangle^{\lambda}|\phi(\xi-\frac{y}{|\xi|})|d\xi\\
&\le C\|\phi\|^2_{\mathbf{L}^{2,\lambda}}\Big(\int_{\mathbb{R}}\langle x\rangle^{-\alpha}dx\Big)^2 \le C\|\phi\|^2_{\mathbf{L}^{2,\lambda}},
\end{aligned}$$

since changing $z = \xi - \frac{x}{|\xi|}$ in the domain $|\xi| \geq 1, |\xi| \geq 2|x|$ we have

$$|\frac{d\xi}{dz}| = \frac{1}{1 + \frac{x}{\xi^2}} \leq 2.$$

Therefore,

$$\int_{|\xi|\geq 1, |\xi| \geq 2|x|} \langle \xi - \frac{x}{|\xi|} \rangle^{2\lambda} \phi^2 (\xi - \frac{x}{|\xi|}) d\xi \leq 2 \int_{\mathbb{R}} \langle z \rangle^{2\lambda} \phi^2(z) dz = C \|\phi\|^2_{\mathbf{L}^{2,\lambda}}.$$

The proof is complete. □

We now estimate the bilinear operator

$$\mathcal{A}_{a,b}(\phi, \psi) = \int_{\mathbb{R}^2} e^{-i\xi(a\eta + b\zeta)} A(\eta, \zeta)(\phi(\xi - \eta) - \phi(\xi))(\psi(\xi - \zeta) - \psi(\xi)) d\eta d\zeta.$$

Lemma 3.2. *Suppose that*

$$|\partial_\eta^k \partial_\zeta^l A(\eta, \zeta)| \leq C(|\eta| + |\zeta|)^{-s-k-l} e^{-C|\eta| - C|\zeta|}$$

for all $\eta, \zeta \in \mathbb{R}$, $k, l = 0, 1$, where $s \geq 1$. Then the estimates are valid

$$\|\mathcal{A}_{a,b}(\phi, \psi)\|_{\mathbf{L}^{p,\theta}} \leq C \|\partial_\xi \phi\|_{\mathbf{L}^{2,\lambda}} \|\partial_\xi \psi\|_{\mathbf{L}^{2,\delta}}$$

if $s < 3$, $a, b \neq 0$, where $\theta = 2 + \lambda + \delta - \epsilon$, $p = 2, \infty$, $\lambda, \delta \in \mathbb{R}$, and

$$\|\mathcal{A}_{a,b}(\phi, \psi)\|_{\mathbf{L}^{2,\sigma}} \leq C \|\phi\|_{\mathbf{L}^{2,\lambda}} \|\partial_\xi \psi\|_{\mathbf{L}^{2,\delta}}$$

if $s < \frac{5}{2}$, $b \neq 0$, where $\sigma = 1 + \lambda + \delta - \epsilon$, $\lambda, \delta \in \mathbb{R}$, $\epsilon > 0$ is small.

Proof. We integrate by parts with respect to η and ζ via identities $e^{-ia\xi\eta} = P\partial_\eta(\eta e^{-ia\xi\eta})$ and $e^{-ib\xi\zeta} = Q\partial_\zeta(\zeta e^{-ib\xi\zeta})$, where

$$P = (1 - ia\xi\eta)^{-1}, Q = (1 - ib\xi\zeta)^{-1},$$

to get

$$\begin{aligned}
\mathcal{A}_{a,b}(\phi, \psi) = \int_{\mathbb{R}^2} & d\eta d\zeta e^{-i\xi(a\eta + b\zeta)} \Big(\eta\zeta PQA \partial_\eta \phi(\xi - \eta) \partial_\zeta \psi(\xi - \zeta) \\
& + \zeta\eta\partial_\eta(PQA)(\phi(\xi - \eta) - \phi(\xi))\partial_\zeta \psi(\xi - \zeta) \\
& + \eta\zeta\partial_\zeta(PQA)(\psi(\xi - \zeta) - \psi(\xi))\partial_\eta \phi(\xi - \eta) \\
& + (\phi(\xi - \eta) - \phi(\xi))(\psi(\xi - \zeta) - \psi(\xi))\eta\partial_\eta\zeta\partial_\zeta(PQA) \Big).
\end{aligned}$$

Using the estimates

$$\begin{gathered}
|P| \leq C\langle \xi\eta \rangle^{-1}, |Q| \leq C\langle \xi\zeta \rangle^{-1}, \\
|\phi(\xi - \eta) - \phi(\xi)| \leq C\langle \xi \rangle^{-\lambda} |\eta|^{\frac{1}{2}} \langle \eta \rangle^{|\lambda|} \|\partial_\xi \phi\|_{\mathbf{L}^{2,\lambda}}, \\
|\psi(\xi - \zeta) - \psi(\xi)| \leq C\langle \xi \rangle^{-\delta} |\zeta|^{\frac{1}{2}} \langle \zeta \rangle^{|\delta|} \|\partial_\xi \psi\|_{\mathbf{L}^{2,\delta}}
\end{gathered}$$

and by the condition of the lemma

$$\begin{aligned}
& |PQA| + |\eta\partial_\eta(PQA)| + |\zeta\partial_\zeta(PQA)| + |\eta\partial_\eta\zeta\partial_\zeta(PQA)| \\
& \leq C\langle \xi\zeta \rangle^{-1} \langle \xi\eta \rangle^{-1} (|\eta| + |\zeta|)^{-s} e^{-C|\eta| - C|\zeta|},
\end{aligned}$$

we obtain

$$\begin{aligned}
& |\mathcal{A}_{a,b}(\phi, \psi)| \\
& \leq C \int_{\mathbb{R}^2} d\eta d\zeta \langle \xi\eta \rangle^{-1} \langle \xi\zeta \rangle^{-1} (|\eta| + |\zeta|)^{-s} e^{-C|\eta| - C|\zeta|}
\end{aligned}$$

$$\times \Big(|\eta\zeta||\partial_\eta\phi(\xi-\eta)||\partial_\zeta\psi(\xi-\zeta)| + \langle\xi\rangle^{-\lambda}\|\partial_\xi\phi\|_{\mathbf{L}^{2,\lambda}}|\eta|^{1/2}|\zeta||\partial_\zeta\psi(\xi-\zeta)|$$
$$+ \langle\xi\rangle^{-\delta}\|\partial_\xi\psi\|_{\mathbf{L}^{2,\delta}}|\eta||\zeta|^{1/2}|\partial_\eta\phi(\xi-\eta)| + \langle\xi\rangle^{-\lambda-\delta}|\eta\zeta|^{1/2}\|\partial_\xi\phi\|_{\mathbf{L}^{2,\lambda}}\|\partial_\xi\psi\|_{\mathbf{L}^{2,\delta}}\Big).$$

Hence by the Cauchy-Schwarz inequality

$$\begin{aligned}
|\mathcal{A}_{a,b}(\phi,\psi)| &\le C\langle\xi\rangle^{-\lambda-\delta}\|\partial_\xi\phi\|_{\mathbf{L}^{2,\lambda}}\|\partial_\xi\psi\|_{\mathbf{L}^{2,\delta}} \\
&\quad\times\Big(\Big(\int_{\mathbb{R}}|\eta|^{2-s}\langle\xi\eta\rangle^{-2}e^{-C|\eta|}d\eta\Big)^{1/2}\Big(\int_{\mathbb{R}}|\zeta|^{2-s}\langle\xi\zeta\rangle^{-2}e^{-C|\zeta|}d\zeta\Big)^{1/2} \\
&\quad+\int_{\mathbb{R}}|\eta|^{\frac{1-s}{2}}\langle\xi\eta\rangle^{-1}e^{-C|\eta|}d\eta\Big(\int_{\mathbf{R}}|\zeta|^{2-s}\langle\xi\zeta\rangle^{-2}e^{-C|\zeta|}d\zeta\Big)^{1/2} \\
&\quad+\int_{\mathbb{R}}|\eta|^{\frac{1-s}{2}}\langle\xi\eta\rangle^{-1}e^{-C|\eta|}d\eta\int_{\mathbb{R}}|\zeta|^{\frac{1-s}{2}}\langle\xi\zeta\rangle^{-1}e^{-C|\zeta|}d\zeta\Big) \\
&\le C\langle\xi\rangle^{-\theta}\|\partial_\xi\phi\|_{\mathbf{L}^{2,\lambda}}\|\partial_\xi\psi\|_{\mathbf{L}^{2,\delta}}
\end{aligned}$$

with $\theta = 2+\lambda+\delta-\epsilon$, $s<3$, $\epsilon>0$. For the case $p=2$ we use the inequality

$$\|\phi(\cdot-\eta)-\phi(\cdot)\|_{\mathbf{L}^{2,\lambda}} \le C|\eta|\langle\eta\rangle^{|\lambda|}\|\partial_\xi\phi\|_{\mathbf{L}^{2,\lambda}}$$

to get

$$\begin{aligned}
&\|\mathcal{A}_{a,b}(\phi,\psi)\|_{\mathbf{L}^{2,\theta}} \\
&\le \sup_{\xi\in\mathbb{R}}\int_{\mathbb{R}^2}d\eta d\zeta\langle\xi\eta\rangle^{-1}\langle\xi\zeta\rangle^{-1}(|\eta|+|\zeta|)^{-s}e^{-C|\eta|-C|\zeta|} \\
&\quad\times\Big(\|\partial_\xi\phi\|_{\mathbf{L}^{2,\lambda}}\langle\xi\rangle^{\theta-\lambda}|\eta\zeta||\partial_\zeta\psi(\xi-\zeta)| + \|\partial_\xi\psi\|_{\mathbf{L}^{2,\delta}}\langle\xi\rangle^{\theta-\delta}|\eta\zeta||\partial_\eta\phi(\xi-\eta)| \\
&\quad+\|\partial_\xi\phi\|_{\mathbf{L}^{2,\lambda}}\|\partial_\xi\psi\|_{\mathbf{L}^{2,\delta}}\langle\xi\rangle^{\theta-\lambda-\delta}|\eta||\zeta|^{1/2}\Big).
\end{aligned}$$

Hence

$$\begin{aligned}
\|\mathcal{A}_{a,b}(\phi,\psi)\|_{\mathbf{L}^{2,\theta}} &\le C\|\partial_\xi\phi\|_{\mathbf{L}^{2,\lambda}}\|\partial_\xi\psi\|_{\mathbf{L}^{2,\delta}}\sup_{\xi\in\mathbb{R}}\langle\xi\rangle^{\theta-\lambda-\delta} \\
&\quad\times\Big(\int_{\mathbb{R}}|\eta|^{1-\frac{2s}{3}}\langle\xi\eta\rangle^{-1}e^{-C|\eta|}d\eta\Big(\int_{\mathbb{R}}|\zeta|^{2-\frac{2s}{3}}\langle\xi\zeta\rangle^{-2}e^{-C|\zeta|}d\zeta\Big)^{1/2} \\
&\quad+\int_{\mathbb{R}}|\zeta|^{1-\frac{2s}{3}}\langle\xi\zeta\rangle^{-1}e^{-C|\zeta|}d\zeta\Big(\int_{\mathbb{R}}|\eta|^{2-\frac{2s}{3}}\langle\xi\eta\rangle^{-2}e^{-C|\eta|}d\eta\Big)^{1/2} \\
&\quad+\int_{\mathbb{R}}|\eta|^{1-\frac{2s}{3}}\langle\xi\eta\rangle^{-1}e^{-C|\eta|}d\eta\int_{\mathbb{R}}|\zeta|^{\frac{1}{2}-\frac{s}{3}}\langle\xi\zeta\rangle^{-1}e^{-C|\zeta|}d\zeta\Big) \\
&\le C\|\partial_\xi\phi\|_{\mathbf{L}^{2,\lambda}}\|\partial_\xi\psi\|_{\mathbf{L}^{2,\delta}}
\end{aligned}$$

with $\theta = 2+\lambda+\delta-\epsilon$, $s<3$, $\epsilon>0$. To prove the second estimate of the lemma as above we integrate by parts with respect to ζ

$$\begin{aligned}
\mathcal{A}_{a,b}(\phi,\psi) = \int_{\mathbb{R}^2}d\eta d\zeta e^{-bi\xi\zeta}\Big(&\zeta QA(\phi(\xi-\eta)-\phi(\xi))\partial_\zeta\psi(\xi-\zeta) \\
&+(\phi(\xi-\eta)-\phi(\xi))(\psi(\xi-\zeta)-\psi(\xi))\zeta\partial_\zeta(QA)\Big).
\end{aligned}$$

Using the estimates

$$|\psi(\xi-\zeta)-\psi(\xi)| \le C\langle\xi\rangle^{-\lambda}|\zeta|^{\frac12}\langle\zeta\rangle^{|\lambda|}\|\partial_\xi\psi\|_{\mathbf{L}^{2,\lambda}}$$

and

$$|\zeta\partial_\zeta(QA)| + |QA| \le C\langle\xi\zeta\rangle^{-1}(|\eta|+|\zeta|)^{-s}e^{-C|\eta|-C|\zeta|},$$

we obtain

$$|\mathcal{A}_{a,b}(\phi,\psi)| \le \int_{\mathbb{R}^2} d\eta d\zeta |(\phi(\xi-\eta)-\phi(\xi))| \langle \xi\zeta\rangle^{-1}(|\eta|+|\zeta|)^{-s} e^{-C|\eta|-C|\zeta|} \\ \times \big(|\zeta||\partial_\zeta \psi(\xi-\zeta)| + \|\partial_\xi \psi\|_{\mathbf{L}^{2,\delta}} \langle \xi\rangle^{-\delta}|\zeta|^{1/2}\big).$$

Hence

$$\begin{aligned} \|\mathcal{A}_{a,b}(\phi,\psi)\|_{\mathbf{L}^{2,\sigma}} &\le C\|\phi\|_{\mathbf{L}^{2,\lambda}} \|\partial_\xi \psi\|_{\mathbf{L}^{2,\delta}} \sup_{\xi\in\mathbb{R}} \langle \xi\rangle^{\sigma-\lambda-\delta} \int_{\mathbb{R}} e^{-C|\eta|} d\eta \\ &\quad \times \Big(\Big(\int_{\mathbb{R}} |\zeta|^2 \langle \xi\zeta\rangle^{-2}(|\eta|+|\zeta|)^{-2s} e^{-C|\zeta|} d\zeta\Big)^{1/2} \\ &\quad + \int_{\mathbb{R}} |\zeta|^{1/2}\langle \xi\zeta\rangle^{-1}(|\eta|+|\zeta|)^{-s} e^{-C|\zeta|} d\zeta\Big) \\ &\le C\|\phi\|_{\mathbf{L}^{2,\lambda}} \|\partial_\xi \psi\|_{\mathbf{L}^{2,\delta}} \end{aligned}$$

if $\sigma = 1+\lambda+\delta-\epsilon$, $s<\frac{5}{2}$, $\epsilon>0$. The proof is complete. □

We next estimate the operator

$$\mathbb{A}(\phi,\psi) \equiv \int_{\mathbb{R}^2} e^{-i\xi(a\eta+b\zeta)} A(\eta,\zeta)\phi(\xi-\eta)\psi(\xi-\zeta) d\eta d\zeta,$$

where $a,b\in\mathbb{R}\backslash\{0\}$. Denote

$$\Lambda(\xi) = \int_{\mathbb{R}^2} e^{-i\xi(a\eta+b\zeta)} A(\eta,\zeta) d\eta d\zeta.$$

Lemma 3.3. *Suppose that*

$$\big|\partial_\eta^k \partial_\zeta^l \Big(A(\eta,\zeta) - \frac{a_1\eta+b_1\zeta}{\eta^2-\zeta\eta+\zeta^2} e^{-\langle\eta\rangle-\langle\zeta\rangle}\Big)\big| \le C(|\eta|+|\zeta|)^{-k-l} e^{-\langle\eta\rangle-\langle\zeta\rangle} \tag{3.1}$$

for all $\eta,\zeta\in\mathbb{R}$, $k,l=0,1,2,3$, *where* $a_1,b_1\in\mathbf{R}$. *Then*

$$\begin{aligned} &\|\mathbb{A}(\phi,\psi) - \Lambda\phi\psi\|_{\mathbf{L}^{p,\theta}} \\ &\le C\|\phi\|_{\mathbf{L}^{p,\alpha}} \|\partial_\xi\psi\|_{\mathbf{L}^{2,\delta}} + C\|\psi\|_{\mathbf{L}^{p,\beta}} \|\partial_\xi\phi\|_{\mathbf{L}^{2,\lambda}} + C\|\partial_\xi\phi\|_{\mathbf{L}^{2,\lambda}} \|\partial_\xi\psi\|_{\mathbf{L}^{2,\delta}}, \end{aligned}$$

with

$$\theta = \min(\frac{3}{2}+\frac{1}{p}+\alpha+\delta, \frac{3}{2}+\frac{1}{p}+\beta+\lambda, 2+\lambda+\delta-\epsilon), \quad p=2,\infty$$

and

$$\|\mathbb{A}(\phi,\psi)\|_{\mathbf{L}^{2,\sigma}} \le C\|\phi\|_{\mathbf{L}^{2,\lambda}} (\|\psi\|_{\mathbf{L}^{\infty,\beta}} + \|\partial_\xi\psi\|_{\mathbf{L}^{2,\delta}})$$

with $\sigma = \min(1+\lambda+\beta, 1+\lambda+\delta-\epsilon)$, $\alpha,\beta,\lambda,\delta\in\mathbb{R}$ $\epsilon>0$ *is small.*

Proof. We represent

$$\mathbb{A}(\phi,\psi) = \Lambda\phi\psi + \phi\mathbb{K}_1\psi + \psi\mathbb{K}_2\phi + \mathcal{A}_{a,b}(\phi,\psi),$$

where the kernels of the operators $\mathbb{K}_1$ and $\mathbb{K}_2$ are

$$K_1(\xi,\zeta) = \int_{\mathbb{R}} e^{-i\xi(a\eta+b\zeta)} A(\eta,\zeta) d\eta, \quad K_2(\xi,\eta) = \int_{\mathbb{R}} e^{-i\xi(a\eta+b\zeta)} A(\eta,\zeta) d\zeta.$$

For $|\xi\zeta|<1$, $\zeta\ne 0$ changing $\eta-\frac{\zeta}{2}=z$ we get

$$K_1(\xi,\zeta) = e^{-\langle\zeta\rangle} \int_{\mathbb{R}} e^{-ia\xi\eta - \langle \eta-\frac{\zeta}{2}\rangle} \frac{a_1\eta+b_1\zeta}{\eta^2-\zeta\eta+\zeta^2} d\eta$$

$$
\begin{aligned}
&+ O\Big(e^{-\langle\zeta\rangle}\int_{\mathbb{R}}((|\eta|+|\zeta|)^{-1}|e^{-\langle\eta\rangle}-e^{-\langle\eta-\frac{\zeta}{2}\rangle}|+e^{-\langle\eta\rangle})d\eta\Big)\\
&= e^{-\frac{i}{2}a\xi\zeta-\langle\zeta\rangle}\int_{\mathbb{R}} e^{-\langle z\rangle}(C_1 iz\sin(a\xi z)+C_2\zeta\cos(a\xi z))\frac{dz}{z^2+\frac{3}{4}\zeta^2}+O(e^{-\langle\zeta\rangle})\\
&= O(e^{-\langle\zeta\rangle}).
\end{aligned}
$$

For $|\xi\zeta|\geq 1$ integrating three times by parts we obtain

$$
\begin{aligned}
|K_1(\xi,\zeta)| &= |(a\xi)^{-3}\int_{\mathbb{R}} e^{-ia\xi\eta}\partial_\eta^3 A(\eta,\zeta)d\eta|\\
&\leq C|\xi|^{-3}e^{-\langle\zeta\rangle}\int_{\mathbb{R}}\frac{d\eta}{(|\eta|+|\zeta|)^4}\\
&\leq C|\xi\zeta|^{-3}e^{-\langle\zeta\rangle}.
\end{aligned}
$$

Hence the estimates are true for all $\xi\in\mathbb{R}$, $\zeta,\eta\in\mathbb{R}\backslash\{0\}$

$$
|K_1(\xi,\zeta)|\leq Ce^{-\langle\zeta\rangle}\langle\xi\zeta\rangle^{-3},\quad |K_2(\xi,\eta)|\leq Ce^{-\langle\eta\rangle}\langle\xi\eta\rangle^{-3}.
$$

Then by Lemma 3.1, we find

$$
\begin{aligned}
\|\psi\mathbb{K}_2\phi\|_{\mathbf{L}^{p,\frac{3}{2}+\frac{1}{p}+\beta+\lambda}} &\leq C\|\psi\|_{\mathbf{L}^{p,\beta}}\|\partial_\xi\phi\|_{\mathbf{L}^{2,\lambda}},\\
\|\phi\mathbb{K}_1\psi\|_{\mathbf{L}^{p,\frac{3}{2}+\frac{1}{p}+\alpha+\delta}} &\leq C\|\phi\|_{\mathbf{L}^{p,\alpha}}\|\partial_\xi\psi\|_{\mathbf{L}^{2,\delta}}.
\end{aligned}
$$

Applying Lemma 3.2 with $s=1$ we have

$$
\|\mathcal{A}_{a,b}(\phi,\psi)\|_{\mathbf{L}^{p,2+\lambda+\delta-\epsilon}}\leq C\|\partial_\xi\phi\|_{\mathbf{L}^{2,\lambda}}\|\partial_\xi\psi\|_{\mathbf{L}^{2,\delta}}
$$

with $\epsilon>0$, $p=2,\infty$. Hence the first estimate of the lemma follows. To prove the second estimate of the lemma we note that by (3.1),

$$
\begin{aligned}
\Lambda(\xi) &= \int_{\mathbb{R}^2} e^{-i\xi(a\eta+b\zeta)}A(\eta,\zeta)d\eta d\zeta\\
&= \int_{\mathbb{R}} d\eta e^{-ia\xi\eta-\langle\eta\rangle}\int_{\mathbb{R}} e^{-ib\xi\zeta}\frac{a_1\eta+b_1\zeta}{\eta^2-\eta\zeta+\zeta^2}d\zeta+O(\langle\xi\rangle^{-2})\\
&= \pi\int_{\mathbb{R}}(C_1+C_2\operatorname{sign}\eta)e^{-i(a+\frac{b}{2})\xi\eta-\frac{\sqrt{3}}{2}|\eta||b\xi|-\langle\eta\rangle}d\eta+O(\langle\xi\rangle^{-2})\\
&= O(\langle\xi\rangle^{-1}).
\end{aligned}
$$

Hence

$$
\|\Lambda\phi\psi\|_{\mathbf{L}^{2,1+\lambda+\beta}}\leq C\|\phi\|_{\mathbf{L}^{2,\lambda}}\|\psi\|_{\mathbf{L}^{\infty,\beta}}.
$$

By virtue of estimates in Lemma 3.1, we get

$$
\begin{aligned}
\|\phi\mathbb{K}_1\psi\|_{\mathbf{L}^{2,\frac{3}{2}+\lambda+\delta}} &\leq C\|\phi\|_{\mathbf{L}^{2,\lambda}}\|\partial_\xi\psi\|_{\mathbf{L}^{2,\delta}},\\
\|\psi\mathbb{K}_2\phi\|_{\mathbf{L}^{2,1+\lambda+\beta}} &\leq C\|\phi\|_{\mathbf{L}^{2,\lambda}}\|\psi\|_{\mathbf{L}^{\infty,\beta}}.
\end{aligned}
$$

Then applying the second estimate of Lemma 3.2 with $s=1$, we have

$$
\|\mathcal{A}_{a,b}(\phi,\psi)\|_{\mathbf{L}^{2,1+\lambda+\delta-\epsilon}}\leq C\|\phi\|_{\mathbf{L}^{2,\lambda}}\|\partial_\xi\psi\|_{\mathbf{L}^{2,\delta}}
$$

with $\epsilon>0$. The proof is complete. □

Next we consider the $\mathbf{L}^2$-estimates of the operator

$$
\mathbb{H}(\phi,\psi)=\int_{\mathbb{R}^2}A(\eta,\zeta)e^{\frac{i}{2}a(\xi-\eta)^2}\phi(\xi-\eta)\partial_\zeta\big(e^{\frac{i}{2}b(\xi-\zeta)^2}\psi(\xi-\zeta)\big)d\eta\, d\zeta.
$$

Lemma 3.4. *Suppose that condition* (3.1) *is fulfilled. Then the estimates*

$$\|\mathbb{H}(\phi,\psi)\|_{\mathbf{L}^2} \le C\|\phi\|_{\mathbf{L}^2}(\|\psi\|_{\mathbf{L}^\infty} + \|\partial_\xi\psi\|_{\mathbf{L}^{2,\lambda}} + \|\partial_\xi^2\psi\|_{\mathbf{L}^2}),$$
$$\|\mathbb{H}(\phi,\psi)\|_{\mathbf{L}^2} \le C\|\psi\|_{\mathbf{L}^2}(\|\phi\|_{\mathbf{L}^\infty} + \|\partial_\xi\phi\|_{\mathbf{L}^2})$$

are true provided that the right-hand sides are finite, $\lambda > 0$.

Proof. To prove the first estimate of the lemma we write

$$\mathbb{H}(\phi,\psi) = e^{\frac{i}{2}(a+b)\xi^2}(\xi\mathbb{A}_1(\phi,\psi) + \mathbb{A}_2(\phi,\psi) + \mathbb{A}_3(\phi,\partial_\xi\psi))$$

where the kernels are

$$A_1(\eta,\zeta) = -ibe^{\frac{i}{2}(a\eta^2+b\zeta^2)}A(\eta,\zeta),$$
$$A_2(\eta,\zeta) = \frac{ib}{2}e^{\frac{i}{2}(a\eta^2+b\zeta^2)}\zeta A(\eta,\zeta),$$
$$A_3(\eta,\zeta) = -e^{\frac{i}{2}(a\eta^2+b\zeta^2)}A(\eta,\zeta)$$

respectively. Then by the second estimate of Lemma 3.3 we obtain

$$\begin{aligned}\|\mathbb{H}(\phi,\psi)\|_{\mathbf{L}^2} &\le \|\xi\mathbb{A}_1(\phi,\psi)\|_{\mathbf{L}^2} + \|\mathbb{A}_2(\phi,\psi)\|_{\mathbf{L}^2} + \|\mathbb{A}_3(\phi,\partial_\xi\psi)\|_{\mathbf{L}^2}\\ &\le C\|\phi\|_{\mathbf{L}^2}\big(\|\psi\|_{\mathbf{L}^\infty} + \|\partial_\xi\psi\|_{\mathbf{L}^{2,\lambda}} + \|\partial_\xi\psi\|_{\mathbf{L}^\infty} + \|\partial_\xi^2\psi\|_{\mathbf{L}^2}\big).\end{aligned}$$

Thus the first estimate of the lemma is true. To prove the second estimate we denote $e^{\frac{i}{2}b\xi^2}\psi(\xi) = \widetilde{\psi}(\xi)$ and represent

$$e^{\frac{i}{2}a(\xi-\eta)^2}\phi(\xi-\eta) = \phi(\xi)e^{\frac{i}{2}a(\xi-\eta)^2} + e^{\frac{i}{2}a(\xi-\eta)^2}(\phi(\xi-\eta)-\phi(\xi)).$$

Then we integrate by parts with respect to ζ,

$$\begin{aligned}\mathbb{H}(\phi,\psi) &= -\int_{\mathbb{R}^2} e^{\frac{i}{2}a(\xi-\eta)^2}\partial_\zeta A(\eta,\zeta)\phi(\xi-\eta)(\widetilde{\psi}(\xi-\zeta)-\widetilde{\psi}(\xi))d\eta d\zeta\\ &= -\phi\mathbb{K}_1\widetilde{\psi} - e^{\frac{i}{2}a\xi^2}\widetilde{\mathcal{A}}_{a,0}(\phi,\widetilde{\psi}),\end{aligned}$$

where the kernels are respectively

$$K_1(\xi,\zeta) = \int_{\mathbb{R}} e^{\frac{i}{2}a(\xi-\eta)^2}\partial_\zeta A(\eta,\zeta)d\eta,$$
$$\widetilde{A}(\eta,\zeta) = e^{\frac{i}{2}a\eta^2}\partial_\zeta A(\eta,\zeta).$$

As above we have the estimate

$$|K_1(\xi,\zeta)| \le C\langle\xi\rangle e^{-\langle\zeta\rangle}\langle\xi\zeta\rangle^{-3}.$$

Thus by Lemma 3.1 we find

$$\|\phi\mathbb{K}_1\widetilde{\psi}\|_{\mathbf{L}^2} \le C\|\phi\|_{\mathbf{L}^\infty}\|\psi\|_{\mathbf{L}^2}.$$

Finally by the second estimate of Lemma 3.2 with $s=2$, $\lambda=0$ we estimate

$$\|e^{\frac{i}{2}a\xi^2}\widetilde{\mathcal{A}}_{a,0}(\phi,\widetilde{\psi})\|_{\mathbf{L}^2} \le C\|\partial_\xi\phi\|_{\mathbf{L}^2}\|\psi\|_{\mathbf{L}^2}$$

Thus the second estimate of the lemma is true. The lemma is proved. □

As a consequence of Lemmas 3.3–3.4 we obtain the estimates of the operators

$$\mathcal{G}_{a,b}(\phi,\psi) = \overline{E}^{a+b}\int_{\mathbf{R}^2} h(t,\eta,\zeta)e^{\frac{a}{2}it(\xi-\eta)^2}\phi(t,\xi-\eta)e^{\frac{b}{2}it(\xi-\zeta)^2}\psi(t,\xi-\zeta)d\eta d\zeta$$

and

$$\mathcal{H}_{a,b}(\phi,\psi) = t^{-1/2}\overline{E}^{a+b}\int_{\mathbb{R}^2} h(t,\eta,\zeta)e^{\frac{a}{2}it(\xi-\eta)^2}\phi(t,\xi-\eta) \times \partial_\zeta(e^{\frac{b}{2}it(\xi-\zeta)^2}\psi(t,\xi-\zeta))d\eta d\zeta.$$

Denote $B_\lambda = \langle \xi\sqrt{t}\rangle^\lambda$ and

$$Q_{a,b}(t,\xi) = \int_{\mathbb{R}^2} e^{-it\xi(a\eta+b\zeta)}h(t,\eta,\zeta)d\eta d\zeta.$$

Lemma 3.5. *Suppose that*

$$|\partial_\eta^k\partial_\zeta^l(h(t,\eta,\zeta) - \frac{\sqrt{t}(a_1\eta+b_1\zeta)}{\eta^2-\zeta\eta+\zeta^2}e^{-\langle\sqrt{t}\eta\rangle-\langle\sqrt{t}\zeta\rangle})| \le C(|\eta|+|\zeta|)^{-k-l}e^{-C\langle\sqrt{t}\eta\rangle-C\langle\sqrt{t}\zeta\rangle} \tag{3.2}$$

for all $t\ge 1$, $\eta,\zeta\in\mathbb{R}$, $k,l=0,1,2,3$. Then the following estimates are true for all $t\ge1$:

$$\begin{aligned}&\|B_\theta(\mathcal{G}_{a,b}(\phi,\psi) - Q_{a,b}\phi\psi)\|_{\mathbf{L}^p}\\ &\le Ct^{-\frac14}\|B_\alpha\phi\|_{\mathbf{L}^p}\|B_\delta\partial_\xi\psi\|_{\mathbf{L}^2} + Ct^{-\frac14}\|B_\beta\psi\|_{\mathbf{L}^p}\|B_\lambda\partial_\xi\phi\|_{\mathbf{L}^2}\\ &\quad + Ct^{-1/2}\|B_\lambda\partial_\xi\phi\|_{\mathbf{L}^2}\|B_\delta\partial_\xi\psi\|_{\mathbf{L}^2},\end{aligned}$$

where

$$\theta = \min(\frac32+\frac1p+\alpha+\delta, \frac32+\frac1p+\beta+\lambda, 2+\lambda+\delta-\epsilon), \quad p=2,\infty$$

and

$$\|B_\sigma\mathcal{G}_{a,b}(\phi,\psi)\|_{\mathbf{L}^2} \le C\|B_\lambda\phi\|_{\mathbf{L}^2}(\|B_\beta\psi\|_{\mathbf{L}^\infty} + t^{-\frac14}\|B_\delta\partial_\xi\psi\|_{\mathbf{L}^2})$$

for $\sigma = \min(1+\lambda+\beta, 1+\lambda+\delta-\epsilon), \alpha,\beta,\lambda,\delta\in\mathbb{R}$, $\epsilon>0$ is small. Also for all $t\ge1$,

$$\|\mathcal{H}_{a,b}(\phi,\psi)\|_{\mathbf{L}^2} \le C\|\phi\|_{\mathbf{L}^2}\big(\|\psi\|_{\mathbf{L}^\infty} + t^{-\frac14}\|B_\lambda\partial_\xi\psi\|_{\mathbf{L}^{2,\lambda}} + t^{-\frac34}\|\partial_\xi^2\psi\|_{\mathbf{L}^2}\big)$$

and

$$\|\mathcal{H}_{a,b}(\phi,\psi)\|_{\mathbf{L}^2} \le C\|\psi\|_{\mathbf{L}^2}\big(\|\phi\|_{\mathbf{L}^\infty} + t^{-\frac14}\|\partial_\xi\phi\|_{\mathbf{L}^2}\big),$$

where $\lambda>0$.

Proof. We make a change $\sqrt{t}\xi = \xi'$, $\sqrt{t}\eta=\eta'$ and $\sqrt{t}\zeta=\zeta'$,

$$\begin{aligned}\mathcal{G}_{a,b}(\phi,\psi) &= \overline{E}^{a+b}\int_{\mathbb{R}^2} h(t,\eta,\zeta)e^{\frac{a}{2}it(\xi-\eta)^2}\phi(t,\xi-\eta)e^{\frac{b}{2}it(\xi-\zeta)^2}\psi(t,\xi-\zeta)d\eta d\zeta\\ &= \int_{\mathbb{R}^2} e^{-i\xi'(a\eta'+b\zeta')}\widetilde{A}(\eta',\zeta')\widetilde{\psi}(t,\xi'-\eta')\widetilde{\phi}(t,\xi'-\zeta')d\eta'd\zeta',\end{aligned}$$

where

$$\widetilde{A}(\eta',\zeta') = t^{-1}e^{\frac{a}{2}i(\eta')^2+\frac{b}{2}i(\zeta')^2}h(t,\frac{\eta'}{\sqrt{t}},\frac{\zeta'}{\sqrt{t}}),$$

$$\widetilde{\phi}(t,\xi') = \phi(t,\frac{\xi'}{\sqrt{t}}), \qquad \widetilde{\psi}(t,\xi') = \psi(t,\frac{\xi'}{\sqrt{t}}),$$

Also we denote

$$\widetilde{Q}(\xi) = \int_{\mathbb{R}^2} e^{-i\xi(a\eta'+b\zeta')}\widetilde{A}(\eta',\zeta')d\eta'd\zeta'.$$

Now application of Lemma 3.3 yields the first two estimates of the lemma. As above we make a change $\sqrt{t}\xi = \xi'$, $\sqrt{t}\eta = \eta'$ and $\sqrt{t}\zeta = \zeta'$

$$\mathcal{H}_{a,b}(\phi,\psi) = e^{\frac{i}{2}(a+b)(\xi')^2} \int_{\mathbb{R}^2} \widetilde{A}(\eta',\zeta') e^{\frac{i}{2}a(\xi'-\eta')^2} \widetilde{\phi}(t,\xi'-\eta') \times \partial_{\zeta'}(e^{\frac{i}{2}b(\xi'-\zeta')^2} \widetilde{\psi}(t,\xi'-\eta'))d\eta' d\zeta',$$

then applying Lemma 3.4 we find the estimates of the lemma. The proof is complete. □

4. Inverse transformation

We consider the transformation

$$\mathcal{I}(v) = v - \overline{E}^3 \mathcal{G}_{-1,-1}(\overline{v},\overline{v}),$$

where $E(t) = e^{\frac{it}{2}\xi^2}$ and

$$\mathcal{G}_{a,b}(\phi,\psi) = \overline{E}^{a+b} \int_{\mathbf{R}^2} h(t,\eta,\zeta) e^{\frac{a}{2}it(\xi-\eta)^2} \phi(t,\xi-\eta) e^{\frac{b}{2}it(\xi-\zeta)^2} \psi(t,\xi-\zeta) d\eta d\zeta.$$

We first give estimates of the operator $\mathcal{G}_{a,b}$ in the norm

$$\|\phi\|_{\mathbf{X}_T^{\alpha,\lambda}} \equiv \sup_{1\le t\le T} (\|B_\alpha \phi(t)\|_{\mathbf{L}^\infty} + t^{-\frac14} \|B_\lambda \partial_\xi \phi(t)\|_{\mathbf{L}^2} + t^{-\frac34} \|\partial_\xi^2 \phi(t)\|_{\mathbf{L}^2}).$$

Lemma 4.1. *Let condition* (3.2) *be fulfilled. Then the estimate*

$$\|E^q(\mathcal{G}_{a,b}(\phi,\psi) - Q_{a,b}\phi\psi)\|_{\mathbf{X}_T^{\theta,\sigma}} \le C\|\phi\|_{\mathbf{X}_T^{\alpha,\lambda}} \|\psi\|_{\mathbf{X}_T^{\beta,\delta}}$$

is true, where $\theta = \min(\frac32 + \alpha + \delta, \frac32 + \beta + \lambda, 2 + \lambda + \delta - \epsilon)$, $\alpha,\beta,\lambda,\delta \in \mathbb{R}$, $\epsilon > 0$ *is small,*

$$\sigma < \min(2+\alpha, 2+\beta, 2+\lambda, 2+\delta, 1+\alpha+\delta, 1+\beta+\lambda, \frac{3}{2}+\lambda+\delta)$$

with an additional condition in the case of $q \ne 0$ *that* $\alpha,\beta,\lambda,\delta$ *are such that*

$$\sigma < \min(\alpha+\delta, \beta+\lambda, \frac{1}{2}+\lambda+\delta),$$

$$\min(\alpha+\delta, \beta+\lambda, \frac{1}{2}+\lambda+\delta) > 1.$$

Proof. By the first estimate of Lemma 3.5 with $p = \infty$ we have

$$\|B_\theta(\mathcal{G}_{a,b}(\phi,\psi) - Q_{a,b}\phi\psi)\|_{\mathbf{L}^\infty} \le C\|\phi\|_{\mathbf{X}_T^{\alpha,\lambda}} \|\psi\|_{\mathbf{X}_T^{\beta,\delta}}.$$

We now estimate the derivative

$$\begin{aligned}
&\partial_\xi(E^q(\mathcal{G}_{a,b}(\phi,\psi) - Q_{a,b}\phi\psi)) \\
&= E^q(\mathcal{G}_{a,b}(\partial_\xi\phi,\psi) - Q_{a,b}\psi\partial_\xi\phi) + E^q(\mathcal{G}_{a,b}(\phi,\partial_\xi\psi) - Q_{a,b}\phi\partial_\xi\psi) \\
&\quad + \sqrt{t}E^q(\widetilde{\mathcal{G}_{a,b}}(\phi,\psi) - \widetilde{Q}_{a,b}\phi\psi) + itq\xi E^q(\mathcal{G}_{a,b}(\phi,\psi) - Q_{a,b}\phi\psi),
\end{aligned}$$

where $\widetilde{\mathcal{G}_{a,b}}$ and $\widetilde{Q}_{a,b}$ are defined by the kernel $\widetilde{h}(t,\eta,\zeta) = -i\sqrt{t}(a\eta + b\zeta)h(t,\eta,\zeta)$. Then by the first estimate of Lemma 3.5 with $p = 2$ we find the estimate

$$\begin{aligned}
&\|B_\sigma \partial_\xi(E^q(\mathcal{G}_{a,b}(\phi,\psi) - Q_{a,b}\phi\psi))\|_{\mathbf{L}^2} \\
&\le C\|B_\sigma(\mathcal{G}_{a,b}(\partial_\xi\phi,\psi) - Q_{a,b}\psi\partial_\xi\phi)\|_{\mathbf{L}^2} \\
&\quad + C\|B_\sigma(\mathcal{G}_{a,b}(\phi,\partial_\xi\psi) - Q_{a,b}\phi\partial_\xi\psi)\|_{\mathbf{L}^2}
\end{aligned}$$

$$
\begin{aligned}
&+ C\sqrt{t}\|B_{-\frac{1}{2}-\epsilon}\|_{\mathbf{L}^2}\|B_{\sigma+\frac{1}{2}+\epsilon}(\widetilde{\mathcal{G}_{a,b}}(\phi,\psi)-\widetilde{Q}_{a,b}\phi\psi)\|_{\mathbf{L}^\infty}\\
&+ Cq\sqrt{t}\|B_{-\frac{1}{2}-\epsilon}\|_{\mathbf{L}^2}\|B_{\sigma+\frac{3}{2}+\epsilon}(\mathcal{G}_{a,b}(\phi,\psi)-Q_{a,b}\phi\psi)\|_{\mathbf{L}^\infty}\\
&\le Ct^{1/4}\|\phi\|_{\mathbf{X}_T^{\alpha,\lambda}}\|\psi\|_{\mathbf{X}_T^{\beta,\delta}}
\end{aligned}
$$

with the additional condition

$$
\sigma < \min\big(2+\alpha, 2+\beta, 2+\lambda, 2+\delta, \alpha+\delta, \beta+\lambda, \frac{1}{2}+\lambda+\delta\big)
$$

for the case of $q \neq 0$. Finally we estimate the second derivative

$$
\begin{aligned}
&\partial_\xi^2(E^q(\mathcal{G}_{a,b}(\phi,\psi)-Q_{a,b}\phi\psi))\\
&= E^q(\mathcal{G}_{a,b}(\partial_\xi^2\phi,\psi)-Q_{a,b}\psi\partial_\xi^2\phi)\\
&+ 2E^q(\mathcal{G}_{a,b}(\partial_\xi\phi,\partial_\xi\psi)-Q_{a,b}\partial_\xi\phi\partial_\xi\psi) + E^q(\mathcal{G}_{a,b}(\phi,\partial_\xi^2\psi)-Q_{a,b}\phi\partial_\xi^2\psi)\\
&+ \sqrt{t}E^q(\widetilde{\mathcal{G}_{a,b}}(\partial_\xi\phi,\psi)-\widetilde{Q}_{a,b}\partial_\xi\phi\psi) + \sqrt{t}E^q(\widetilde{\mathcal{G}_{a,b}}(\phi,\partial_\xi\psi)-\widetilde{Q}_{a,b}\phi\partial_\xi\psi)\\
&+ tE^q(\widetilde{\widetilde{\mathcal{G}_{a,b}}}(\phi,\psi)-\widetilde{\widetilde{Q}}_{a,b}\phi\psi) + iqt(1+iqt\xi^2)E^q(\mathcal{G}_{a,b}(\phi,\psi)-Q_{a,b}\phi\psi)\\
&+ itq\xi E^q(\mathcal{G}_{a,b}(\partial_\xi\phi,\psi)-Q_{a,b}\psi\partial_\xi\phi) + itq\xi E^q(\mathcal{G}_{a,b}(\phi,\partial_\xi\psi)-Q_{a,b}\phi\partial_\xi\psi)\\
&+ 2iqt^{\frac{3}{2}}\xi E^q(\widetilde{\mathcal{G}_{a,b}}(\phi,\psi)-\widetilde{Q}_{a,b}\phi\psi),
\end{aligned}
$$

where $\widetilde{\widetilde{\mathcal{G}_{a,b}}}$ and $\widetilde{\widetilde{Q}}_{a,b}$ are defined by the kernel

$$
\widetilde{\widetilde{h}}(t,\eta,\zeta) = -i\sqrt{t}(a\eta+b\zeta)\widetilde{h}(t,\eta,\zeta).
$$

Then by Lemma 3.5 we find the estimates

$$
\begin{aligned}
&\|\partial_\xi^2(E^q(\mathcal{G}_{a,b}(\phi,\psi)-Q_{a,b}\phi\psi))\|_{\mathbf{L}^2}\\
&\le C\|\mathcal{G}_{a,b}(\partial_\xi^2\phi,\psi)-Q_{a,b}\psi\partial_\xi^2\phi\|_{\mathbf{L}^2}\\
&\quad + C\|\mathcal{G}_{a,b}(\partial_\xi\phi,\partial_\xi\psi)-Q_{a,b}\partial_\xi\phi\partial_\xi\psi\|_{\mathbf{L}^2} + C\|\mathcal{G}_{a,b}(\phi,\partial_\xi^2\psi)-Q_{a,b}\phi\partial_\xi^2\psi\|_{\mathbf{L}^2}\\
&\quad + C\sqrt{t}\|\widetilde{\mathcal{G}_{a,b}}(\partial_\xi\phi,\psi)-\widetilde{Q}_{a,b}\psi\partial_\xi\phi\|_{\mathbf{L}^2} + C\sqrt{t}\|\widetilde{\mathcal{G}_{a,b}}(\phi,\partial_\xi\psi)-\widetilde{Q}_{a,b}\phi\partial_\xi\psi\|_{\mathbf{L}^2}\\
&\quad + Ct\|B_{-\frac{1}{2}-\epsilon}\|_{\mathbf{L}^2}\|B_{\frac{1}{2}+\epsilon}(\widetilde{\widetilde{\mathcal{G}_{a,b}}}(\phi,\psi)-\widetilde{\widetilde{Q}}_{a,b}\phi\psi)\|_{\mathbf{L}^\infty}\\
&\quad + Cq\sqrt{t}\|B_1(\mathcal{G}_{a,b}(\partial_\xi\phi,\psi)-Q_{a,b}\psi\partial_\xi\phi)\|_{\mathbf{L}^2}\\
&\quad + Cq\sqrt{t}\|B_1(\mathcal{G}_{a,b}(\phi,\partial_\xi\psi)-Q_{a,b}\phi\partial_\xi\psi)\|_{\mathbf{L}^2}\\
&\quad + Cqt\|B_{-\frac{1}{2}-\epsilon}\|_{\mathbf{L}^2}\|B_{\frac{3}{2}+\epsilon}(\widetilde{\mathcal{G}_{a,b}}(\phi,\psi)-\widetilde{Q}_{a,b}\phi\psi)\|_{\mathbf{L}^\infty}\\
&\quad + Cqt\|B_{-\frac{1}{2}-\epsilon}\|_{\mathbf{L}^2}\|B_{\frac{5}{2}+\epsilon}(\mathcal{G}_{a,b}(\phi,\psi)-Q_{a,b}\phi\psi)\|_{\mathbf{L}^\infty}\\
&\le Ct^{3/4}\|\phi\|_{\mathbf{X}_T^{\alpha,\lambda}}\|\psi\|_{\mathbf{X}_T^{\beta,\delta}}
\end{aligned}
$$

with the additional condition

$$
\min\big(\alpha+\delta, \beta+\lambda, \frac{1}{2}+\lambda+\delta\big) > 1
$$

for the case of $q \neq 0$. Lemma 4.1 is proved. □

Now let us find the inverse transformation $\mathcal{I}^{-1}$. We consider the equation

$$
\phi = \mathcal{I}(v). \tag{4.1}
$$

We look for the solution of (4.1) in the form $v = \phi + \psi_1 + \overline{E}^3 \psi_2$ and substitute it into (4.1), then we find

$$\psi_1 + \overline{E}^3 \psi_2 = \overline{E}^3 \mathcal{G}_{-1,-1}(\overline{\phi} + \overline{\psi_1}, \overline{\phi} + \overline{\psi_1}) + 2\mathcal{G}_{-1,2}(\overline{\phi} + \overline{\psi_1}, \overline{\psi_2}) + E^3 \mathcal{G}_{2,2}(\overline{\psi_2}, \overline{\psi_2}). \quad (4.2)$$

Comparing in (4.2) the terms with the same oscillating exponents like $\overline{E}^3$ we find a system of equations

$$\begin{gathered} \psi_1 = \mathcal{G}_{-1,2}(\overline{\phi} + \overline{\psi_1}, \overline{\psi_2}) + E^3 \mathcal{G}_{2,2}(\overline{\psi_2}, \overline{\psi_2}), \\ \psi_2 = \mathcal{G}_{-1,-1}(\overline{\phi} + \overline{\psi_1}, \overline{\phi} + \overline{\psi_1}). \end{gathered} \quad (4.3)$$

In the next lemma we solve this system in the space

$$\mathbf{Z}_\varepsilon = \left\{ (\psi_1, \psi_2) \in (\mathbf{C}([1,T]; \mathbf{C}^2(\mathbf{R})))^2 : \|(\psi_1, \psi_2)\|_{\mathbf{Z}} \le C\varepsilon^2 \right\}$$

with the norm

$$\|(\psi_1, \psi_2)\|_{\mathbf{Z}} \equiv \|\psi_1\|_{\mathbf{X}_T^{2,1+\frac{\gamma}{3}}} + \|\psi_2\|_{\mathbf{X}_T^{1,1+\frac{\gamma}{2}}},$$

where $\gamma \in (0, \frac{1}{2})$.

Lemma 4.2. *Suppose that $\phi \in \mathbf{C}([1,T]; \mathbf{C}^2(\mathbb{R}))$ and the estimate is true*

$$\|\phi\|_{\mathbf{X}_T^{0,\gamma}} \le \varepsilon,$$

where $\varepsilon > 0$ is sufficiently small. Then there exists unique solutions $\psi_1, \psi_2 \in \mathbf{C}([1,T]; \mathbf{C}^2(\mathbb{R}))$ of a system (4.3) such that $\|(\psi_1, \psi_2)\|_{\mathbf{Z}} \le C\varepsilon^2$.

Proof. We solve equations (4.3) by the contraction mapping principle in the set $\mathbf{Z}_\varepsilon$. Define the transformation $\mathcal{M}(\psi_1, \psi_2) = (\mathcal{M}_1, \mathcal{M}_2)$, where

$$\begin{gathered} \mathcal{M}_1 = \mathcal{G}_{-1,2}(\overline{\phi} + \overline{\psi_1}, \overline{\psi_2}) + E^3 \mathcal{G}_{2,2}(\overline{\psi_2}, \overline{\psi_2}), \\ \mathcal{M}_2 = \mathcal{G}_{-1,-1}(\overline{\phi} + \overline{\psi_1}, \overline{\phi} + \overline{\psi_1}) \end{gathered} \quad (4.4)$$

for $(\psi_1, \psi_2) \in \mathbf{Z}_\varepsilon$. Applying Lemma 4.1, in view of the fact that $\|\phi\|_{\mathbf{X}_T^{0,\gamma}} \le \varepsilon$ and $\|(\psi_1, \psi_2)\|_{\mathbf{Z}} \le C\varepsilon^2$, we obtain, by (4.4),

$$\begin{aligned} \|\mathcal{M}_1\|_{\mathbf{X}_T^{2,1+\frac{\gamma}{3}}} &\le \|\mathcal{G}_{-1,2}(\overline{\phi} + \overline{\psi_1}, \overline{\psi_2}) - Q_{-1,2}(\overline{\phi} + \overline{\psi_1})\overline{\psi_2}\|_{\mathbf{X}_T^{2,1+\frac{\gamma}{3}}} \\ &\quad + \|E^3(\mathcal{G}_{2,2}(\overline{\psi_2}, \overline{\psi_2}) - Q_{2,2}\overline{\psi_2}^2)\|_{\mathbf{X}_T^{2,1+\frac{\gamma}{3}}} \\ &\quad + \|Q_{-1,-1}(\overline{\phi} + \overline{\psi_1})\overline{\psi_2}\|_{\mathbf{X}_T^{2,1+\frac{\gamma}{3}}} + \|E^3 Q_{2,2}\overline{\psi_2}^2\|_{\mathbf{X}_T^{2,1+\frac{\gamma}{3}}} \\ &\le C\|\psi_2\|_{\mathbf{X}_T^{1,1+\frac{\gamma}{2}}} (\|\phi\|_{\mathbf{X}_T^{0,\gamma}} + \|\psi_1\|_{\mathbf{X}_T^{0,\gamma}} + \|\psi_2\|_{\mathbf{X}_T^{1,1+\frac{\gamma}{2}}}) \le C\varepsilon^3. \end{aligned}$$

In the same manner we have

$$\begin{aligned} \|\mathcal{M}_2\|_{\mathbf{X}_T^{1,1+\frac{\gamma}{2}}} &\le \|\mathcal{G}_{-1,-1}(\overline{\phi} + \overline{\psi_1}, \overline{\phi} + \overline{\psi_1}) - Q_{-1,-1}(\overline{\phi} + \overline{\psi_1})^2\|_{\mathbf{X}_T^{1,1+\frac{\gamma}{2}}} \\ &\quad + \|Q_{-1,-1}(\overline{\phi} + \overline{\psi_1})^2\|_{\mathbf{X}_T^{1,1+\frac{\gamma}{2}}} \\ &\le C(\|\phi\|_{\mathbf{X}_T^{0,\gamma}} + \|\psi_1\|_{\mathbf{X}_T^{0,\gamma}})^2 \le C\varepsilon^2. \end{aligned}$$

Thus the mapping $\mathcal{M}(\psi_1, \psi_2)$ transforms the set $\mathbf{Z}_\varepsilon$ into itself. In the same manner we find

$$\|\mathcal{M}(\psi_1, \psi_2) - \mathcal{M}(\widetilde{\psi_1}, \widetilde{\psi_2})\|_{\mathbf{Z}} \le \frac{1}{2}\|(\psi_1, \psi_2) - (\widetilde{\psi_1}, \widetilde{\psi_2})\|_{\mathbf{Z}}.$$

Therefore $(\mathcal{M}_1, \mathcal{M}_2)$ is a contraction mapping in $\mathbf{Z}_\varepsilon$. Hence there exist a unique solution $(\psi_1, \psi_2) \in \mathbf{Z}_\varepsilon$ of a system of integral equations (4.3). The proof is complete. □

5. Estimates for derivatives

We now make a change $\phi = \mathcal{I}(v)$ in equation (2.5)

$$\begin{gathered} \mathcal{L}\phi = t^{-1}\mathcal{P}, \\ \phi(1,\xi) = \phi_0(\xi), \end{gathered} \tag{5.1}$$

where $\mathcal{L} = i\partial_t + \frac{1}{2t^2}\partial_\xi^2$, $E = e^{\frac{i}{2}t\xi^2}$,

$$\mathcal{P} = \overline{E}^3 \widetilde{\mathcal{G}}_{-1,-1}(\overline{v}, \overline{v}) - \mathcal{H}_{-1,2}(\overline{v}, v^2)$$

with $v = \mathcal{I}^{-1}(\phi)$ and the operator

$$\begin{aligned} \mathcal{H}_{a,b}(\phi,\psi) = t^{-1/2}\overline{E}^{a+b} \int_{\mathbb{R}^2} h(t,\eta,\zeta) e^{\frac{a}{2}it(\xi-\eta)^2} \phi(t,\xi-\eta) \\ \times \partial_\zeta \big(e^{\frac{b}{2}it(\xi-\zeta)^2} \psi(t,\xi-\zeta)\big) d\eta d\zeta. \end{aligned}$$

Define the norms

$$\begin{gathered} \|\phi\|_{\mathbf{V}_T^\alpha} \equiv \sup_{1\le t\le T} \|B_\alpha \phi(t)\|_{\mathbf{L}^\infty}, \\ \|\phi\|_{\mathbf{W}_T^\gamma} \equiv \sup_{1\le t\le T} (t^{-\frac14}\|B_\gamma\phi(t)\|_{\mathbf{L}^2} + t^{-\frac34}\|\partial_\xi\phi(t)\|_{\mathbf{L}^2}). \end{gathered}$$

First we state the local existence result for equation (5.1). Denote $\mathbf{Y} = \{\phi \in \mathbf{L}^\infty, \phi' \in \mathbf{H}^{1,1}\}$.

Theorem 5.1. *Assume that the initial data $\phi_0 \in \mathbf{Y}$. Then for some time $T > 1$ there exists a unique solution $\phi \in \mathbf{C}([1,T];\mathbf{Y})$ of the Cauchy problem* (5.1).

In the next lemma we give a representation for the derivatives of the operator $\mathcal{H}_{a,b}$.

Lemma 5.2. *Let condition* (3.2) *be fulfilled. Then the estimate is true*

$$\|\partial_\xi(E^q\mathcal{H}_{a,b}(\phi,\psi)) - iqbt^{\frac32}\xi^2E^qQ_{a,b}\phi\psi\|_{\mathbf{W}_T^\rho} \le C\|\phi\|_{\mathbf{X}_T^{\alpha,\lambda}}\|\psi\|_{\mathbf{X}_T^{\beta,\delta}},$$

where

$$\rho = \min\big(\alpha+\delta, \beta+\lambda, 1+\alpha, 1+\beta, \frac12+\alpha+\beta-\epsilon, \lambda+\delta-\epsilon, 1+\delta-\epsilon, 1+\lambda-\epsilon\big)$$

$\alpha,\beta,\lambda,\delta \in \mathbb{R}$, $\epsilon > 0$ *is small. Also in the case of $q \neq 0$ we assume that $\alpha, \beta \in \mathbb{R}$, $\lambda, \delta > 0$ are such that $\rho \ge 1$.*

Proof. We note that

$$\mathcal{H}_{a,b}(\phi,\psi) = ib\sqrt{t}\xi\mathcal{G}_{a,b}(\phi,\psi) + t^{-1/2}\mathcal{G}_{a,b}(\phi,\partial_\xi\psi) + \mathcal{G}_{a,b}^{(1)}(\phi,\psi),$$

where $\mathcal{G}_{a,b}^{(1)}$ and $Q^{(1)}$ are defined by the kernel $h^{(1)}(t,\eta,\zeta) = \sqrt{t}b\zeta h(t,\eta,\zeta)$. Then we obtain by Lemma 3.5

$$\begin{aligned} &\|B_\rho\xi tE^q(\mathcal{H}_{a,b}(\phi,\psi) - ib\sqrt{t}\xi Q_{a,b}\phi\psi)\|_{\mathbf{L}^2} \\ &\le \sqrt{t}\|B_{\rho+1}(\mathcal{H}_{a,b}(\phi,\psi) - ib\sqrt{t}\xi Q_{a,b}\phi\psi)\|_{\mathbf{L}^2} \\ &\le C\sqrt{t}\|B_{2+\rho}(\mathcal{G}_{a,b}(\phi,\psi) - Q_{a,b}\phi\psi)\|_{\mathbf{L}^2} \end{aligned}$$

$$
\begin{aligned}
&+ C\|B_{\rho+1}\mathcal{G}_{a,b}(\phi,\partial_\xi\psi)\|_{\mathbf{L}^2} + C\sqrt{t}\|B_{\rho+1}\mathcal{G}^{(1)}_{a,b}(\phi,\psi)\|_{\mathbf{L}^2}\\
&\le Ct^{1/4}\|\phi\|_{\mathbf{X}_T^{\alpha,\lambda}}\|\psi\|_{\mathbf{X}_T^{\beta,\delta}}.
\end{aligned}
$$

We now estimate by Lemma 3.5 the derivative,

$$
\begin{aligned}
&\|B_\rho\partial_\xi(\mathcal{H}_{a,b}(\phi,\psi) - ib\sqrt{t}Q_{a,b}\phi\psi)\|_{\mathbf{L}^2}\\
&\le C\sqrt{t}\|B_\rho\partial_\xi\xi(\mathcal{G}_{a,b}(\phi,\psi) - Q_{a,b}\phi\psi)\|_{\mathbf{L}^2}\\
&\quad + Ct^{-1/2}\|B_\rho\partial_\xi\mathcal{G}_{a,b}(\phi,\partial_\xi\psi)\|_{\mathbf{L}^2} + C\|B_\sigma\partial_\xi\mathcal{G}^{(1)}_{a,b}(\phi,\psi)\|_{\mathbf{L}^2}\\
&\le Ct^{1/4}\|\phi\|_{\mathbf{X}_T^{\alpha,\lambda}}\|\psi\|_{\mathbf{X}_T^{\beta,\delta}}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\|\partial_\xi\xi tE^q(\mathcal{H}_{a,b}(\phi,\psi) - ib\sqrt{t}\xi Q_{a,b}\phi\psi)\|_{\mathbf{L}^2}\\
&\le Ct\|\mathcal{H}_{a,b}(\phi,\psi) - ib\sqrt{t}\xi Q_{a,b}\phi\psi\|_{\mathbf{L}^2}\\
&\quad + Cqt\|B_2(\mathcal{H}_{a,b}(\phi,\psi) - ib\sqrt{t}\xi Q_{a,b}\phi\psi)\|_{\mathbf{L}^2}\\
&\quad + C\sqrt{t}\|B_1\partial_\xi(\mathcal{H}_{a,b}(\phi,\psi) - ib\sqrt{t}\xi Q_{a,b}\phi\psi)\|_{\mathbf{L}^2}\\
&\le Ct^{\frac{3}{4}}\|\phi\|_{\mathbf{X}_T^{\alpha,\lambda}}\|\psi\|_{\mathbf{X}_T^{\beta,\delta}}
\end{aligned}
$$

if $\rho \ge 1$ for the case $q \neq 0$. Also by Lemma 3.5 we have

$$
\begin{aligned}
&\|B_\rho E^q\partial_\xi\mathcal{H}_{a,b}(\phi,\psi)\|_{\mathbf{L}^2}\\
&\le C\|B_\rho\partial_\xi(\mathcal{H}_{a,b}(\phi,\psi) - ib\sqrt{t}\xi Q_{a,b}\phi\psi)\|_{\mathbf{L}^2} + C\sqrt{t}\|B_\rho\partial_\xi(\xi Q_{a,b}\phi\psi)\|_{\mathbf{L}^2}\\
&\le Ct^{1/4}\|\phi\|_{\mathbf{X}_T^{\alpha,\lambda}}\|\psi\|_{\mathbf{X}_T^{\beta,\delta}}
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\partial_\xi E^q\partial_\xi\mathcal{H}_{a,b}(\phi,\psi)\|_{\mathbf{L}^2}\\
&\le \|\partial_\xi^2\mathcal{H}_{a,b}(\phi,\psi)\|_{\mathbf{L}^2} + Cq\sqrt{t}\|B_1E^q\partial_\xi\mathcal{H}_{a,b}(\phi,\psi)\|_{\mathbf{L}^2}\\
&\le Ct^{3/4}\|\phi\|_{\mathbf{X}_T^{\alpha,\lambda}}\|\psi\|_{\mathbf{X}_T^{\beta,\delta}}
\end{aligned}
$$

if $\rho \ge 1$ in the case of $q \neq 0$. The lemma is proved. □

We now substitute the inverse transformation $v = \mathcal{I}^{-1}(\phi) = \phi_1 + \overline{E}^3\psi_2$ with $\phi_1 = \phi + \psi_1$ into the operator $\mathcal{P}$ to get

$$
\begin{aligned}
\mathcal{P} &= \overline{E}^3\mathcal{H}_{2,2}(\overline{\psi_2},\phi_1^2) + 2E^3\mathcal{H}_{-1,-1}(\overline{\phi_1},\phi_1\psi_2) + E^6\mathcal{H}_{-1,-4}(\overline{\phi_1},\psi_2^2)\\
&\quad + \overline{E}^3\widetilde{\mathcal{G}}_{-1,-1}(\overline{\phi_1},\overline{\phi_1}) + \mathcal{H}_{-1,2}(\overline{\phi_1},\phi_1^2) + 2\widetilde{\mathcal{G}}_{-1,2}(\overline{\phi_1},\overline{\psi_2})\\
&\quad + E^3\widetilde{\mathcal{G}}_{2,2}(\overline{\psi_2},\overline{\psi_2}) + 2\mathcal{H}_{2,-1}(\overline{\psi_2},\phi_1\psi_2) + E^3\mathcal{H}_{2,-4}(\overline{\psi_2},\psi_2^2).
\end{aligned}
$$

Denote $\varepsilon = \|\phi\|_{\mathbf{X}_T^{0,\gamma}}$. Since $\psi_2 = \mathcal{G}_{-1,-1}(\overline{\phi} + \overline{\psi_1}, \overline{\phi} + \overline{\psi_1})$ then by Lemma 3.5 we have

$$
\|t\xi(\psi_2 - Q_{-1,-1}\overline{\phi}^2)\|_{\mathbf{W}_T^{1+\gamma}} \le C\varepsilon^2
$$

Then by virtue of Lemma 5.2 with $\rho = 1$ we have

$$
\begin{aligned}
&\|\partial_\xi\overline{E}^3\mathcal{H}_{2,2}(\overline{\psi_2},\phi_1^2) + 6it^{\frac{3}{2}}\xi^2\overline{E}^3Q_{2,2}\overline{Q_{-1,-1}}\phi^4\|_{\mathbf{W}_T^1}\\
&\le \|\partial_\xi(\overline{E}^3\mathcal{H}_{2,2}(\overline{\psi_2},\phi_1^2)) + 6it^{\frac{3}{2}}\xi^2\overline{E}^3Q_{2,2}\overline{\psi_2}\phi_1^2\|_{\mathbf{W}_T^1}\\
&\quad + C\|t^{\frac{3}{2}}\xi^2\overline{E}^3Q_{2,2}\overline{\psi_2}\phi_1^2 - t^{\frac{3}{2}}\xi^2\overline{E}^3Q_{2,2}\overline{Q_{-1,-1}}\phi^4\|_{\mathbf{W}_T^1}
\end{aligned}
$$

$$\le C\|\psi_2\|_{\mathbf{X}_T^{1,1+\frac{\gamma}{2}}}\|\phi_1^2\|_{\mathbf{X}_T^{0,\gamma}} \le C\varepsilon^2.$$

In the same manner

$$\|\partial_\xi E^3\mathcal{H}_{-1,-1}(\overline{\phi_1},\phi_1\psi_2) + 3it^{\frac{3}{2}}\xi^2E^3Q^2_{-1,-1}\phi\overline{\phi}^3\|_{\mathbf{W}_T^1} \le C\varepsilon^2,$$
$$\|\partial_\xi E^6\mathcal{H}_{-1,-4}(\overline{\phi_1},\psi_2^2) + 24it^{\frac{3}{2}}\xi^2E^6Q^2_{-1,-1}Q_{-1,-4}\overline{\phi}^5\|_{\mathbf{W}_T^1} \le C\varepsilon^2.$$

All the other terms in the derivative $\partial_\xi\mathcal{P}$ can be estimated in the norm $\mathbf{W}_T^1$, the worst term is $\mathcal{H}_{-1,2}(\overline{\phi_1},\phi_1^2)$ which yields the restriction $\gamma < \frac{1}{2} - \epsilon$. Therefore we can represent $\partial_\xi\mathcal{P}$ in the form

$$\partial_\xi\mathcal{P} = t^{1/2}\sum_{j=1}^{3}E^{\omega_j}\Omega_j\mathcal{N}_j + \mathcal{R}, \tag{5.2}$$

where $\mathcal{N}_1 = \phi^4$, $\mathcal{N}_2 = \phi\overline{\phi}^3$, $\mathcal{N}_3 = \overline{\phi}^5$, $\omega_1 = -3$, $\omega_2 = 3$, $\omega_3 = 6$,

$$\Omega_1 = -6it\xi^2\overline{Q_{-1,-1}}Q_{2,2}, \quad \Omega_2 = -3it\xi^2Q^2_{-1,-1}, \quad \Omega_3 = -24it\xi^2Q^2_{-1,-1}Q_{-1,-4}$$

with the estimate of the remainder $\|\mathcal{R}\|_{\mathbf{W}_T^\gamma} \le C\varepsilon^2$ and

$$t^{-\frac{k}{2}+l}|\partial_\xi^k\partial_t^l\Omega_j(t,\xi)| \le C\langle\xi\sqrt{t}\rangle^{-l-k}$$

for all $t \ge 1$, $\xi \in \mathbb{R}$, $j = 1,2,3$, $k,l = 0,1,2$.

Lemma 5.3. *Let the initial data $\phi_0 \in \mathbf{Y}$ and $\|\phi_0\|_{\mathbf{Y}} \le \varepsilon$, where $\varepsilon > 0$ is sufficiently small. Assume that representation (5.2) is valid with the estimate of the remainder*

$$\|\mathcal{R}\|_{\mathbf{W}_T^\gamma} \le C\varepsilon^2.$$

Suppose that

$$\|\phi\|_{\mathbf{V}_T^0} \le \varepsilon. \tag{5.3}$$

Then the solutions $\phi \in \mathbf{C}([1,T];\mathbf{Y})$ of (5.1) satisfy the estimate

$$\|\partial_\xi\phi\|_{\mathbf{W}_T^\gamma} < 10\varepsilon. \tag{5.4}$$

Proof. We prove estimate (5.4) by contradiction. By the continuity of ϕ we can find a maximal time $\widetilde{T} \in (1,T]$ such that

$$\|\partial_\xi\phi\|_{\mathbf{W}_{\widetilde{T}}^\gamma} \le 10\varepsilon. \tag{5.5}$$

Thus we have $\|\phi\|_{\mathbf{X}_{\widetilde{T}}^{0,\gamma}} \le 10\varepsilon$. By a direct calculation we have

$$\mathcal{L}(E^{\omega_j}\chi_j) = t^{-1}E^{\omega_j}\big(\frac{i\omega_j}{2A_j}\chi_j + i\omega_j\xi\partial_\xi\chi_j + t\mathcal{L}\chi_j\big) \tag{5.6}$$

with $A_j = (1+(1+\omega_j)it\xi^2)^{-1}$. This identity is useful in the case of $\omega \ne -1$. Then we get, from (5.1) and (5.2),

$$\begin{aligned}&\mathcal{L}\big(\phi_\xi + \sum_{j=1}^{3}E^{\omega_j}\chi_j\big)\\ &= t^{-1}\sum_{j=1}^{3}E^{\omega_j}\big(t^{1/2}\Omega_j\mathcal{N}_j + \frac{i\omega_j}{2A_j}\chi_j\big) + t^{-1}\sum_{j=1}^{3}E^{\omega_j}(i\omega_j\xi\partial_\xi\chi_j + t\mathcal{L}\chi_j) + t^{-1}\mathcal{R}.\end{aligned} \tag{5.7}$$

To eliminate the first summand in the right-hand side of (5.7) we choose $\chi_j = \frac{2i}{\omega_j} t^{1/2}\Omega_j A_j \mathcal{N}_j$ and denote $\Phi = \phi_\xi + \sum_{j=1}^{3} E^{\omega_j}\chi_j$. By the identities

$$\mathcal{L}(uv) = v\mathcal{L}u + \frac{1}{t^2}u_\xi v_\xi + u\mathcal{L}v,$$

$$\mathcal{L}\overline{\phi} = -\overline{\mathcal{L}\phi} + \frac{1}{t^2}\overline{\phi}_{\xi\xi},$$

$$\mathcal{L}\mathcal{N}_j = \mathcal{N}_{j\phi}\mathcal{L}\phi - \mathcal{N}_{j\overline{\phi}}\overline{\mathcal{L}\phi} + \frac{1}{2t^2}(\mathcal{N}_{j\phi\phi}\phi_\xi^2 + 2\mathcal{N}_{j\phi\overline{\phi}}|\phi_\xi|^2 + \mathcal{N}_{j\overline{\phi}\overline{\phi}}(\overline{\phi_\xi})^2) + \frac{1}{t^2}\mathcal{N}_{j\overline{\phi}}\overline{\phi_{\xi\xi}},$$

in view of (5.1), we obtain

$$\begin{aligned}
t\mathcal{L}\chi_j &= \frac{2i}{\omega_j}\mathcal{N}_j t\mathcal{L}(t^{1/2}\Omega_j A_j) + t^{-1/2}\partial_\xi(\Omega_j A_j)\partial_\xi\mathcal{N}_j \\
&\quad + \frac{i}{\omega_j}t^{-1/2}\Omega_j A_j(\mathcal{N}_{j\phi\phi}\phi_\xi^2 + 2\mathcal{N}_{j\phi\overline{\phi}}|\phi_\xi|^2 + \mathcal{N}_{j\overline{\phi}\overline{\phi}}(\overline{\phi_\xi})^2) \\
&\quad + \frac{2i}{\omega_j}t^{1/2}\Omega_j A_j(\mathcal{N}_{j\phi}\mathcal{P} - \mathcal{N}_{j\overline{\phi}}\overline{\mathcal{P}}) \\
&\quad + \frac{2i}{\omega_j}t^{-1/2}\Omega_j A_j\mathcal{N}_{j\overline{\phi}}\Big(\overline{\Phi}_\xi + 2it^{\frac12}\sum_{j=1}^{3}\frac{1}{\omega_j}\partial_\xi(\overline{E^{\omega_j}}\,\overline{\Omega_j A_j\mathcal{N}_j})\Big)
\end{aligned}$$

Therefore, from (5.7),

$$\mathcal{L}\Phi = 2it^{-3/2}\sum_{j=1}^{3}\frac{1}{\omega_j}E^{\omega_j}\Omega_j A_j\mathcal{N}_{j\overline{\phi}}\overline{\Phi_\xi} + t^{-1}\mathcal{R}_1, \tag{5.8}$$

where in view of (5.1) we have

$$\begin{aligned}
\mathcal{R}_1 &= \mathcal{R} - 2\sum_{j=1}^{3}E^{\omega_j}\big(\xi\partial_\xi(\Omega_j A_j\mathcal{N}_j) - \frac{i}{\omega_j}\mathcal{N}_j t\mathcal{L}(t^{1/2}\Omega_j A_j)\big) \\
&\quad + \sum_{j=1}^{3}E^{\omega_j}t^{-1/2}\partial_\xi(\Omega_j A_j)\partial_\xi\mathcal{N}_j \\
&\quad + \sum_{j=1}^{3}E^{\omega_j}\frac{i}{\omega_j}t^{-1/2}\Omega_j A_j(\mathcal{N}_{j\phi\phi}\phi_\xi^2 + 2\mathcal{N}_{j\phi\overline{\phi}}|\phi_\xi|^2 + \mathcal{N}_{j\overline{\phi}\overline{\phi}}\overline{\phi_\xi}^2) \\
&\quad + \sum_{j=1}^{3}E^{\omega_j}\frac{2i}{\omega_j}t^{1/2}\Omega_j A_j\mathcal{N}_j(\mathcal{N}_{j\phi}\mathcal{P} - \mathcal{N}_{j\overline{\phi}}\overline{\mathcal{P}}) \\
&\quad - \sum_{j,l=1}^{3}E^{\omega_j}\frac{4}{\omega_j\omega_l}\Omega_j A_j\mathcal{N}_{j\overline{\phi}}\partial_\xi(\overline{E^{\omega_l}}\,\overline{\Omega_l A_l\mathcal{N}_l}).
\end{aligned}$$

Since

$$|\Omega_j A_j| + |\xi\partial_\xi(\Omega_j A_j)| + |t\mathcal{L}(\Omega_j A_j)| \le CB_{-2}, t^{-1/2}|\partial_\xi(\Omega_j A_j)| \le CB_{-3},$$

it follows that $\mathcal{R}_1$ satisfies

$$\|\mathcal{R}_1\|_{\mathbf{W}_{\tilde{T}}^{\gamma}} \le C\varepsilon^2.$$

We multiply (5.8) by $(M + t\xi^2)^{\gamma/2}$ and use the commutator

$$\mathcal{L}((M+t\xi^2)^{\gamma/2}\Phi) = (M+t\xi^2)^{\gamma/2}\mathcal{L}\Phi + \Phi\mathcal{L}(M+t\xi^2)^{\gamma/2} + 2\gamma t^{-1}\xi(M+t\xi^2)^{\frac{\gamma}{2}-1}\partial_\xi\Phi$$

then we get $\mathcal{L}((M+t\xi^2)^{\gamma/2}\Phi) = \mathcal{R}_2$, where

$$\begin{aligned}\mathcal{R}_2 &= \Phi\mathcal{L}(M+t\xi^2)^{\gamma/2} + 2\gamma t^{-1}\xi(M+t\xi^2)^{\frac{\gamma}{2}-1}\Phi_\xi \\ &\quad + 2it^{-3/2}\sum_{j=1}^{3}\frac{1}{\omega_j}E^{\omega_j}B_\gamma\Omega_j A_j\mathcal{N}_{j\overline{\phi}}\overline{\Phi_\xi} + t^{-1}(M+t\xi^2)^{\gamma/2}\mathcal{R}_1.\end{aligned}$$

Since

$$|\mathcal{L}(M+t\xi^2)^{\gamma/2}| \le \frac{\gamma}{2t}\big(1+\frac{1}{M}\big)(M+t\xi^2)^{\gamma/2},$$

by (5.5) choosing M sufficiently large we have

$$\|\mathcal{R}_2\|_{\mathbf{L}^2} \le \frac{\gamma}{2t}\|(M+t\xi^2)^{\gamma/2}\Phi\|_{\mathbf{L}^2} + C\varepsilon^{\frac{5}{4}}t^{-\frac{3}{4}}.$$

Then we apply the energy method to estimate the $\mathbf{L}^2$-norm of $B_\gamma\Phi$,

$$\frac{d}{dt}\|(M+t\xi^2)^{\gamma/2}\Phi\|_{\mathbf{L}^2} \le \frac{\gamma}{2t}\|(M+t\xi^2)^{\gamma/2}\Phi\|_{\mathbf{L}^2} + C\varepsilon^{\frac{5}{4}}t^{-\frac{3}{4}}.$$

Hence integration with respect to time yields

$$\|(M+t\xi^2)^{\gamma/2}\Phi\|_{\mathbf{L}^2} \le \varepsilon + C\varepsilon^{\frac{5}{4}}t^{1/4}$$

for all $t \in [1,\widetilde{T}]$ if $0<\gamma<\frac{1}{2}$. Therefore, $\|B_\gamma\phi_\xi\|_{\mathbf{L}^2} \le \varepsilon + C\varepsilon^{\frac{5}{4}}t^{1/4}$ for all $t\in[1,\widetilde{T}]$.

We now differentiate (5.8) with respect to ξ to get

$$\mathcal{L}\Phi_\xi = 2it^{-3/2}\sum_{j=1}^{3}\frac{1}{\omega_j}E^{\omega_j}\Omega_j A_j\mathcal{N}_{j\overline{\phi}}\overline{\Phi_{\xi\xi}} + \mathcal{R}_3, \tag{5.9}$$

where

$$\mathcal{R}_3 = 2it^{-3/2}\sum_{j=1}^{3}\frac{1}{\omega_j}\overline{\Phi_\xi}\partial_\xi(E^{\omega_j}\Omega_j A_j\mathcal{N}_{j\overline{\phi}}) + t^{-1}\partial_\xi\mathcal{R}_1.$$

By (5.5) we see that $\|\mathcal{R}_3\|_{\mathbf{L}^2} \le C\varepsilon^2 t^{-\frac{1}{4}}$. Then to estimate the $\mathbf{L}^2$-norm of Φ_ξ we apply the energy method to (5.9)

$$\frac{d}{dt}\|\Phi_\xi\|_{\mathbf{L}^2}^2 \le Ct^{-3/2}\sum_{j=1}^{3}\Big|\int_{\mathbb{R}}\Omega_j A_j E^{\omega_j}\mathcal{N}_{j\overline{\phi}}\overline{\Phi_{\xi\xi}\Phi_\xi}d\xi\Big| + C\varepsilon^3 t^{\frac{1}{2}}.$$

Hence integrating by parts with respect to ξ we avoid the derivative loss and obtain $\|\Phi_\xi\|_{\mathbf{L}^2} \le \varepsilon + C\varepsilon^{\frac{3}{2}}t^{\frac{3}{4}}$ for all $t\in[1,\widetilde{T}]$. Therefore

$$\|\partial_\xi^2\phi\|_{\mathbf{L}^2} < \varepsilon + C\varepsilon^{\frac{3}{2}}t^{\frac{3}{4}}$$

for all $t\in[1,\widetilde{T}]$. Thus we have $\|\partial_\xi\phi\|_{\mathbf{W}_T^\gamma} < 10\varepsilon$. The contradiction proves estimate (5.4). Lemma 5.3 is proved. □

6. Estimates in the uniform norm

We now estimate ϕ in the norm $\|\phi\|_{\mathbf{V}_T^0} = \sup_{1\le t\le T}\|\phi(t)\|_{\mathbf{L}^\infty}$.

Lemma 6.1. *Let the initial data $\phi_0 \in \mathbf{Y}$ and $\|\phi_0\|_{\mathbf{Y}} \le \varepsilon$, where $\varepsilon > 0$ is sufficiently small. Suppose that*

$$\|\partial_\xi\phi\|_{\mathbf{W}_T^\gamma} < 10\varepsilon. \tag{6.1}$$

Then the solutions $\phi \in \mathbf{C}([1,T];\mathbf{Y})$ of (5.1) satisfy the estimate

$$\|\phi\|_{\mathbf{V}_T^0} < 10\varepsilon. \tag{6.2}$$

Proof. We prove estimate (6.2) by the contradiction. By the continuity of ϕ we can find a maximal time $\widetilde{T} \in (1, T]$ such that

$$\|\phi\|_{\mathbf{V}^0_{\widetilde{T}}} \le 10\varepsilon. \tag{6.3}$$

Now (6.3) along with (6.1) imply that

$$\|\phi\|_{\mathbf{X}^{0,\gamma}_{\widetilde{T}}} \le C\varepsilon. \tag{6.4}$$

Denote $w(t) = \mathcal{V}(-t)\phi(t)$, where

$$\mathcal{V}(-t) = \mathcal{F}\overline{M}\mathcal{F}^{-1} = \sqrt{t}\int_{\mathbb{R}} d\eta e^{-\frac{it}{2}(\xi-\eta)^2}.$$

Applying operator $\mathcal{V}(-t)$ to equation (5.1), we have

$$\begin{gathered} iw_t = t^{-1}\mathcal{V}(-t)\mathcal{P}, \\ w(1) = \mathcal{V}(-1)\phi_0, \end{gathered} \tag{6.5}$$

where $\mathcal{P} = \overline{E}^3 \widetilde{\mathcal{G}}_{-1,-1}(\overline{v}, \overline{v}) - \mathcal{H}_{-1,2}(\overline{v}, v^2)$, and $E = e^{\frac{i}{2}t\xi^2}$. Note that $\mathcal{P}$ has the form

$$\mathcal{P} = \overline{E}\partial_\xi(\mathbb{G}_0(\overline{v}, \overline{v}) + \mathbb{G}_1(\overline{v}, v^2)),$$

where

$$\begin{aligned} \mathbb{G}_0(\overline{v}, \overline{v}) &= \int_{\mathbb{R}^2} \widetilde{h}_1(t, \eta, \zeta) e^{-\frac{1}{2}it(\xi-\eta)^2} \overline{v}(\xi - \eta) e^{-\frac{1}{2}it(\xi-\zeta)^2} \overline{v}(\xi - \zeta) d\eta d\zeta \\ \mathbb{G}_1(\overline{v}, v^2) &= \int_{\mathbb{R}^2} h_1(t, \eta, \zeta) e^{-\frac{1}{2}it(\xi-\eta)^2} \overline{v}(\xi - \eta) e^{it(\xi-\zeta)^2} v^2(\xi - \zeta) d\eta d\zeta \end{aligned}$$

with kernels

$$\begin{aligned} \widetilde{h}_1(t, \eta, \zeta) &= -\frac{\sqrt{t}}{\pi\sqrt{3}}(2 + t\partial_t)t\partial_t K_0\Big(\sqrt{\frac{4t}{3i}(\eta^2 - \eta\zeta + \zeta^2)}\Big), \\ h_1(t, \eta, \zeta) &= \frac{\sqrt{t}}{\pi\sqrt{3}}(1 + t\partial_t)\partial_\zeta K_0\Big(\sqrt{\frac{4t}{3i}(\eta^2 - \eta\zeta + \zeta^2)}\Big). \end{aligned}$$

Then by the identities

$$\begin{aligned} \mathcal{V}(-t)\overline{E}\partial_\xi &= \mathcal{F}\overline{M}\mathcal{F}^{-1}(\partial_\xi + it\xi)\overline{E} \\ &= \mathcal{F}\overline{M}(-ix + t\partial_x)\mathcal{F}^{-1}\overline{E} \\ &= t\mathcal{F}\partial_x\overline{M}\mathcal{F}^{-1}\overline{E} = it\xi\mathcal{V}(t)\overline{E}, \end{aligned}$$

we find

$$\mathcal{V}(-t)\mathcal{P} = it\xi\mathcal{V}(-t)\overline{E}(\mathbb{G}_0(\overline{v}, \overline{v}) + \mathbb{G}_1(\overline{v}, v^2)).$$

Integration by parts yields

$$\begin{aligned} &|(B_\gamma\mathcal{V}(-t) - \mathcal{V}(-t)B_\gamma)\phi| \\ &= \sqrt{t}\Big|\int_{\mathbb{R}} e^{-\frac{it}{2}(\xi-\eta)^2}(\langle\xi\sqrt{t}\rangle^\gamma - \langle\eta\sqrt{t}\rangle^\gamma)\phi(\eta)d\eta\Big| \\ &= t^{-1/2}\Big|\int_{\mathbb{R}} e^{-\frac{it}{2}(\xi-\eta)^2}(\xi - \eta)\partial_\eta\Big(\frac{\langle\xi\sqrt{t}\rangle^\gamma - \langle\eta\sqrt{t}\rangle^\gamma}{1 + it(\xi - \eta)^2}\phi(\eta)\Big)d\eta\Big| \\ &\le C\|\phi\|_{\mathbf{L}^\infty} + Ct^{-1/4}\|\partial_\xi\phi\|_{\mathbf{L}^2} \\ &\le C\|\phi\|_{\mathbf{X}^{0,\gamma}_{\widetilde{T}}}. \end{aligned}$$

Hence

$$\begin{aligned}\|B_\gamma(w-\phi)\|_{\mathbf{L}^\infty} &\le \|(B_\gamma\mathcal{V}(-t)-\mathcal{V}(-t)B_\gamma)\phi\|_{\mathbf{L}^\infty}+\|(\mathcal{V}(-t)-1)B_\gamma\phi\|_{\mathbf{L}^\infty}\\ &\le C\|\phi\|_{\mathbf{X}^{0,\gamma}_{\widetilde{T}}}+Ct^{-\frac14}\|\partial_\xi B_\gamma\phi\|_{\mathbf{L}^2}\le C\|\phi\|_{\mathbf{X}^{0,\gamma}_{\widetilde{T}}}.\end{aligned}$$

Thus by the estimates of Section 4 we see that

$$v=\phi+O(\varepsilon B_{-1})=w+O(\varepsilon B_{-\gamma}).$$

Then by Lemma 3.5 we get the representation for the operator $\mathcal{P}$,

$$\begin{aligned}t^{-1}\mathcal{V}(-t)\mathcal{P}&=i\xi\mathcal{V}(-t)\overline{E}(\mathbb{G}_0(\overline{v},\overline{v})+\mathbb{G}_1(\overline{v},v^2))\\ &=-it^{-1/2}\xi\widetilde{Q}|w|^2w+O(\varepsilon^2t^{-1/2}\xi B_{-1-\gamma}),\end{aligned}$$

where

$$\begin{aligned}\widetilde{Q}&=\int_{\mathbb{R}^2}e^{it\xi(\eta-2\zeta)}h_1(t,\eta,\zeta)d\eta d\zeta\\ &=\frac{\sqrt{t}}{\pi\sqrt{3}}\int_{\mathbb{R}^2}e^{it\xi(\eta-2\zeta)}\frac{\zeta-\frac{\eta}{2}}{\eta^2-\eta\zeta+\zeta^2}e^{-\langle\eta\sqrt{t}\rangle}d\eta d\zeta+O(\langle\xi\sqrt{t}\rangle^{-2})\\ &=\frac{2i}{\sqrt{3}(1+\xi\sqrt{3t})}+O(\langle\xi\sqrt{t}\rangle^{-2}).\end{aligned}$$

Thus we can write (6.5) in the form

$$w_t=i\Theta|w|^2w+O(\varepsilon^2t^{-\frac12}\xi B_{-1-\gamma}). \tag{6.6}$$

where

$$\Theta(t,\xi)=-\frac{2\xi}{\sqrt{3t}(1+\xi\sqrt{3t})}.$$

The first term in the right-hand side of (6.5) is divergent, so we eliminate it by the change $w(t,\xi)=\varphi(t,\xi)\mathcal{E}_w$, where

$$\mathcal{E}_w=\exp\Big(i\int_1^t\Theta(\tau,\xi)|w(\tau,\xi)|^2d\tau\Big).$$

Then we get from (6.6)

$$\varphi_t=O(\varepsilon^2t^{-1/2}\xi B_{-1-\gamma}). \tag{6.7}$$

Integrating in time

$$\begin{aligned}|\varphi|&\le\varepsilon+C\varepsilon^2\int_1^t\frac{|\xi|d\tau}{\sqrt{\tau}(1+|\xi|\sqrt{\tau})^{1+\gamma}}\\ &\le\varepsilon+C\varepsilon^2\int_{|\xi|}^{|\xi|\sqrt{t}}\frac{dz}{(1+z)^{1+\gamma}}\le\varepsilon+C\varepsilon^2.\end{aligned}$$

Hence $\|\phi\|_{\mathbf{V}^0_{\widetilde{T}}}<10\varepsilon$. This contradiction proves (6.2)and completes the proof. □

7. Proof of Theorem 1.1

By Lemma 5.3 we see that the a priori estimate of $\|\phi\|_{\mathbf{V}_T^0}$ implies the a priori estimate of $\|\partial_\xi\phi\|_{\mathbf{W}_T^\gamma}$. Vice versa by Lemma 6.1 the a priori estimate of $\|\partial_\xi\phi\|_{\mathbf{W}_T^\gamma}$ yields the a priori estimate of $\|\phi\|_{\mathbf{V}_T^0}$. Therefore the global existence of solution $v = \mathcal{I}^{-1}(\phi) \in \mathbf{C}([1,\infty);\mathbf{Y})$ of the Cauchy problem (2.5) satisfying a priori estimate

$$\|\phi\|_{\mathbf{X}_\infty^{0,\gamma}} \le C\varepsilon$$

follows by a standard continuation argument from Lemma 5.3, Lemma 6.1 and the local existence Theorem 5.1. This yields the solution of the Cauchy problem (1.1). Theorem 1.1 is proved.

8. Proof of Theorem 1.2

Existence of a self-similar solution of (2.1) of the form $\frac{1}{\sqrt{t}}MS(\frac{x}{\sqrt{t}})$ follows from Appendix 10 since

$$u(t) = \mathcal{U}(t)\mathcal{F}^{-1}w(t) = \mathcal{D}_t E(t)v(t), w(t) = \mathcal{V}(-t)v(t)$$

and $w(t)$ has the form $w(t,\xi) = MS(\xi\sqrt{t})$. We now prove the stability of solutions in the neighborhood of a self-similar solution of the equation (2.1). We consider the difference $r(t,\xi) = \phi_1(t,\xi) - \phi_2(t,\xi)$ and may assume that $\mathcal{V}(-t)(\phi_1(t) - \phi_2(t)) = 0$ at $\xi = 0$. Define the norm

$$\|r\|_{\mathbf{W}_T^{\mu,\nu}} = \sup_{1\le t\le T} t^{\frac{\mu}{2}}\Big(t^{-\frac14}\|B_\nu r(t)\|_{\mathbf{L}^2} + t^{-\frac34}\|\partial_\xi r(t)\|_{\mathbf{L}^2}\Big).$$

Lemma 8.1. *Let the initial data $\phi_j \in \mathbf{Y}$ and $\|\phi_j\|_{\mathbf{Y}} \le \varepsilon$, where $\varepsilon > 0$ is sufficiently small. Suppose that $\mathcal{V}(-t)(\phi_1(t) - \phi_2(t)) = 0$ at $\xi = 0$. Assume that representation (5.2) is valid. Suppose that*

$$\|\phi_j\|_{\mathbf{X}_T^{0,\gamma}} \le \varepsilon.$$

Then the solutions $\phi_j \in \mathbf{C}([1,T];\mathbf{Y})$ of (5.1) satisfy the estimate

$$\|\partial_\xi(\phi_1 - \phi_2)\|_{\mathbf{W}_T^{\mu,\nu}} < C\varepsilon$$

for $0 < \mu < \gamma$ and $\frac12 - \gamma \le \nu < \frac12 - \mu$.

Proof. Denote $r(t) = \phi_1(t) - \phi_2(t)$, then by (5.1) we have

$$\mathcal{L}r = t^{-1}(\mathcal{P}(\phi_1) - \mathcal{P}(\phi_2)). \tag{8.1}$$

We prove the estimate of the lemma by contradiction. By the continuity of $r(t)$ we can find a maximal time $\widetilde{T} \in (1,T]$ such that

$$\|\partial_\xi(\phi_1 - \phi_2)\|_{\mathbf{W}_{\widetilde{T}}^{\mu,\nu}} \le C\varepsilon \tag{8.2}$$

Since $\mathcal{V}(-t)r(t) = w_1(t,\xi) - w_2(t,\xi)$ vanishes at $\xi = 0$ for all $t \ge 1$, we can estimate the norm

$$\begin{aligned}|\mathcal{V}(-t)r(t)| &= \Big|\int_0^\xi B_{-\nu}B_\nu\partial_\xi\mathcal{V}(-t)r(t)d\xi\Big| \\ &\le C\sqrt{|\xi|}B_{-\nu}\|B_\nu\partial_\xi\mathcal{V}(-t)r(t)\|_{\mathbf{L}^2} \\ &\le |\xi|^\mu\langle\sqrt{t}\xi\rangle^{\frac12-\nu-\mu}t^{\frac{\mu}{2}-\frac14}\|B_\nu\partial_\xi r(t)\|_{\mathbf{L}^2} \\ &\le C\varepsilon|\xi|^\mu\langle\sqrt{t}\xi\rangle^{\frac12-\nu-\mu}.\end{aligned}$$

Also we have

$$|(1-\mathcal{V}(-t))r(t)| \le t^{-\frac{1}{4}}\|\partial_\xi r(t)\|_{\mathbf{L}^2} \le C\varepsilon t^{-\mu/2}.$$

Hence

$$\begin{aligned}|r(t)| &\le |(1-\mathcal{V}(-t))r(t)| + |\mathcal{V}(-t)r(t)|\\ &\le C\varepsilon t^{-\mu/2} + C\varepsilon|\xi|^{\mu}\langle\sqrt{t}\xi\rangle^{\frac{1}{2}-\nu-\mu}.\end{aligned}$$

Then we get from (8.1) in view of (5.6),

$$\begin{aligned}\mathcal{L}\big(r_\xi + \sum_{j=1}^{3} E^{\omega_j}\chi_j\big) &= t^{-1}\sum_{j=1}^{3} E^{\omega_j}\big(t^{1/2}\Omega_j\mathcal{M}_j + \frac{i\omega_j}{2A_j}\chi_j\big)\\ &\quad + t^{-1}\sum_{j=1}^{3} E^{\omega_j}(i\omega_j\xi\partial_\xi\chi_j + t\mathcal{L}\chi_j) + t^{-1}(\mathcal{R}_1-\mathcal{R}_2),\end{aligned} \tag{8.3}$$

where we denote $\mathcal{M}_j = \mathcal{N}_j(\phi_1) - \mathcal{N}_j(\phi_2)$. To eliminate the first summand in the right-hand side of (8.3) we choose $\chi_j = \frac{2i}{\omega_j}t^{1/2}\Omega_j A_j\mathcal{M}_j$ and denote $\Phi = r_\xi + \sum_{j=1}^{3} E^{\omega_j}\chi_j$. Therefore, from (8.3) we obtain

$$\mathcal{L}\Phi = 2it^{-3/2}\sum_{j=1}^{3}\frac{1}{\omega_j}E^{\omega_j}\Omega_j A_j\mathcal{M}_{j\overline{r}}\overline{\Phi_\xi} + t^{-1}\mathcal{R}_3,$$

where in view of (8.1), we have

$$\begin{aligned}\mathcal{R}_3 &= \mathcal{R}_1 - \mathcal{R}_2 - 2\sum_{j=1}^{3} E^{\omega_j}(\xi\partial_\xi(\Omega_j A_j\mathcal{M}_j) - \frac{i}{\omega_j}\mathcal{M}_j t\mathcal{L}(t^{1/2}\Omega_j A_j))\\ &\quad + \sum_{j=1}^{3} E^{\omega_j}t^{-1/2}\partial_\xi(\Omega_j A_j)\partial_\xi\mathcal{M}_j\\ &\quad + \sum_{j=1}^{3} E^{\omega_j}\frac{i}{\omega_j}t^{-1/2}\Omega_j A_j(\mathcal{M}_{jrr}r_\xi^2 + 2\mathcal{M}_{jr\overline{r}}|r_\xi|^2 + \mathcal{M}_{j\overline{r}\overline{r}}\overline{r_\xi}^2)\\ &\quad + \sum_{j=1}^{3} E^{\omega_j}\frac{2i}{\omega_j}t^{1/2}\Omega_j A_j\mathcal{M}_j(\mathcal{M}_{jr}(\mathcal{P}_1-\mathcal{P}_2) - \mathcal{M}_{j\overline{r}}(\overline{\mathcal{P}_1}-\overline{\mathcal{P}_2}))\\ &\quad - \sum_{j,l=1}^{3} E^{\omega_j}\frac{4}{\omega_j\omega_l}\Omega_j A_j\mathcal{M}_{j\overline{r}}\partial_\xi(\overline{E}^{\omega_l}\overline{\Omega_l A_l\mathcal{M}_l}).\end{aligned}$$

Since

$$|\Omega_j A_j| + |\xi\partial_\xi(\Omega_j A_j)| + |t\mathcal{L}(\Omega_j A_j)| \le CB_{-2}, t^{-1/2}|\partial_\xi(\Omega_j A_j)| \le CB_{-3},$$

then $\mathcal{R}_3$ can be estimated as

$$\|\mathcal{R}_3\|_{\mathbf{W}_T^{\mu,\nu}} \le C\varepsilon^2$$

if $\gamma \ge \frac{1}{2}-\nu$. Then as in the proof of Lemma 5.3 we multiply (5.8) by $(M+t\xi^2)^{\nu/2}$ and use the commutator

$$\mathcal{L}((M+t\xi^2)^{\nu/2}\Phi) = (M+t\xi^2)^{\nu/2}\mathcal{L}\Phi + \Phi\mathcal{L}(M+t\xi^2)^{\nu/2} + 2\nu t^{-1}\xi(M+t\xi^2)^{\frac{\nu}{2}-1}\partial_\xi\Phi$$

then we get

$$\mathcal{L}((M+t\xi^2)^{\nu/2}\Phi) = \mathcal{R}_4,$$

where

$$\mathcal{R}_4 = \Phi\mathcal{L}(M+t\xi^2)^{\nu/2} + 2\nu t^{-1}\xi(M+t\xi^2)^{\frac{\nu}{2}-1}\Phi_\xi + 2it^{-3/2}\sum_{j=1}^{3}\frac{1}{\omega_j}E^{\omega_j}B_\nu\Omega_j A_j\mathcal{N}_{j\overline{\phi}}\overline{\Phi_\xi} + t^{-1}(M+t\xi^2)^{\nu/2}\mathcal{R}_3.$$

Since

$$|\mathcal{L}(M+t\xi^2)^{\nu/2}| \le \frac{\nu}{2t}(1+\frac{1}{M})(M+t\xi^2)^{\nu/2},$$

by (5.5) choosing M sufficiently large we have

$$\|\mathcal{R}_4\|_{\mathbf{L}^2} \le \frac{\nu}{2t}\|(M+t\xi^2)^{\nu/2}\Phi\|_{\mathbf{L}^2} + C\varepsilon^{\frac{5}{4}}t^{\frac{\mu}{2}-\frac{3}{4}}.$$

Then we apply the energy method to estimate the $\mathbf{L}^2$ - norm of $B_\gamma\Phi$

$$\frac{d}{dt}\|(M+t\xi^2)^{\nu/2}\Phi\|_{\mathbf{L}^2} \le \frac{\nu}{2t}\|(M+t\xi^2)^{\frac{\nu}{2}}\Phi\|_{\mathbf{L}^2} + C\varepsilon^{\frac{5}{4}}t^{\frac{\mu}{2}-\frac{3}{4}}.$$

Hence the integration with respect to time yields

$$\|(M+t\xi^2)^{\nu/2}\Phi\|_{\mathbf{L}^2} \le \varepsilon + C\varepsilon^{\frac{5}{4}}t^{\frac{1}{4}-\frac{\mu}{2}}$$

for all $t \in [1,\widetilde{T}]$ if $0 < \nu < \frac{1}{2} - \mu$. Therefore $\|B_\nu r_\xi\|_{\mathbf{L}^2} \le \varepsilon + C\varepsilon^{\frac{5}{4}}t^{\frac{1}{4}-\frac{\mu}{2}}$ for all $t \in [1,\widetilde{T}]$.

We now differentiate (8.3) with respect to ξ to get

$$\mathcal{L}\Phi_\xi = 2it^{-3/2}\sum_{j=1}^{3}\frac{1}{\omega_j}E^{\omega_j}\Omega_j A_j\mathcal{M}_{j\overline{r}}\overline{\Phi_{\xi\xi}} + \mathcal{R}_5,$$

where

$$\mathcal{R}_5 = 2it^{-3/2}\sum_{j=1}^{3}\frac{1}{\omega_j}\overline{\Phi_\xi}\partial_\xi(E^{\omega_j}\Omega_j A_j\mathcal{M}_{j\overline{r}}) + t^{-1}\partial_\xi\mathcal{R}_3.$$

By (8.2) we see that $\|\mathcal{R}_5\|_{\mathbf{L}^2} \le C\varepsilon^2 t^{-\frac{\nu}{2}-\frac{1}{4}}$. Then to estimate the $\mathbf{L}^2$ - norm of Φ_ξ we apply the energy method to (5.9)

$$\frac{d}{dt}\|\Phi_\xi\|^2_{\mathbf{L}^2} \le Ct^{-3/2}\sum_{j=1}^{3}|\int_{\mathbb{R}}\Omega_j A_j E^{\omega_j}\mathcal{M}_{j\overline{r}}\overline{\Phi_{\xi\xi}\Phi_\xi}d\xi| + C\varepsilon^3 t^{\frac{1}{2}-\nu}.$$

Hence integrating by parts with respect to ξ we avoid the derivative loss and obtain $\|\Phi_\xi\|_{\mathbf{L}^2} \le \varepsilon + C\varepsilon^{\frac{3}{2}}t^{\frac{3}{4}-\frac{\nu}{2}}$ for all $t \in [1,\widetilde{T}]$. Therefore

$$\|\partial_\xi^2 r\|_{\mathbf{L}^2} < \varepsilon + C\varepsilon^{\frac{3}{2}}t^{\frac{3}{4}-\frac{\nu}{2}}$$

for all $t \in [1,\widetilde{T}]$. Thus we have $\|\partial_\xi r\|_{\mathbf{W}_T^{\mu,\nu}} < C\varepsilon$. This contradiction proves estimate of the lemma. Lemma 8.1 is proved. □

To study the asymptotic behavior we make a change as in the proof of Lemma 6.1 $w = \overline{\mathcal{V}}\phi = \varphi\mathcal{E}_\varphi$ and $s = \overline{\mathcal{V}}S = g\mathcal{E}_s$, where

$$\mathcal{E}_s = \exp\Big(i\int_1^t \Theta(\tau,\xi)|s(\tau,\xi)|^2 d\tau\Big).$$

Then from (6.5) we get for the difference $f = \varphi - g$

$$\begin{aligned} if_t &= \overline{\mathcal{E}_\varphi}t^{-1}\overline{\mathcal{V}}\mathcal{P}(\phi) - i\Theta|\varphi|^2\varphi - \overline{\mathcal{E}_g}t^{-1}\overline{\mathcal{V}}\mathcal{P}(S) + i\Theta|g|^2 g \\ &= (\overline{\mathcal{E}_\varphi} - \overline{\mathcal{E}_g})(t^{-1}\overline{\mathcal{V}}\mathcal{P}(\mathcal{V}s) - i\Theta|s|^2 s) + t^{-1}\overline{\mathcal{E}_\varphi}(\overline{\mathcal{V}} - 1)(\mathcal{P}(\phi) - \mathcal{P}(S)) \end{aligned}$$

$$
\begin{aligned}
&- i\Theta\overline{\mathcal{E}_\varphi}\Big(\overline{((\overline{\mathcal{V}}-1)\phi)}(\overline{\mathcal{V}}\phi)^2 - \overline{((\overline{\mathcal{V}}-1)S)}(\overline{\mathcal{V}}S)^2 \\
&+ \overline{\phi}((\overline{\mathcal{V}}-1)\phi)(\overline{\mathcal{V}}+1)\phi - \overline{S}((\overline{\mathcal{V}}-1)S)(\overline{\mathcal{V}}+1)S\Big) \\
&+ \overline{\mathcal{E}_\varphi}(t^{-1}\mathcal{P}(\phi) - i\Theta|\phi|^2\phi - t^{-1}\mathcal{P}(S) + i\Theta|S|^2S) \\
&= O(\varepsilon^2 t^{-\frac{\mu}{2}-1}).
\end{aligned}
$$

Then we get

$$
|f(t)-f(t_1)| \le C\varepsilon^2 \int_{t_1}^t \tau^{-\frac{\mu}{2}-1} d\tau \le \varepsilon^2 {t_1}^{-\mu/2}.
$$

Also by (6.7),

$$
|\varphi(t)-\varphi(t_1)| \le C\varepsilon^2 \int_{t_1}^t \tau^{-1/2}\xi\langle\xi\sqrt{\tau}\rangle^{-1-\gamma} d\tau \le \frac{C\varepsilon^2}{\langle\xi\sqrt{t_1}\rangle^{\gamma}}
$$

for all $t > t_1 > 1$. Therefore the limits exist

$$
F(\xi) = \lim_{t\to\infty} f(t,\xi),\ \Phi(\xi) = \lim_{t\to\infty}\varphi(t,\xi) \text{ and } G(\xi) = \lim_{t\to\infty} g(t,\xi)
$$

with the estimates $|F(\xi)| + |\Phi(\xi)| + |G(\xi)| \le C\varepsilon$. Then using the estimates

$$
|f(t,\xi)-F(\xi)| \le \varepsilon^2 t^{-\mu/2}, \quad |\varphi(t,\xi)-\Phi(\xi)| \le \frac{C\varepsilon^2}{\langle\xi\sqrt{t}\rangle^{\gamma}},
$$

$$
|g(t,\xi)-G(\xi)| \le \frac{C\varepsilon^2}{\langle\xi\sqrt{t}\rangle^{\gamma}}, \quad |1-\mathcal{E}_g\overline{\mathcal{E}_\varphi}| \le C\varepsilon^2 t^{-\mu/2}\langle\xi\sqrt{t}\rangle^{\gamma},
$$

we obtain

$$
\begin{aligned}
\phi &= S + \mathcal{E}_\varphi(f + (1-\mathcal{E}_g\overline{\mathcal{E}_\varphi})g) + (1-\overline{\mathcal{V}})r \\
&= S + \mathcal{E}_\varphi(F + (1-\mathcal{E}_g\overline{\mathcal{E}_\varphi})G) + O(\varepsilon^2 t^{-\frac{\mu}{2}}).
\end{aligned}
$$

Note that

$$
|\varphi(t,\xi)|^2 = |\Phi(\xi)|^2 + O(\varepsilon^2\langle\xi\sqrt{t}\rangle^{-\gamma}).
$$

We now denote

$$
\Psi_1(t) = -i\int_1^t (|\varphi(\tau,\xi)|^2 - |\varphi(t,\xi)|^2)\Theta d\tau,
$$

$$
\Psi_2(t) = -i\int_1^t (|g(\tau,\xi)|^2 - |\varphi(\tau,\xi)|^2 - |g(t,\xi)|^2 + |\varphi(t,\xi)|^2)\Theta d\tau.
$$

We then get

$$
\Psi_1(t)-\Psi_1(t_1) = -i\int_{t_1}^t (|\varphi(\tau,\xi)|^2 - |\varphi(t,\xi)|^2)\Theta d\tau + i(|\varphi(t,\xi)|^2 - |\varphi(t_1,\xi)|^2)\int_1^{t_1}\Theta d\tau
$$

and

$$
\begin{aligned}
\Psi_2(t)-\Psi_2(t_1) &= -i\int_{t_1}^t (|g(\tau,\xi)|^2 - |\varphi(\tau,\xi)|^2 - |g(t,\xi)|^2 + |\varphi(t,\xi)|^2)\Theta d\tau \\
&\quad + i(|g(t,\xi)|^2 - |\varphi(t,\xi)|^2 - |g(t_1,\xi)|^2 + |\varphi(t_1,\xi)|^2)\int_1^{t_1}\Theta d\tau
\end{aligned}
$$

for all $1 < t_1 < \tau < t$. Using these estimates we get

$$
|\Psi_j(t)-\Psi_j(t_1)| \le C\varepsilon^2\langle\xi\sqrt{t_1}\rangle^{-\gamma}, \quad j = 1,2.
$$

Therefore, there exist unique functions $\Phi_{j+2} \in \mathbf{L}^\infty$, such that

$$i\Phi_{j+2} = \lim_{t\to\infty} \Psi_j(t)$$

and

$$|i\Phi_{j+2} - \Psi_j(t)| \le C\varepsilon^2 \langle \xi\sqrt{t}\rangle^{-\gamma}, j = 1, 2.$$

Then we find

$$-i\int_1^t |\varphi(\tau)|^2 \Theta d\tau = i|\Phi|^2 \log(\xi\sqrt{t}) + i\Phi_3 + O(\varepsilon^2 t^{-\mu/2})$$

and

$$-i\int_1^t (|g(\tau)|^2 - |\varphi(\tau)|^2)\Theta d\tau = i(|G|^2 - |\Phi|^2) \log(\xi\sqrt{t}) + i\Phi_4 + O(\varepsilon^2 t^{-\mu/2})$$

with some functions Φ_3, $\Phi_4 \in \mathbf{L}^\infty$. Hence

$$\mathcal{E}_\varphi = \exp\Big(- i|\Phi|^2 \log(\frac{1+|\xi|\sqrt{t}}{1+|\xi|}) - i\Phi_3 + O(\varepsilon^2 t^{-\mu/2})\Big)$$

and

$$\mathcal{E}_g\overline{\mathcal{E}_\varphi} = \exp\Big(- i(|G|^2 - |\Phi|^2) \log(\frac{1+|\xi|\sqrt{t}}{1+|\xi|}) - i\Phi_4 + O(\varepsilon^2 t^{-\mu/2})\Big).$$

Thus

$$\begin{aligned}\phi &= S + e^{-i|\Phi|^2 \log(\frac{1+|\xi|\sqrt{t}}{1+|\xi|}) - i\Phi_3}\Big(F + G\big(1 - e^{-i(|G|^2-|\Phi|^2)\log(\frac{1+|\xi|\sqrt{t}}{1+|\xi|}) - i\Phi_4}\big)\Big) \\ &\quad + O(\varepsilon^3 t^{-\gamma}) \\ &= S + \sum_{j=1}^2 H_j e^{iB_j \log(\frac{1+|\xi|\sqrt{t}}{1+|\xi|})} + O(\varepsilon^2 t^{-\mu/2}).\end{aligned}$$

The asymptotic formula follows now from the inverse transformation [11] of $u(t,x) = t^{-1/2} e^{\frac{it}{2}\xi^2} v(t,\xi)$ and $\xi = x/t$,

$$u(t,x) = t^{-1/2} E v(t) = t^{-\frac{1}{2}} ES + t^{-1/2} E \sum_{j=1}^2 H_j e^{iB_j \log(\frac{1+\frac{|x|}{\sqrt{t}}}{1+\frac{|x|}{t}})} + O(t^{-\frac{1}{2}-\frac{\mu}{2}}).$$

Theorem 1.2 is proved.

9. Appendix 1, Hope-Cole transformation

We consider the quadratic nonlinear Schrödinger equation

$$\begin{gathered} iu_t + \frac{1}{2} u_{xx} = \partial_x (u^2), \quad x \in \mathbb{R}, t \in \mathbb{R}, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}. \end{gathered} \tag{9.1}$$

There exists a self-similar solution of the form $s(t,x) = \frac{1}{\sqrt{t}}\varphi(\xi)$, with $\xi = \frac{x}{\sqrt{t}}$. Indeed if we substitute $\frac{1}{\sqrt{t}}\varphi(\xi)$ into (9.1) we get the ordinary differential equation for the function $\varphi(\xi)$,

$$-\frac{i}{2}(\xi\varphi)' + \frac{1}{2}\varphi'' = (\varphi^2)',$$

hence integrating with respect to ξ and choosing the integration constant as zero, we obtain the Bernoulli equation $\varphi' = i\xi\varphi + 2\varphi^2$, which has a solution

$$\varphi(\xi) = e^{\frac{i}{2}\xi^2}\Big(C - \sqrt{2\pi i}\,\mathrm{erf}(\xi\sqrt{-\frac{i}{2}})\Big)^{-1},$$

with the error function $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}dy$. We suppose that the constant C have a sufficiently large absolute value. Now we represent a solution $u(t,x) = s(t,x) + w(t,x)$ and choose a constant C such that $\int s(t,x)dx = \int \varphi(\xi)d\xi = \int u(t,x)dx$, then

$$\int w(t,x)dx = 0,$$

so we can consider w as a full derivative: $w = v_x$. Thus we have

$$iv_t + \frac{1}{2}v_{xx} = 2sv_x + (v_x)^2.$$

Now by the Hopf-Cole anzatz $v = -\frac{1}{2}\log(1+\phi)$ we linearize the above equation

$$i\phi_t + \frac{1}{2}\phi_{xx} = 2s\phi_x.$$

We can eliminate the right-hand side of the above equation by virtue of the change of the dependent variable $\phi(t,x) = \exp(2\int_0^x s(t,y)dy)\ \psi(t,x)$,

$$\begin{gathered} i\psi_t + \frac{1}{2}\psi_{xx} = 0, \quad x \in \mathbb{R},\ t \in \mathbb{R}, \\ \psi(0,x) = \psi_0(x), \quad x \in \mathbb{R}, \end{gathered}$$

where $\psi_0(x) = (\exp(-2\int_0^x (u_0(y) - s(0,y))dy) - 1)\exp(-2\int_0^x s(0,y)dy)$. Thus the solution of (9.1) have a form

$$u(t,x) = s(t,x) - \frac{1}{2}\partial_x \log(1 + \psi(t,x)\exp(2\int_0^x s(t,y)dy)).$$

10. Appendix 2, Existence of a self-similar solution

Here we study a self-similar solution of the form $v(t,\xi) = w(\xi\sqrt{t})$

$$v_t(t,\xi) = \xi \int_{\mathbb{R}} e^{itS}\overline{v(t,\eta-\xi)v(t,-\eta)}d\eta,$$

where

$$S = \frac{1}{2}(\xi^2 + (\xi-\eta)^2 + \eta^2)$$

and the solution $u(t,x)$ of (1.1) can be obtained from $v(t,\xi)$ through the formula

$$u(t) = \mathcal{U}(t)\mathcal{F}^{-1}v(t).$$

For $w(\xi\sqrt{t})$ we get

$$w'(\xi) = 2\int_{\mathbb{R}} e^{iS}\overline{w(\eta-\xi)w(-\eta)}d\eta.$$

We use the identity $e^{iS} = (\partial_\xi \xi + \partial_\eta \eta)(Ke^{iS}) + Fe^{iS}$, where

$$K = \frac{2+iS}{2(1+iS)^2}, \quad F = \frac{1-iS}{(1+iS)^3}.$$

Then we get

$$\begin{aligned}&\frac{d}{d\xi}(w(\xi)+2\int_{\mathbb{R}}\xi Ke^{iS}\overline{w(\eta-\xi)w(-\eta)}d\eta)\\&=2\int_{\mathbb{R}}Fe^{iS}\overline{w(\eta-\xi)w(-\eta)}d\eta-\int_{\mathbb{R}}K\eta e^{iS}\frac{d}{d\eta}(\overline{w(\eta-\xi)w(-\eta)})d\eta.\end{aligned}$$

Denote

$$\phi(\xi)=w(\xi)+2\int_{\mathbb{R}}\xi K(\xi,\eta)e^{iS}\overline{w(\eta-\xi)w(-\eta)}d\eta, \tag{10.1}$$

then in view of the symmetry $\eta-\zeta\leftrightarrow\zeta$ we have

$$\frac{d\phi}{d\xi}=2\int_{\mathbb{R}}Fe^{iS}\overline{w(\eta-\xi)w(-\eta)}d\eta-4\int_{\mathbb{R}}d\eta e^{iS}K\eta\overline{w(\eta-\xi)}\int_{\mathbb{R}}e^{-iQ}w(\eta-y)w(y)dy \tag{10.2}$$

with

$$Q=\frac{1}{2}(\eta^2+(\eta-y)^2+y^2).$$

We will see later that the right-hand side of (10.2) has the asymptotic

$$\frac{16i\pi}{3\langle\xi\rangle}|\phi(\xi)|^2\phi(\xi)+O(\langle\xi\rangle^{\gamma-2}).$$

To exclude the first divergent term we make a change $\phi=\varphi E$, where

$$E=\exp\Big(\frac{16}{3}i\int_0^{\xi}|\varphi(\xi)|^2\frac{d\xi}{\langle\xi\rangle}\Big),$$

then we get from (10.2),

$$\begin{aligned}\frac{d\varphi}{d\xi}&=2\overline{E}\int_{\mathbb{R}}Fe^{iS}\overline{w(\eta-\xi)w(-\eta)}d\eta-\frac{16i\pi}{3\langle\xi\rangle}|\varphi(\xi)|^2\varphi(\xi)\\&\quad-4\overline{E}\int_{\mathbb{R}}d\eta e^{iS}K\eta\overline{w(\eta-\xi)}\int_{\mathbb{R}}e^{-iQ}w(\eta-y)w(y)dy.\end{aligned} \tag{10.3}$$

Integrating the above expression, we obtain

$$\begin{aligned}\varphi(\xi)&=\varphi(0)+\int_0^{\xi}2\overline{E}\int_{\mathbb{R}}Fe^{iS}\overline{w(\eta-\xi)w(-\eta)}d\eta d\xi\\&\quad-\int_0^{\xi}\Big(\frac{16i\pi}{3\langle\xi\rangle}|\varphi(\xi)|^2\varphi(\xi)\\&\quad+4\overline{E}\int_{\mathbb{R}}d\eta e^{iS}K\eta\overline{w(\eta-\xi)}\int_{\mathbb{R}}e^{-iQ}w(\eta-y)w(y)dy\Big)d\xi.\end{aligned} \tag{10.4}$$

We solve integral equation (10.4) by the contraction mapping principle. Define the transformation

$$\begin{aligned}\mathcal{A}(\varphi)(\xi)&=\varphi(0)+\int_0^{\xi}2\overline{E}\int_{\mathbb{R}}Fe^{iS}\overline{w(\eta-\xi)w(-\eta)}d\eta d\xi\\&\quad-\int_0^{\xi}\Big(\frac{16i\pi}{3\langle\xi\rangle}|\varphi(\xi)|^2\varphi(\xi)\\&\quad+4\overline{E}\int_{\mathbb{R}}d\eta e^{iS}K\eta\overline{w(\eta-\xi)}\int_{\mathbb{R}}e^{-iQ}w(\eta-y)w(y)dy\Big)d\xi.\end{aligned} \tag{10.5}$$

We now find the inverse transformation to (10.1). Changing the variable of integration $\eta = \eta' + \frac{\xi}{2}$ we rewrite (10.1) as (the prime we will omit)

$$w(\xi) = \phi(\xi) - e^{\frac{3}{4}i\xi^2} \int_{\mathbb{R}} e^{i\eta^2} K_1(\xi,\eta) \overline{w(\eta - \frac{\xi}{2}) w(-\eta - \frac{\xi}{2})} d\eta \tag{10.6}$$

where

$$K_1(\xi,\eta) = \xi K(\xi, \eta + \frac{\xi}{2}) = \frac{\xi(2+iS_1)}{(1+iS_1)^2}, \quad S_1 = \frac{3}{4}\xi^2 + \eta^2.$$

We substitute the representation

$$w(\xi) = \phi(\xi) + \psi_1(\xi) - e^{\frac{3}{4}i\xi^2}\psi_2(\xi) \tag{10.7}$$

into (10.6). In view of the symmetry $\eta \to -\eta$, changing in the second integral $\eta = \frac{3}{2}\xi - \eta'$ we find

$$\begin{aligned}
\psi_1(\xi) - e^{\frac{3}{4}i\xi^2}\psi_2(\xi) &= -e^{\frac{3}{4}i\xi^2} \int_{\mathbb{R}} e^{i\eta^2} K_1(\xi,\eta) \overline{\psi_3(\eta - \frac{\xi}{2})\psi_3(-\eta - \frac{\xi}{2})} d\eta \\
&\quad + 2\int_{\mathbb{R}} e^{\frac{1}{4}i\eta^2} K_2(\xi,\eta) \overline{\psi_3(\xi - \eta)\psi_2(\eta - 2\xi)} d\eta \\
&\quad - e^{\frac{3}{8}i\xi^2} \int_{\mathbb{R}} e^{-\frac{1}{2}i\eta^2} K_1(\xi,\eta) \overline{\psi_2(-\eta - \frac{\xi}{2})\psi_2(-\eta - \frac{\xi}{2})} d\eta,
\end{aligned}$$

where $\psi_3(\xi) = \phi(\xi) + \psi_1(\xi)$ and $K_2(\xi,\eta) = K_1(\xi, \frac{3}{2}\xi - \eta)$. Thus we obtain the system of integral equations for the functions $\psi_1(\xi)$ and $\psi_2(\xi)$

$$\begin{aligned}
\psi_1(\xi) &= 2\int_{\mathbb{R}} e^{\frac{1}{4}i\eta^2} K_2(\xi,\eta) \overline{\psi_2(\eta - 2\xi)\psi_3(\xi - \eta)} d\eta \\
&\quad + e^{-\frac{3}{8}i\xi^2} \int_{\mathbb{R}} e^{-\frac{1}{2}i\eta^2} K_1(\xi,\eta) \overline{\psi_2(\eta - \frac{\xi}{2})\psi_2(-\eta - \frac{\xi}{2})} d\eta, \\
\psi_2(\xi) &= \int_{\mathbb{R}} e^{i\eta^2} K_1(\xi,\eta) \overline{\psi_3(\eta - \frac{\xi}{2})\psi_3(-\eta - \frac{\xi}{2})} d\eta.
\end{aligned} \tag{10.8}$$

Denote the norm

$$\|\psi\|_{\mathbf{Z}} \equiv \sup_{\xi \in \mathbb{R}} (\langle \xi \rangle^{1-\gamma} |\psi(\xi)| + \langle \xi \rangle^{1-2\gamma} |\psi'(\xi)|).$$

Lemma 10.1. *Let $\phi \in \mathbf{C}^1(\mathbb{R})$ satisfy the estimate*

$$\sup_{\xi \in \mathbb{R}} (|\phi(\xi)| + \langle \xi \rangle^{1-\gamma} |\phi'(\xi)|) \le \varepsilon, \tag{10.9}$$

where $\varepsilon > 0$ is sufficiently small and $\gamma \in (0,1)$. Then there exist unique solutions $\psi_1, \psi_2 \in \mathbf{C}^1(\mathbb{R})$ of a system of integral equations (10.8) such that

$$\|\psi_1\|_{\mathbf{Z}} + \|\psi_2\|_{\mathbf{Z}} \le C\varepsilon^2. \tag{10.10}$$

Proof. We solve equations (10.8) by the contraction mapping principle in the set

$$\mathbf{Z}_\varepsilon = \{(\psi_1,\psi_2) \in (\mathbf{C}^1(\mathbb{R}))^2 : \|\psi_1\|_{\mathbf{Z}} + \|\psi_2\|_{\mathbf{Z}} \le C\varepsilon^2\}.$$

Define the transformation

$$\begin{aligned}
\mathcal{M}_1(\psi_1,\psi_2)(\xi) &= 2\int_{\mathbb{R}} e^{\frac{1}{4}i\eta^2} K_2(\xi,\eta) \overline{\psi_2(\eta - 2\xi)\psi_3(\xi - \eta)} d\eta \\
&\quad + e^{-\frac{3}{8}i\xi^2} \int_{\mathbb{R}} e^{-\frac{1}{2}i\eta^2} K_1(\xi,\eta) \overline{\psi_2(\eta - \frac{\xi}{2})\psi_2(-\eta - \frac{\xi}{2})} d\eta,
\end{aligned}$$

$$\mathcal{M}_2(\psi_1,\psi_2)(\xi) = \int_{\mathbb{R}} e^{i\eta^2} K_1(\xi,\eta)\overline{\psi_3(\eta-\frac{\xi}{2})\psi_3(-\eta-\frac{\xi}{2})}d\eta$$

for $(\psi_1,\psi_2) \in \mathbf{Z}_\varepsilon$. Via (10.9) and by the fact that $\|\psi_1\|_{\mathbf{Z}} + \|\psi_2\|_{\mathbf{Z}} \le C\varepsilon^2$ we obtain the estimate

$$\begin{aligned}
&|\mathcal{M}_1(\psi_1,\psi_2)(\xi)| \\
&\le C\varepsilon^3 \int_{\mathbb{R}} (1+|\xi|+|\eta|)^{-1}\big(\langle \eta-2\xi\rangle^{\gamma-1} + \langle \eta-\frac{\xi}{2}\rangle^{\gamma-1}\langle \eta+\frac{\xi}{2}\rangle^{\gamma-1}\big)d\eta \\
&\le C\varepsilon^3\langle\xi\rangle^{\gamma-1}.
\end{aligned}$$

Integrating by parts with respect to η via the identity $e^{i\eta^2} = A\partial_\eta(\eta e^{i\eta^2})$ with $A = (1+2i\eta^2)^{-1}$ we get

$$\begin{aligned}
|\mathcal{M}_2(\psi_1,\psi_2)(\xi)| &= \Big| \int_{\mathbb{R}} e^{i\eta^2}\overline{\psi_3(\eta-\frac{\xi}{2})\psi_3(-\eta-\frac{\xi}{2})}\eta\partial_\eta(AK_1)d\eta \\
&\quad + 2\int_{\mathbb{R}} e^{i\eta^2}\eta AK_1\overline{\psi_3(\eta-\frac{\xi}{2})\psi_3'(-\eta-\frac{\xi}{2})}d\eta\Big| \\
&\le C\varepsilon^2 \int_{\mathbb{R}}\Big(\langle\xi\rangle^{-1}\langle\eta\rangle^{-2} + (1+|\xi|+|\eta|)^{-2}\langle\eta\rangle^{-1} \\
&\quad + (1+|\xi|+|\eta|)^{-1}\langle\eta\rangle^{-1}\langle 2\eta+\xi\rangle^{2\gamma-1}\Big)d\eta \\
&\le C\varepsilon^2\langle\xi\rangle^{-1}.
\end{aligned}$$

We now estimate the derivatives

$$\begin{aligned}
&\frac{d}{d\xi}\mathcal{M}_1(\psi_1,\psi_2)(\xi) \\
&= 2\int_{\mathbb{R}} e^{\frac14 i\eta^2}\overline{\psi_2(\eta-2\xi)\psi_3(\xi-\eta)}\partial_\xi K_2 d\eta \\
&\quad + 2\int_{\mathbb{R}} e^{\frac14 i\eta^2}(\overline{\psi_2(\eta-2\xi)\psi_3'(\xi-\eta)} - 2\overline{\psi_2'(\eta-2\xi)\psi_3(\xi-\eta)})K_2 d\eta \\
&\quad - e^{-\frac38 i\xi^2}\int_{\mathbb{R}} e^{-\frac12 i\eta^2}K_1\overline{\psi_2'(\eta-\frac{\xi}{2})\psi_2(-\eta-\frac{\xi}{2})}d\eta \\
&\quad + e^{-\frac38 i\xi^2}\int_{\mathbb{R}} e^{-\frac12 i\eta^2}\overline{\psi_2(\eta-\frac{\xi}{2})\psi_2(-\eta-\frac{\xi}{2})}(\partial_\xi K_1 - \frac34 i\xi K_1)d\eta.
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{d\xi}\mathcal{M}_2(\psi_1,\psi_2)(\xi) &= -\int_{\mathbb{R}} e^{i\eta^2}K_1\overline{\psi_3'(\eta-\frac{\xi}{2})\psi_3(-\eta-\frac{\xi}{2})}d\eta \\
&\quad + \int_{\mathbb{R}} e^{i\eta^2}\overline{\psi_3(\eta-\frac{\xi}{2})\psi_3(-\eta-\frac{\xi}{2})}\partial_\xi K_1 d\eta.
\end{aligned}$$

Then we find the estimates

$$\begin{aligned}
|\frac{d}{d\xi}\mathcal{M}_1(\psi_1,\psi_2)(\xi)| &\le C\varepsilon^3\int_{\mathbb{R}}(\langle\eta-2\xi\rangle^{2\gamma-1} + \langle\xi-\eta\rangle^{2\gamma-1})(1+|\xi|+|\eta|)^{-1}d\eta \\
&\quad + C\varepsilon^4\int_{\mathbb{R}}\langle\eta-\frac{\xi}{2}\rangle^{\gamma-1}\langle\eta+\frac{\xi}{2}\rangle^{\gamma-1}d\eta \le C\varepsilon^3\langle\xi\rangle^{2\gamma-1}
\end{aligned}$$

and

$$|\frac{d}{d\xi}\mathcal{M}_2(\psi_1,\psi_2)(\xi)| \le C\varepsilon^2\int_{\mathbb{R}}(1+|\xi|+|\eta|)^{-1}\langle\eta-\frac{\xi}{2}\rangle^{2\gamma-1}d\eta + C\varepsilon^2\int_{\mathbb{R}}(1+|\xi|+|\eta|)^{-2}d\eta \le C\varepsilon^2\langle\xi\rangle^{2\gamma-1}.$$

Thus the mapping $(\mathcal{M}_1,\mathcal{M}_2)$ transforms the set $\mathbf{Z}$ into itself. In the same manner we find

$$\|\mathcal{M}_j(\psi_1,\psi_2)-\mathcal{M}_j(\widetilde{\psi_1},\widetilde{\psi_2})\|_{\mathbf{Z}} \le \frac{1}{2}\|\psi_1-\widetilde{\psi_1}\|_{\mathbf{Z}}+\frac{1}{2}\|\psi_2-\widetilde{\psi_2}\|_{\mathbf{Z}}.$$

Therefore, $(\mathcal{M}_1,\mathcal{M}_2)$ is a contraction mapping in $\mathbf{Z}$. Hence there exist unique solutions $\psi_1,\psi_2\in\mathbf{Z}$ of a system of integral equations (10.8), which satisfy estimate (10.10). Lemma 10.1 is proved. □

We now evaluate the asymptotic form of the integral

$$\mathcal{I}\equiv\int_{\mathbb{R}}e^{\alpha i(\eta-\mu\xi)^2}A(\xi,\eta)\phi_1(\eta-\xi)\Phi d\eta,$$

where

$$\Phi=\int_{\mathbb{R}}e^{\beta iy^2}\phi_2(a_1\eta-b_1y)\phi_3(a_2\eta-b_2y)dy,$$

with $\alpha,\beta,\mu,a_1,a_2,b_1,b_2\in\mathbf{R}\backslash\{0\}$. Also we assume that $a_1b_2-a_2b_1\neq0$.

Lemma 10.2. *Suppose that*

$$\|\phi_j\|_{\mathbf{Y}}\equiv\sup_{\xi\in\mathbb{R}}(|\phi_j(\xi)|+\langle\xi\rangle^{1-2\gamma}|\phi_j'(\xi)|)\le C,$$
$$|A(\xi,\eta)|\le C(1+|\xi|+|\eta|)^{-1},$$
$$|\partial_\eta A(\xi,\eta)|\le C(1+|\xi|+|\eta|)^{-2}.$$

Then the asymptotic form is true

$$\mathcal{I}=\frac{i\pi}{\sqrt{\alpha\beta}}A(\xi,\mu\xi)\phi_1((\mu-1)\xi)\phi_2(a_1\mu\xi)\phi_3(a_2\mu\xi)+O(\langle\xi\rangle^{\gamma-2}). \tag{10.11}$$

Proof. We first integrate by parts with respect to y via identity $e^{\beta iy^2}=B\partial_y(ye^{\beta iy^2})$ with $B=(1+2i\beta y^2)^{-1}$ we get $\Phi=\Phi_1+\Phi_2+\Phi_3$, where

$$\Phi_1=2\int_{\mathbb{R}}e^{\beta iy^2}\phi_2(a_1\eta-b_1y)\phi_3(a_2\eta-b_2y)B(B-1)dy,$$
$$\Phi_2=b_1\int_{\mathbb{R}}e^{\beta iy^2}\phi_2'(a_1\eta-b_1y)\phi_3(a_2\eta-b_2y)yB\,dy,$$
$$\Phi_3=b_2\int_{\mathbb{R}}e^{\beta iy^2}\phi_2(a_1\eta-b_1y)\phi_3'(a_2\eta-b_2y)yB\,dy,$$

Now in the integral

$$I=\int_{\mathbb{R}}e^{\alpha i(\eta-\mu\xi)^2}A(\xi,\eta)\phi_1(\eta-\xi)\Phi_1(\eta)d\eta$$

we integrate by parts with respect to η via identity

$$e^{\alpha i(\eta-\mu\xi)^2}=H\partial_\eta((\eta-\mu\xi)e^{\alpha i(\eta-\mu\xi)^2})$$

with $H = (1+2i\alpha(\eta-\mu\xi)^2)^{-1}$ to obtain $I = I_1 + \cdots + I_5$, where

$$\begin{aligned}
I_1 &= 2\phi_1((\mu-1)\xi)\Phi_1(\mu\xi)A(\xi,\mu\xi)\int_{\mathbb{R}} e^{\alpha i(\eta-\mu\xi)^2}H(H-1)d\eta,\\
I_2 &= \int_{\mathbb{R}} e^{\alpha i(\eta-\mu\xi)^2}\phi_1(\eta-\xi)\Phi_1(\eta)(\eta-\mu\xi)H\partial_\eta A d\eta\\
I_3 &= \int_{\mathbb{R}} e^{\alpha i(\eta-\mu\xi)^2}\Big(\phi_1(\eta-\xi)\Phi_1(\eta)A(\xi,\eta)\\
&\quad - \phi_1((\mu-1)\xi)\Phi_1(\mu\xi)A(\xi,\mu\xi)\Big)H(H-1)d\eta\\
I_4 &= \int_{\mathbb{R}} e^{\alpha i(\eta-\mu\xi)^2}\phi_1(\eta-\xi)\Phi_1'(\eta)A(\eta-\mu\xi)Hd\eta\\
I_5 &= \int_{\mathbb{R}} e^{\alpha i(\eta-\mu\xi)^2}\phi_1'(\eta-\xi)\Phi_1(\eta)A(\eta-\mu\xi)Hd\eta
\end{aligned}$$

We prove that the integral I_1 is the main term. Since

$$2\int_{\mathbb{R}} e^{\beta i y^2}B(B-1)dy = \frac{\sqrt{i\pi}}{\sqrt{\beta}},$$

and $\langle y\rangle|\phi_j(a_i\mu\xi - b_i y) - \phi_j(a_i\mu\xi)| \le C\langle\xi\rangle^{\gamma-1}\langle y\rangle^{-\gamma}$, we have

$$\begin{aligned}
\Phi_1(\mu\xi) &= 2\phi_2(a_1\mu\xi)\phi_3(a_2\mu\xi)\int_{\mathbb{R}} e^{\beta i y^2}B(B-1)dy\\
&\quad + 2\int_{\mathbb{R}} e^{\beta i y^2}\Big(\phi_2(a_1\mu\xi-b_1y)\phi_3(a_2\mu\xi-b_2y)\\
&\quad - \phi_2(a_1\mu\xi)\phi_3(a_2\mu\xi)\Big)B(B-1)dy\\
&= \frac{\sqrt{i\pi}}{\sqrt{\beta}}\phi_2(a_1\mu\xi)\phi_3(a_2\mu\xi) + O\Big(\langle\xi\rangle^{\gamma-1}\int_{\mathbb{R}}\langle y\rangle^{-1-\gamma}dy\Big).
\end{aligned}$$

By a direct calculation,

$$2\int_{\mathbb{R}} e^{\alpha i(\eta-\mu\xi)^2}H(H-1)d\eta = \frac{1}{\sqrt{\alpha}}\int_{\mathbb{R}} e^{i\eta^2}d\eta = \frac{\sqrt{i\pi}}{\sqrt{\alpha}}.$$

Therefore,

$$I_1 = \frac{i\pi}{\sqrt{\alpha\beta}}A(\xi,\mu\xi)\phi_1((\mu-1)\xi)\phi_2(a_1\mu\xi)\phi_3(a_2\mu\xi) + O\big(\langle\xi\rangle^{\gamma-2}\big).$$

We have

$$\begin{aligned}
|\Phi_1(\eta)| &\le C\int_{\mathbb{R}}\langle a_1\eta-b_1y\rangle^{-\gamma_1}\langle a_2\eta-b_2y\rangle^{-\gamma_2}\langle y\rangle^{-2}dy \le C\langle\eta\rangle^{-\gamma_1-\gamma_2},\\
|\Phi_1'(\eta)| &\le C\int_{\mathbb{R}}\langle a_1\eta-b_1y\rangle^{\gamma-1}\langle a_2\eta-b_2y\rangle^{-\gamma_2}\langle y\rangle^{-2}dy\\
&\quad + C\int_{\mathbb{R}}\langle a_1\eta-b_1y\rangle^{-\gamma_1}\langle a_2\eta-b_2y\rangle^{\gamma-1}\langle y\rangle^{-2}dy \le C\langle\eta\rangle^{\gamma-1}
\end{aligned}$$

Hence

$$|I_2| \le C\langle\xi\rangle^{\gamma-2}\int_{\mathbb{R}}\langle\eta-\mu\xi\rangle^{-1}\langle\eta\rangle^{-\gamma}d\eta \le C\langle\xi\rangle^{\gamma-2}.$$

Since

$$\langle\eta-\mu\xi\rangle^{-1}|\phi(\eta)-\phi(\mu\xi)| \le C\langle\xi\rangle^{\gamma-1}\langle\eta\rangle^{-\gamma}$$

and

$$\langle\eta-\mu\xi\rangle^{-1}|A(\xi,\eta)-A(\xi,\mu\xi)|\leq C\langle\xi\rangle^{\gamma-2}\langle\eta\rangle^{-\gamma}$$

we find

$$|I_3|\leq C\langle\xi\rangle^{\gamma-2}\int_{\mathbb{R}}\langle\eta\rangle^{-\gamma}\langle\eta-\mu\xi\rangle^{-1}d\eta\leq C\langle\xi\rangle^{\gamma-2}.$$

We also have

$$|I_4|+|I_5|\leq C\langle\xi\rangle^{-1}\int_{\mathbb{R}}(\langle\eta\rangle^{\gamma-1}+\langle\eta-\xi\rangle^{\gamma-1})\langle\eta-\mu\xi\rangle^{-1}d\eta\leq C\langle\xi\rangle^{\gamma-2}.$$

In the integral Φ_2 we change $a_1\eta-b_1y=y'$ (the prime we will omit) then with

$$\widetilde{a_2}=\frac{1}{b_1}(a_2b_1-a_1b_2)\neq 0,\quad \widetilde{b_2}=\frac{b_2}{b_1},$$
$$\widetilde{B}_1=(1+\frac{2\beta}{b_1^2}i(a_1\eta-y)^2)^{-1},\quad \beta_1=\frac{\beta}{b_1^2}$$

we have

$$\Phi_2=C\int_{\mathbb{R}}e^{i\beta_1(a_1\eta-y)^2}\phi_2'(y)\phi_3(\widetilde{a_2}\eta-\widetilde{b_2}y)(a_1\eta-y)\widetilde{B}dy.$$

Then with $Q=\beta_1(a_1\eta-y)^2+\alpha(\eta-\mu\xi)^2$. we define

$$I_6=C\int_{\mathbb{R}}dy\phi_2'(y)\int_{\mathbb{R}}A(\xi,\eta)\phi_1(\eta-\xi)e^{iQ}\phi_3(\widetilde{a_2}\eta-\widetilde{b_2}y)(a_1\eta-y)\widetilde{B}d\eta\,.$$

Then we integrate by parts with respect to η via identity $e^{iQ}=H\partial_\eta(\widetilde{\eta}e^{iQ})$ with $H=(1+i\widetilde{\eta}Q_\eta)^{-1}$, where

$$\widetilde{\eta}=\eta-\widetilde{b}y-\widetilde{a}\xi,\quad \widetilde{b}=\frac{a_1\beta_1}{\alpha+\beta_1a_1^2},\quad \widetilde{a}=\frac{\alpha\mu}{\alpha+\beta_1a_1^2},$$

if $\alpha+\beta_1a_1^2\neq 0$ and $\widetilde{\eta}=\eta$ if $\alpha+\beta_1a_1^2=0$ we get

$$I_6=C\int_{\mathbb{R}}dy\phi_2'(y)\int_{\mathbb{R}}e^{iQ}\widetilde{\eta}\partial_\eta HA(\xi,\eta)\phi_1(\eta-\xi)\phi_3(\widetilde{a_2}\eta-\widetilde{b_2}y)(a_1\eta-y)\widetilde{B}d\eta$$

since $\langle\widetilde{\eta}Q_\eta\rangle^{-1}\leq C\langle\widetilde{\eta}\rangle^{-2}$ if $\alpha+\beta_1a_1^2\neq 0$ and

$$\langle\widetilde{\eta}Q_\eta\rangle^{-1}\leq C|\widetilde{\eta}|^{\gamma-1}|a_1\beta_1y+\alpha\mu\xi|^{\gamma-1}$$

if $\alpha+\beta_1a_1^2=0$ we obtain

$$\begin{aligned}|I_6|&\leq C\langle\xi\rangle^{-1}\int_{\mathbb{R}}dy\langle y\rangle^{\gamma-1}\int_{\mathbb{R}}\langle\eta-\frac{y}{a_1}\rangle^{-1}\langle\eta-\widetilde{b}y-\widetilde{a}\xi\rangle^{-1}d\eta\\&\leq C\langle\xi\rangle^{-1}\int_{\mathbb{R}}dy\langle y\rangle^{\gamma-1}\langle a_3y-\widetilde{a}\xi\rangle^{\gamma-1}\leq C\langle\xi\rangle^{2\gamma-2}\end{aligned}$$

if $\alpha+\beta_1a_1^2\neq 0$, since a_1, $\widetilde{a}$, $a_3\equiv\frac{1-a_1\widetilde{b}}{a_1}\neq 0$, and

$$\begin{aligned}|I_6|\leq C\langle\xi\rangle^{\gamma-1}\int_{\mathbb{R}}dy\langle y\rangle^{\gamma-1}|a_1\beta_1y+\alpha\mu\xi|^{\gamma-1}\int_{\mathbb{R}}\langle a_1\eta-y\rangle^{-1}\Big(|\eta|^{\gamma-1}\\+\langle\eta-\xi\rangle^{\gamma-1}+\langle\widetilde{a_2}\eta-\widetilde{b_2}y\rangle^{\gamma-1}+\langle a_1\eta-y\rangle^{-1}\Big)d\eta\leq C\langle\xi\rangle^{2\gamma-2}\end{aligned}$$

if $\alpha+\beta_1a_1^2=0$. In the same manner we estimate the integral with Φ_3. Thus we have the asymptotic form (10.11). Lemma 10.2 is proved. □

Now we substitute (10.7) $w(\xi)=\psi_3(\xi)-e^{\frac34 i\xi^2}\psi_2(\xi)$ into the second summand in the right-hand side of (10.2) as above changing the variables of integration $y=y'+\frac{\eta}{2}$ or $y=2\eta-y'$ we find

$$\begin{aligned}\int_{\mathbb{R}} e^{-iQ}w(\eta-y)w(y)dy &= e^{-\frac34 i\eta^2}\int_{\mathbb{R}} e^{-iy^2}\psi_3(\frac{\eta}{2}-y)\psi_3(\frac{\eta}{2}+y)dy\\ &\quad -2\int_{\mathbb{R}} e^{-\frac14 iy^2}\psi_3(y-\eta)\psi_2(2\eta-y)dy\\ &\quad + e^{-\frac38 i\eta^2}\int_{\mathbb{R}} e^{\frac12 iy^2}\psi_2(\frac{\eta}{2}-y)\psi_2(\frac{\eta}{2}+y)dy.\end{aligned}$$

So we have

$$\begin{aligned}&\int_{\mathbb{R}} d\eta e^{iS}K\eta\overline{w(\eta-\xi)}\int_{\mathbb{R}} e^{-iQ}w(\eta-y)w(y)dy\\ &= \int_{\mathbb{R}} e^{\frac14 i(\eta-2\xi)^2}K\eta\overline{\psi_3(\eta-\xi)}\Psi_1(\eta)d\eta\\ &\quad + 2e^{\frac34 i\xi^2}\int_{\mathbb{R}} e^{i(\eta-\frac12\xi)^2}K\eta\overline{\psi_3(\eta-\xi)}\Psi_2(\eta)d\eta\\ &\quad + e^{\frac35 i\xi^2}\int_{\mathbb{R}} e^{\frac58 i(\eta-\frac45\xi)^2}K\eta\overline{\psi_3(\eta-\xi)}\Psi_3(\eta)d\eta\\ &\quad + e^{\frac38 i\xi^2}\int_{\mathbb{R}} e^{-\frac12 i(\eta-\frac12\xi)^2}K\eta\overline{\psi_2(\eta-\xi)}\Psi_1(\eta)d\eta\\ &\quad + 2\int_{\mathbb{R}} e^{\frac14 i(\eta+\xi)^2}K\eta\overline{\psi_2(\eta-\xi)}\Psi_2(\eta)d\eta\\ &\quad + e^{\frac34 i\xi^2}\int_{\mathbb{R}} e^{-\frac18 i(\eta-2\xi)^2}K\eta\overline{\psi_2(\eta-\xi)}\Psi_3(\eta)d\eta\end{aligned}$$

where

$$\begin{aligned}\Psi_1(\eta) &= \int_{\mathbb{R}} e^{-iy^2}\psi_3(\frac{\eta}{2}-y)\psi_3(\frac{\eta}{2}+y)dy,\\ \Psi_2(\eta) &= \int_{\mathbb{R}} e^{-\frac14 iy^2}\psi_3(y-\eta)\psi_2(2\eta-y)dy,\\ \Psi_3(\eta) &= \int_{\mathbb{R}} e^{\frac12 iy^2}\psi_2(\frac{\eta}{2}-y)\psi_2(\frac{\eta}{2}+y)dy,\end{aligned}$$

Applying Lemma 10.2 we obtain the asymptotic form

$$\begin{aligned}&\int_{\mathbb{R}} d\eta e^{iS}K\eta\overline{w(\eta-\xi)}\int_{\mathbb{R}} e^{-iQ}w(\eta-y)w(y)dy\\ &= \int_{\mathbb{R}} e^{\frac14 i(\eta-2\xi)^2}K\eta\overline{\psi_3(\eta-\xi)}\int_{\mathbb{R}} e^{-iy^2}\psi_3(\frac{\eta}{2}-y)\psi_3(\frac{\eta}{2}+y)dyd\eta + O(\langle\xi\rangle^{\gamma-2})\\ &= -\frac{4i\pi}{3\langle\xi\rangle}\overline{\psi_3(\xi)}\psi_3(\xi)\psi_3(\xi) + O(\langle\xi\rangle^{\gamma-2})\\ &= -\frac{4i\pi}{3\langle\xi\rangle}|\phi(\xi)|^2\phi(\xi) + O(\langle\xi\rangle^{\gamma-2}).\end{aligned}$$

Thus we can estimate the right-hand side of (10.5)

$$\frac{d}{d\xi}\mathcal{A}(\varphi)(\xi) = 2\overline{E}\int_{\mathbb{R}} Fe^{iS}\overline{w(\eta-\xi)w(-\eta)}d\eta d\xi - \frac{16i\pi}{3\langle\xi\rangle}|\varphi(\xi)|^2\varphi(\xi)$$

$$
\begin{aligned}
&- 4\overline{E}\int_{\mathbb{R}} d\eta e^{iS} K\eta\overline{w(\eta-\xi)}\int_{\mathbb{R}} e^{-iQ} w(\eta-y)w(y)dy \\
&= O(\varepsilon^3\langle\xi\rangle^{\gamma-2}).
\end{aligned}
$$

We solve equation (10.4) by the contraction mapping principle in the set

$$\mathbf{X}_\varepsilon = \{\varphi \in \mathbf{C}^1(\mathbb{R}) : \|\varphi\|_{\mathbf{X}} \le C\varepsilon^2\}$$

where the norm

$$\|\varphi\|_{\mathbf{X}} \equiv \sup_{\xi\in\mathbb{R}} \big(|\varphi(\xi)| + \langle\xi\rangle^{2-2\gamma}|\varphi'(\xi)|\big).$$

When $\varphi \in \mathbf{X}_\varepsilon$, then $\|\phi\|_{\mathbf{Y}} \equiv \sup_{\xi\in\mathbb{R}}(|\phi(\xi)| + \langle\xi\rangle^{1-2\gamma}|\phi'(\xi)|) \le C\varepsilon$ and by Lemma 10.1

$$\|\psi_1\|_{\mathbf{Z}} + \|\psi_2\|_{\mathbf{Z}} + \|\psi_3\|_{\mathbf{Z}} \le C\varepsilon.$$

Then by Lemma 10.2

$$|\frac{d}{d\xi}\mathcal{A}(\varphi)(\xi)| \le C\varepsilon\langle\xi\rangle^{2\gamma-2}.$$

And integrating we have

$$|\mathcal{A}(\varphi)(\xi)| \le C\varepsilon\int_0^\xi \langle\xi\rangle^{2\gamma-2}d\xi + |\varphi(0)| \le C\varepsilon.$$

In the same manner we can estimate the difference

$$|\mathcal{A}(\varphi_1) - \mathcal{A}(\varphi_2)| \le C\varepsilon\|\varphi_1 - \varphi_2\|_{\mathbf{X}}.$$

Therefore, $\mathcal{A}$ is a contraction mapping in $\mathbf{X}$. Hence there exists a unique solution $\varphi \in \mathbf{X}$ of integral equation (10.4).

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