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BIFURCATION AND MULTIPLICITY RESULTS FOR A NONHOMOGENEOUS SEMILINEAR ELLIPTIC PROBLEM

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ABSTRACT. In this article we consider the problem

$$-\Delta u(x) + u(x) = \lambda(a(x)u^p + h(x)) \quad \text{in } \mathbb{R}^N,$$
$$u \in H^1(\mathbb{R}^N), \quad u > 0 \quad \text{in } \mathbb{R}^N,$$

where λ is a positive parameter. We assume there exist $\mu > 2$ and $C > 0$ such that $a(x) - 1 \geq -Ce^{-\mu|x|}$ for all $x \in \mathbb{R}^N$. We prove that there exists a positive λ^* such that there are at least two positive solutions for $\lambda \in (0, \lambda^*)$ and a unique positive solution for $\lambda = \lambda^*$. Also we show that $(\lambda^*, u(\lambda^*))$ is a bifurcation point in $C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$.

1. Introduction

We consider the existence and properties of multiple positive solutions for the nonhomogeneous semilinear elliptic problem

$$\begin{gathered} -\Delta u(x) + u(x) = \lambda(a(x)u^p + h(x)) \quad \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \quad u > 0 \quad \text{in } \mathbb{R}^N, \end{gathered} \tag{1.1}$$

where $\lambda > 0$, $1 < p < \frac{N+2}{N-2}$ $(N \geq 3)$, $1 < p < \infty$ $(N = 1, 2)$, $h(x) \in H^{-1}(\mathbb{R}^N)$, $0 \not\equiv h(x) \geq 0$ in $\mathbb{R}^N$, $h(x) \to 0$ as $|x| \to \infty$, and $a(x)$ satisfies the following conditions:

- (A1) $a(x) \in C(\mathbb{R}^N)$; $a(x) \in (0, 1]$ for all $x \in \mathbb{R}^N$; $a(x) \to 1$ as $|x| \to \infty$; $a(x) \not\equiv 1$;
- (A2) there exist $\mu > 2$ and $C > 0$ such that

$$a(x) - 1 \geq -Ce^{-\mu|x|} \quad \text{for all } x \in \mathbb{R}^N.$$

Here the constant μ corresponds to a convergent rate (from below), and the condition $\mu > 2$ play an important role in our existence result.

The main result is as follows.

Theorem 1.1. *Assume* (A1)–(A2) *hold.*

(1) *There exists λ^*, $0 < \lambda^* < \infty$, such that* (1.1) *has at least two distinct positive solutions $u(\lambda)$, $U(\lambda)$ for any $\lambda \in (0, \lambda^*)$, $u(\lambda), U(\lambda) \in C^{2,\alpha}(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ for any $\lambda \in (0, \lambda^*)$, and a unique positive solution $u(\lambda^*)$ for $\lambda = \lambda^*$; moreover, $u(\lambda)$ is the minimal positive solution of* (1.1)*, $u(\lambda)$ is strictly*

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increasing in λ *for* $\lambda \in (0, \lambda^*)$, *and* $\lambda \to u(\lambda) \in C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$ *is continuous.*

(2) *Denote by* $Q = \{(\lambda, u) : u solves\ (1.1)\}$, *the set of positive solutions of* (1.1), *for any* $(\lambda, u) \in Q$, $u(x)$ *and* $|\nabla u(x)|$ *have uniform limit zero as* $|x| \to \infty$.

(3) $(\lambda^*, u(\lambda^*)) \in C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$ *is a bifurcation point for problem* (1.1). *Consequently* Q *is unbounded in* $(0, \lambda^*) \times C^{2,\alpha}(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ *but is bounded in* $[\epsilon, \lambda^*] \times C^{1,\alpha}(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ *for any* $\epsilon > 0$.

We write $u_\pm(x) = \max\{\pm u(x), 0\}$, $\|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)dx$, and we define for given $a(x)$ and $h(x)$,

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{p+1}\int_{\mathbb{R}^N} a(x)u_+^{p+1}dx - \lambda\int_{\mathbb{R}^N} h(x)u\,dx.$$

Many authors have studied the existence of positive solutions of the semilinear elliptic boundary value problem

$$\begin{gathered} -\Delta u + u = g(x, u), \quad x \in \Omega \subset \mathbb{R}^N, \\ u|_{\partial\Omega} = 0, \end{gathered} \tag{1.2}$$

a problem that occurs in varies branches of geometry and mathematical physics. There are many results about the existence of the positive solutions of (1.2) when $g(x, u)$ is a "homogeneous" function (i.e. $g(x, 0) \equiv 0$), see [5], [6], [11], [17]. For the "nonhomogeneous" function (i.e. $g(x, 0) \not\equiv 0$), some existence of two solutions have been obtained in [12] when $g(x, u)$ is less than critical growth in the sense that $\lim_{u\to\infty} \frac{g(x,u)}{u^q} = 0$ with $q = \frac{N+2}{N-2}$ and Ω is bounded. A substantial difference between the problems on bounded domain and on unbounded domain is the lack of compactness for Sobolev embedding when we deal with the latter. Thus there seems to be little progress on the existence theory for the "nonhomogeneous" function of (1.2) when Ω is unbounded. Zhu [19], Zhu-Zhou [20], Cao-Zhou [8], Chen [9], and Jeanjean [15] showed the existence of multiple positive solutions in unbounded domains under the assumption:

$$a(x) \geq 1 \quad \text{for all } x \in \mathbb{R}^N \tag{1.3}$$

or

$$g(x, u) \geq g^\infty(u) \quad \Big(= \lim_{|x|\to\infty} g(x, u)\Big) \quad \text{for all } x \in \mathbb{R}^N \text{ and } u \geq 0. \tag{1.4}$$

Now we will show the existence of solutions without assuming (1.3), (1.4).

In Section 2, we assert that there exists λ^*, $\infty > \lambda^* > 0$, such that (1.1) has a minimal positive solution for all $\lambda \in (0, \lambda^*)$ by the implicit function theorem and the barrier method. In Section 3, by the variational method and the concentration-compactness principle, we establish the existence of second positive solution, the following energy estimate (1.5) plays an important role. We find $R_0 \geq 1$ such that

$$\sup_{t\geq 0} J(t\omega(x - Re)) < I^\infty(\omega) \quad \text{for all } R \geq R_0, \tag{1.5}$$

where J is the energy functional of (3.1), e is a fixed unit vector in $\mathbb{R}^N$, $\omega(x)$ is a ground state solution of the limit problem (3.2) and I^∞ is the energy functional of (3.2). To obtain the energy estimate, assumption (A2) plays an important role. In Section 4, we discuss problem (1.1) has at least two distinct positive solutions for any $\lambda \in (0, \lambda^*)$, a unique positive solution for $\lambda = \lambda^*$, $u(\lambda^*)$ is a bifurcation point in $C^{2,\alpha}(\mathbb{R}^N)\cap H^2(\mathbb{R}^N)$, and further analyzes of the set of positive solutions are made.

2. Existence of minimal positive solution

In this section, the implicit function theorem and the barrier method enable us to deduce the existence of a minimal positive solution $u(\lambda)$ of (1.1) for all λ in some finite interval $(0,\lambda^*)$.

Lemma 2.1. *For each $h\in C^\alpha(\mathbb{R}^N)\cap L^2(\mathbb{R}^N)$, the linear problem*
$$-\Delta u+u=h,\quad u\in H^1(\mathbb{R}^N) \tag{2.1}$$
has a solution $u\in C^{2,\alpha}(\mathbb{R}^N)\cap H^2(\mathbb{R}^N)$.

Proof. Consider the functional $\Phi(u)=\frac12\|u\|^2-\int_{\mathbb{R}^N} hudx$, $u\in H^1(\mathbb{R}^N)$. From the Hölder's and Young's inequality we have, for any $\epsilon>0$ that
$$\begin{aligned}\Phi(u)&\ge \frac12\|u\|^2-\|h\|_{L^2}\|u\|_{L^2}\\ &\ge (\frac12-\epsilon)\|u\|^2-C_\epsilon\|h\|^2_{L^2}\\ &\ge -C_\epsilon\|h\|^2_{L^2}\end{aligned} \tag{2.2}$$
if we choose ϵ small.

Let $\{u_n\}\subset H^1(\mathbb{R}^N)$ be a minimizing sequence of the variational problem $d=\inf\{\Phi(u):u\in H^1(\mathbb{R}^N)\}$. From (2.2),
$$\big(\frac12-\epsilon\big)\|u_n\|^2\le\Phi(u_n)+C_\epsilon\|h\|^2_{L^2}=d+C_\epsilon\|h\|^2_{L^2}+o(1)\quad\text{as } n\to\infty,$$
we can also deduce that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ if we choose ϵ small. Thus we may assume that
$$u_n\to u\quad\text{weakly in } H^1(\mathbb{R}^N)\quad\text{as } n\to\infty,$$
$$u_n\to u\quad\text{a.e. in } \mathbb{R}^N\quad\text{as } n\to\infty.$$
By Fatou's lemma,
$$\|u\|^2\le\liminf_{n\to\infty}\|u_n\|^2.$$
The weak convergence implies $\int_{\mathbb{R}^N} hu_ndx\to\int_{\mathbb{R}^N} hudx$ as $n\to\infty$, therefore $\Phi(u)\le\lim_{n\to\infty}\Phi(u_n)=d$ and hence $\Phi(u)=d$.

By the assumption $h\in C^\alpha(\mathbb{R}^N)\cap L^2(\mathbb{R}^N)$, from [18, Proposition 4.3] that $u\in H^2(\mathbb{R}^N)$; the standard elliptic regular theorem yield $u\in C^{2,\alpha}(\mathbb{R}^N)$. □

Theorem 2.2. *Assume* (A1)–(A2) *hold. Then there exists $\lambda^*>0$ such that for any $\lambda\in(0,\lambda^*)$,* (1.1) *has a minimal positive solution $u(\lambda)$; $u(\lambda)$ is strictly increasing in λ for $\lambda\in(0,\lambda^*)$; $\lambda\to u(\lambda)\in C^{2,\alpha}(\mathbb{R}^N)\cap H^2(\mathbb{R}^N)$ is continuous.*

Proof. Define $F:\mathbb{R}\times C^{2,\alpha}(\mathbb{R}^N)\cap H^2(\mathbb{R}^N)\to C^\alpha(\mathbb{R}^N)\cap L^2(\mathbb{R}^N)$ by
$$F(\lambda,u)=-\Delta u+u-\lambda(a(x)u^p+h(x)),$$
by Lemma 2.1, $F_u(0,0)u=-\Delta u+u$ is an isomorphism of $C^{2,\alpha}(\mathbb{R}^N)\cap H^2(\mathbb{R}^N)$ onto $C^\alpha(\mathbb{R}^N)\cap L^2(\mathbb{R}^N)$. Applying the implicit function theorem to F, we can find that the solutions of $F(\lambda,u)=0$ near $(0,0)$ are given by a continuous curve $(\lambda,u(\lambda))$ with $u(0)=0$.

Denote $\lambda^* = \sup\{\lambda > 0$:there exists $u(\lambda)$ such that $u(\lambda)$ is a unique continuous function of λ into $C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$, $F(\lambda, u(\lambda)) = 0$ and $F_u(\lambda, u(\lambda))$ is nonsingular$\}$, then $0 < \lambda^* \le +\infty$; by the maximum principle, $u(\lambda) > 0$ for all $\lambda \in (0, \lambda^*)$.

Let λ, $\lambda' \in (0, \lambda^*)$, $\lambda < \lambda'$, $u(\lambda)$, $u(\lambda')$ are corresponding positive solutions, then $u(\lambda')$ is a supersolution of (1.1). From $h(x) \ge 0$ we deduce that 0 is obviously a subsolution of (1.1); the standard barrier method and the maximun principle enable us to find a solution $v(\lambda)$ such that $0 < v(\lambda) < u(\lambda')$. Moreover, we may select $v(\lambda)$ as a minimal positive solution of (1.1). In the same way one finds a minimal positive solution $v(\lambda)$ of (1.1) for each $\lambda \in (0, \lambda^*)$, $v(\lambda)$ is strictly increasing in λ for $\lambda \in (0, \lambda^*)$. The uniqueness of $u(\lambda)$ yields $u(\lambda) = v(\lambda)$ for $\lambda \in (0, \lambda^*)$. □

We establish the decay estimate for positive solutions of (1.1) which we will use later on.

Theorem 2.3. *Assume* (A1) *holds. If* $u \in H^1(\mathbb{R}^N)$ *is a positive solution of* (1.1), *then*

(1) $u \in L^q(\mathbb{R}^N)$ *for* $q \in [2, \infty)$;
(2) $u(x)$ *and* $|\nabla u(x)|$ *have uniform limit zero as* $|x| \to \infty$.

Proof. (1) follows by the classical regularity theory based on a result of Brezis-Kato [7]. We will write it in detail for the reader's convenience. For $s \ge 0$, $l \ge 1$, let $\varphi = \varphi_{s,l} = u \min\{|u|^{2s}, l^2\} \in H^1(\mathbb{R}^N)$. Testing (1.1) with φ, then

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla u|^2 \min\{|u|^{2s}, l^2\} dx + 2s \int_{\{|u|^s \le l\}} |\nabla u|^2 |u|^{2s} dx \\ &\le C \int_{\mathbb{R}^N} |u|^2 dx + C \int_{\mathbb{R}^N} |u|^{2+2s} dx + C \int_{\mathbb{R}^N} |u|^{p+1} dx \\ &\quad + C \int_{\mathbb{R}^N} |u|^{2s+p+1} dx + C \int_{\mathbb{R}^N} |h||u| \min\{|u|^{2s}, l^2\} dx. \end{aligned} \tag{2.3}$$

Suppose $u \in L^{2s+p+1}(\mathbb{R}^N)$, then by (2.3) and applying the H ölder's inequality and the Sobolev embedding, we have

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla (u \min\{|u|^s, l\})|^2 dx \\ &\le \int_{\mathbb{R}^N} |\nabla u|^2 \min\{|u|^{2s}, l^2\} dx + (s^2 + 2s) \int_{\{|u|^s \le l\}} |\nabla u|^2 |u|^{2s} dx \\ &\le C + C \int_{\mathbb{R}^N} |h||u| dx + C \int_{\mathbb{R}^N} |h||u|^2 \min\{|u|^{2s}, l^2\} dx \\ &\le C + CK \int_{\mathbb{R}^N} |u|^{2s+2} dx + C \int_{\{|h| \ge K\}} |h||u \min\{|u|^s, l\}|^2 dx \\ &\le C(1+K) + C\Big(\int_{\{|h| \ge K\}} |h|^{\frac{N}{2}} dx \Big)^{2/N} \Big(\int_{\mathbb{R}^N} |u \min\{|u|^s, l\}|^{\frac{2N}{N-2}} dx \Big)^{\frac{N-2}{N}} \\ &\le C(1+K) + C\varepsilon(K) \int_{\mathbb{R}^N} |\nabla (u \min\{|u|^s, l\})|^2 dx, \end{aligned}$$

where

$$\varepsilon(K) = \Big(\int_{\{|h| \ge K\}} |h|^{\frac{N}{2}} dx \Big)^{2/N} \to 0 \quad \text{as } K \to \infty.$$

Fix K such that $\varepsilon(K) = 1/2C$ and observe that for this choice of K (and s as above) we now may conclude that

$$\int_{\{|u|^s \le l\}} |\nabla(|u|^{s+1})|^2 dx \le C \int_{\mathbb{R}^N} |\nabla(u \min\{|u|^s, l\})|^2 dx \le C(1+K)$$

for any $l \ge 1$. Hence we may let $l \to \infty$ to derive that

$$|u|^{s+1} \in H^1(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N).$$

It is easy to see that $u \in L^{\frac{(s+1)2N}{N-2}}(\mathbb{R}^N)$.

Now iterate, letting $s_0 = 0$, $2s_i + p + 1 = (s_{i-1} + 1)\frac{2N}{N-2}$ for $i \ge 1$, then $u \in L^{2s_{i-1}+p+1}(\mathbb{R}^N)$ implies $u \in L^{2s_i+p+1}(\mathbb{R}^N)$. It is easily to see $s_i \to \infty$ as $i \to \infty$. Therefore $u \in L^q(\mathbb{R}^N)$ for $2 \le q < \infty$.

(2) Obviously u satisfies the linear problem

$$-\Delta u + u = F(x) = \lambda(a(x)u^p + h(x)), x \in \mathbb{R}^N, u \in H^1(\mathbb{R}^N).$$

Choose $q > \max\{\frac{Np}{2}, \frac{2N}{N-2}\}$, by the Hölder's inequality in $B_2(x)$ we get

$$\|u\|_{L^2(B_2(x))} \le C\|u\|_{L^q(B_2(x))},$$

then

$$\|F\|_{L^{\frac{q}{p}}(B_2(x))} \le C\left(\|u\|_{L^q(B_2(x))} + \|h\|_{L^q(B_2(x))}\right).$$

It is deduced by elliptic regular theory that $u \in C^{2,\alpha}(\mathbb{R}^N)$. By [14, Theorem 8.24], we have

$$\|u\|_{C^\alpha(B_1(x))} \le C\left(\|u\|_{L^q(B_2(x))} + \|h\|_{L^q(B_2(x))}\right), \tag{2.4}$$

then $u(x) \to 0$ as $|x| \to \infty$ since $u \in L^q(\mathbb{R}^N)$.

By [14, Theorem 8.32],

$$\|u\|_{C^{1,\alpha}(B_1(x))} \le C\left(\|u\|_{C^\alpha(B_2(x))} + \|h\|_{L^\infty(B_2(x))}\right). \tag{2.5}$$

Then (2.4), (2.5) give $|\nabla u(x)| \to 0$ as $|x| \to \infty$. $\square$

We will verify that λ^* is finite by considering linear eigenvalue problems related to decaying positive solutions of (1.1).

Theorem 2.4. *Assume* (A1)-(A2) *hold. Then*

(1) *the minimization problem*

$$\begin{aligned}\mu_\lambda &= \mu_\lambda(u(\lambda))\\ &= \inf\Big\{\int_{\mathbb{R}^N}(|\nabla v|^2 + v^2)dx : v \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} pa(x)u(\lambda)^{p-1}v^2 dx = 1\Big\},\end{aligned}$$

where $u(\lambda)$ is the minimal positive solution of (1.1)*, can be achieved by some $\overline{v}_\lambda > 0$;*

(2) *$\mu_\lambda > \lambda$, μ_λ is strictly decreasing in λ for $\lambda \in (0, \lambda^*)$;*

(3) *λ^* is finite.*

Proof. It is easy to see that $0 < \mu_\lambda < \infty$. Let $\{v_n\} \subset H^1(\mathbb{R}^N)$ be a minimizing sequence of μ_λ, then $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Without loss of generality (at least by choosing a subsequence) we can assume that, for some $\overline{v}_\lambda \in H^1(\mathbb{R}^N)$,

$$\begin{gathered} v_n \rightharpoonup \overline{v}_\lambda \quad \text{weakly in } H^1(\mathbb{R}^N), \\ v_n \to \overline{v}_\lambda \quad \text{a.e. in } \mathbb{R}^N, \\ v_n \to \overline{v}_\lambda \quad \text{strongly in } L^s_{\rm loc}(\mathbb{R}^N) \quad \text{for } 2 \le s < \frac{2N}{N-2}. \end{gathered}$$

By Theorem 2.3 and (A1), we have $pa(x)u(\lambda)^{p-1} \to 0$ as $|x| \to \infty$, hence for any $\varepsilon > 0$, there exists $R > 0$ such that for $x \in \mathbb{R}^N$, $|x| \ge R$, $|pa(x)u(\lambda)^{p-1}| < \varepsilon$. Consequently, there exists a constant $C > 0$ such that $|pa(x)u(\lambda)^{p-1}| \le C$ for $x \in \mathbb{R}^N$. Then

$$\begin{aligned} &\int_{\mathbb{R}^N} pa(x)u(\lambda)^{p-1}|v_n - \overline{v}_\lambda|^2 dx \\ &= \int_{B_R} pa(x)u(\lambda)^{p-1}|v_n - \overline{v}_\lambda|^2 dx + \int_{\mathbb{R}^N\setminus B_R} pa(x)u(\lambda)^{p-1}|v_n - \overline{v}_\lambda|^2 dx \\ &\le C\int_{B_R} |v_n - \overline{v}_\lambda|^2 dx + \varepsilon\int_{\mathbb{R}^N\setminus B_R} |v_n - \overline{v}_\lambda|^2 dx. \end{aligned}$$

Since $v_n \to \overline{v}_\lambda$ strongly in $L^s_{\rm loc}(\mathbb{R}^N)$ for $2 \le s < 2N/N-2$, $\{v_n\}$ is a bounded sequence in $H^1(\mathbb{R}^N)$, taking $n \to \infty$, then $\varepsilon \to 0$, we obtain $\int_{\mathbb{R}^N} pa(x)u(\lambda)^{p-1}\overline{v}_\lambda^2 = 1$. Therefore $\overline{v}_\lambda$ achieves μ_λ. Clearly $|\overline{v}_\lambda|$ also achieves μ_λ. Hence we may assume $\overline{v}_\lambda \ge 0$ in $\mathbb{R}^N$ and $\overline{v}_\lambda$ satisfies $-\Delta v + v = \mu_\lambda pa(x)u(\lambda)^{p-1}v$. Once again, by the maximum principle we deduce that $\overline{v}_\lambda > 0$ in $\mathbb{R}^N$.

To prove (2). Setting $0 < \lambda < \lambda'$ and $\lambda,\ \lambda' \in (0, \lambda^*)$, by Theorem 2.2, $u(\lambda') > u(\lambda)$ as $\lambda' > \lambda$. Noting that $\lambda' > \lambda$, $h(x) \ge 0$, $a(x)u(\lambda)^p > 0$, we get

$$\begin{aligned} &-\Delta(u(\lambda') - u(\lambda)) + (u(\lambda') - u(\lambda)) \\ &= \lambda' a(x)u(\lambda')^p - \lambda a(x)u(\lambda)^p + (\lambda' - \lambda)h(x) \\ &= (\lambda' - \lambda)a(x)u(\lambda)^p + \lambda'(a(x)u(\lambda')^p - a(x)u(\lambda)^p) + (\lambda' - \lambda)h(x) \\ &> \lambda pa(x)u(\lambda)^{p-1}(u(\lambda') - u(\lambda)). \end{aligned} \tag{2.6}$$

Multiplying (2.6) by $\overline{v}_\lambda$ and integrating it over $\mathbb{R}^N$, we get

$$\mu_\lambda \int_{\mathbb{R}^N} pa(x)u(\lambda)^{p-1}(u(\lambda') - u(\lambda))\overline{v}_\lambda dx > \lambda \int_{\mathbb{R}^N} pa(x)u(\lambda)^{p-1}(u(\lambda') - u(\lambda))\overline{v}_\lambda dx,$$

which implies that $\mu_\lambda > \lambda$, $\lambda \in (0, \lambda^*)$.

Furthermore, let $\lambda,\ \lambda' \in (0, \lambda^*)$, $\lambda < \lambda'$; by Theorem 2.2,

$$\int_{\mathbb{R}^N} pa(x)u(\lambda')^{p-1}\overline{v}_\lambda^2 dx > \int_{\mathbb{R}^N} pa(x)u(\lambda)^{p-1}\overline{v}_\lambda^2 dx = 1,$$

so that there exists $0 < t < 1$ such that $\int_{\mathbb{R}^N} pa(x)u(\lambda')^{p-1}t^2\overline{v}_\lambda^2 dx = 1$, therefore $\mu_{\lambda'} \le t^2\|\overline{v}_\lambda\|^2 < \|\overline{v}_\lambda\|^2 = \mu_\lambda$; that is μ_λ is strictly decreasing in λ for $\lambda \in (0, \lambda^*)$.

Finally, we show that λ^* is finite. Fix any $\lambda_0 \in (0, \lambda^*)$, then by (2), $\lambda < \mu_\lambda < \mu_{\lambda_0} < \infty$ for all $\lambda \in (\lambda_0,\ \lambda^*)$ and this implies $\lambda^* < +\infty$. $\square$

Remark 2.5. Let

$$\mu_\lambda(u) = \inf\Big\{\int_{\mathbb{R}^N} (|\nabla v|^2 + v^2)dx : v \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} pa(x)u^{p-1}v^2 dx = 1\Big\},$$

where u is any positive solution of (1.1), then from the proof of Theorem 2.4(1), $\mu_\lambda(u)$ also can be achieved.

3. Existence of second positive solution

When $\lambda \in (0, \lambda^*)$, we have shown that (1.1) has the minimal positive solution $u(\lambda)$ by Theorem 2.2. In this section, we want to prove that (1.1) has another positive solution in the form of $U(\lambda) = u(\lambda) + v_\lambda$, where v_λ is a positive solution of the following auxiliary problem:

$$\begin{gathered} -\Delta v + v = \lambda a(x)((u(\lambda) + v)^p - u(\lambda)^p) \quad \text{in } \mathbb{R}^N, \\ v \in H^1(\mathbb{R}^N), \quad v > 0 \quad \text{in } \mathbb{R}^N. \end{gathered} \tag{3.1}$$

For this equation, we define the energy functional $J : H^1(\mathbb{R}^N) \to \mathbb{R}$ as follows:

$$J(v) = \frac{1}{2}\|v\|^2 - \frac{\lambda}{p+1}\int_{\mathbb{R}^N} a(x)\left[(u(\lambda) + v_+)^{p+1} - u(\lambda)^{p+1} - (p+1)u(\lambda)^p v_+\right] dx.$$

We know that critical points of $J(v)$ correspond to positive solutions of 3.1. The existence of nontrivial critical points will be deduced by the mountain pass lemma of Ambrosetti and Rabinowitz [3].

In this section, firstly several technical results will be established. Let us recall that a sequence $\{v_n\} \subset H^1(\mathbb{R}^N)$ is called a $(PS)_c$-sequence if $J(v_n) \to c$ and $J'(v_n) \to 0$ as $n \to \infty$. If any $(PS)_c$-sequence possesses a convergent subsequence, we say $(PS)_c$-condition is satisfied.

Let us now introduce the problem at infinity associated with (1.1) is

$$\begin{gathered} -\Delta u + u = \lambda u^p \quad \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \quad u > 0 \quad \text{in } \mathbb{R}^N. \end{gathered} \tag{3.2}$$

We state here some known results for (3.2). First of all, we recall that Lions [17] has studied the following minimization problem closely related to (3.2):

$$S^\infty = \inf\left\{I^\infty(u) : u \in H^1(\mathbb{R}^N), u \neq 0, I^{\infty\prime}(u) = 0\right\} > 0, \tag{3.3}$$

where $I^\infty(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{p+1}\int_{\mathbb{R}^N} u_+^{p+1} dx$. For future reference note also that a minimum exists and is realized by a ground state $\omega > 0$ in $\mathbb{R}^N$ such that $S^\infty = I^\infty(\omega) = \sup_{s\geq 0} I^\infty(s\omega)$. Gidas-Ni-Nirenberg [13] showed that there exist a_1, $a_2 > 0$ such that for all $x \in \mathbb{R}^N$,

$$a_1(|x| + 1)^{\frac{-(N-1)}{2}} e^{-|x|} \leq \omega(x) \leq a_2(|x| + 1)^{\frac{-(N-1)}{2}} e^{-|x|}. \tag{3.4}$$

Secondly we study the break down of the Palais-Smale condition for J. The ground state solution ω of (3.2) play an important role to describe an asymptotic behavior of (PS)-sequence for J.

Proposition 3.1. *Assume* (A1), (A2) *hold. Let $\{v_n\} \subset H^1(\mathbb{R}^N)$ be a (PS)-sequence for J. Then there exist a subsequence (still denoted $\{v_n\}$) for which the following holds: there exist an integer $m \geq 0$, sequence of points $\{y_n^i\} \subset \mathbb{R}^N$ for $1 \leq i \leq m$, a solution v_λ of* (3.1) *and solutions v^i, for $1 \leq i \leq m$, of* (3.2) *such* *that as $n \to \infty$,*

$$
\begin{gathered}
v_n \rightharpoonup v_\lambda \quad \textit{weakly in } H^1(\mathbb{R}^N),\\
v_n - v_\lambda - \sum_{i=1}^m v^i(x - y_n^i) \to 0 \quad \textit{strongly in } H^1(\mathbb{R}^N),\\
J(v_n) \to J(v_\lambda) + \sum_{i=1}^m I^\infty(v^i),\\
|y_n^i| \to \infty, |y_n^i - y_n^j| \to \infty, \quad \textit{for } 1 \le i \ne j \le m,
\end{gathered}
$$

where we agree that in the case $m = 0$ the above holds without v^i, y_n^i.

Proof. This is a standard result that we give here without proof (see [4], [5], [17] for analogous statements). □

Thirdly, we show that $J(u)$ possesses the mountain pass structure.

Lemma 3.2. *Assume* (A1), (A2) *hold. Then there exist small $\beta > 0$ and $\alpha > 0$ such that $J(v) \ge \alpha > 0$ for all $v \in S_\beta = \{u \in H^1(\mathbb{R}^N) : \|u\| = \beta\}$.*

Proof. For each $v \in H^1(\mathbb{R}^N)$, using Taylor's formula, the definition of μ_λ, $a(x) \in (0, 1]$, and the Sobolev embedding, we obtain

$$
\begin{aligned}
J(v) &= \frac{1}{2}\|v\|^2 - \frac{1}{2}\lambda p \int_{\mathbb{R}^N} a(x)u(\lambda)^{p-1}v_+^2\,dx \\
&\quad - \lambda \int_{\mathbb{R}^N} \int_0^{v_+} a(x)\left[(u(\lambda) + s)^p - u(\lambda)^p - pu(\lambda)^{p-1}s\right] ds dx \\
&\ge \frac{1}{2}\Big[\|v\|^2 - \lambda p \int_{\mathbb{R}^N} a(x)u(\lambda)^{p-1}v_+^2\,dx\Big] - \lambda \int_{\mathbb{R}^N} a(x)\Big[\frac{\epsilon}{2}v_+^2 + \frac{C_\varepsilon}{p+1}v_+^{p+1}\Big]dx \\
&\ge \big(\frac{1}{2} - \frac{\lambda}{2\mu_\lambda} - \frac{1}{2}\lambda\epsilon\big)\|v\|^2 - \lambda C\|v\|^{p+1}
\end{aligned}
$$

for all $\epsilon > 0$. Since $\mu_\lambda > \lambda$ by Theorem 2.4, we may choose ε small enough such that $\frac{1}{2} - \frac{\lambda}{2\mu_\lambda} - \frac{1}{2}\lambda\epsilon > 0$. If we fix $\varepsilon = (\mu_\lambda - \lambda)/(2\lambda\mu_\lambda)$, then

$$
J(v) \ge \frac{1}{4\mu_\lambda}(\mu_\lambda - \lambda)\|v\|^2 - \lambda C\|v\|^{p+1}.
$$

Hence, there exist small $\beta > 0$, $\alpha > 0$ such that $J(v) \ge \alpha > 0$ for all $v \in S_\beta = \{u \in H^1(\mathbb{R}^N) : \|u\| = \beta\}$. □

Let e be a fixed unit vector in $\mathbb{R}^N$. The following estimates are important to find a path which lies below the first level of the break down of $(PS)_c$-condition. Here we use an interaction phenomenon between 0 and $\omega(x - R_0e)$.

Proposition 3.3. *Assume* (A1)-(A2) *hold. Then*

1. *there exists $t_0 > 0$ such that $J(t\omega(x - Re)) < 0$ for all $t \ge t_0$ uniformly in $R \ge 1$;*
2. *there exists $R_0 > 1$ such that $\sup_{t\ge 0} J(t\omega(x - Re)) < S^\infty$ for all $R \ge R_0$.*

To proof the above proposition, we need the following result.

Lemma 3.4. *There exist some constants C_1, C_2 and $C_3 > 0$ independent of $R \ge 1$ such that*

(I) $\int_{|x|\le 1}\omega(x-Re)^2dx \ge C_1R^{-(N-1)}e^{-2R}$ *for* $R\ge 1$,
(II) $\int_{\mathbb{R}^N} e^{-\mu|x|}\omega(x-Re)^2dx \le C_2R^{-(N-1)}e^{-2R}$ *for* $R\ge 1$,
(III) $\int_{\mathbb{R}^N} e^{-\mu|x|}\omega(x-Re)^{p+1}dx \le C_3e^{-\mu R}$ *for* $R\ge 1$.

In what follows, we denote various positive constants independent of $R\ge 1$ by C_1, C_2 and C_3.

Proof. (I) From (3.4), for $R\ge 1$, we have

$$\begin{aligned}\int_{|x|\le 1}\omega(x-Re)^2dx &\ge \int_{|x|\le 1} a_1^2(|x-Re|+1)^{-(N-1)}e^{-2|x-Re|}dx\\ &\ge a_1^2(R+2)^{-(N-1)}e^{-2R-2}\int_{|x|\le 1}dx\\ &\ge C_1R^{-(N-1)}e^{-2R}.\end{aligned}$$

(II) From (3.4) again,

$$\begin{aligned}\int_{\mathbb{R}^N} e^{-\mu|x|}\omega(x-Re)^2dx &\le a_2^2\int_{\mathbb{R}^N} e^{-\mu|x|}(|x-Re|+1)^{-(N-1)}e^{-2|x-Re|}dx\\ &= a_2^2\int_{\mathbb{R}^N} e^{-(\mu-2)|x|}(|x-Re|+1)^{-(N-1)}e^{-2(|x-Re|+|x|)}dx\\ &\le a_2^2R^{-(N-1)}e^{-2R}\int_{\mathbb{R}^N} e^{-(\mu-2)|x|}(\frac{R}{|x-Re|+1})^{N-1}dx.\end{aligned}$$

We estimate the last integral. First, we observe that

$$e^{-(\mu-2)|x|}(\frac{R}{|x-Re|+1})^{N-1}\to e^{-(\mu-2)|x|}\quad \text{as } R\to\infty \quad \text{for all } x\in\mathbb{R}^N.$$

For $|x|\le \frac{R}{2}$,

$$e^{-(\mu-2)|x|}(\frac{R}{|x-Re|+1})^{N-1}\le e^{-(\mu-2)|x|}(\frac{R}{\frac{R}{2}+1})^{N-1}\le 2^{N-1}e^{-(\mu-2)|x|}.$$

For $|x|\ge \frac{R}{2}$,

$$e^{-(\mu-2)|x|}(\frac{R}{|x-Re|+1})^{N-1}\le e^{-(\mu-2)|x|}R^{N-1}\le 2^{N-1}e^{-(\mu-2)|x|}|x|^{N-1}.$$

Thus,

$$e^{-(\mu-2)|x|}(\frac{R}{|x-Re|+1})^{N-1}\le 2^{N-1}e^{-(\mu-2)|x|}\max\{1,|x|^{N-1}\}\in L^1(\mathbb{R}^N).$$

Therefore, we can apply the Lebesgue dominated convergence theorem and we obtain

$$\int_{\mathbb{R}^N} e^{-\mu|x|}\omega(x-Re)^2dx \le a_2^2R^{-(N-1)}e^{-2R}\Big(\int_{\mathbb{R}^N} e^{-(\mu-2)|x|}dx+o(1)\Big)$$

as $R\to\infty$. Thus we have

$$\int_{\mathbb{R}^N} e^{-\mu|x|}\omega(x-Re)^2dx \le C_2R^{-(N-1)}e^{-2R}\quad \text{for } R\ge 1.$$

(III) Since $|x+Re| \geq R-|x|$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N} e^{-\mu|x|}\omega(x-Re)^{p+1}dx \\
&\leq a_2^{p+1}\int_{\mathbb{R}^N} e^{-\mu|x|}(|x-Re|+1)^{-(p+1)(N-1)/2}e^{-(p+1)|x-Re|}dx \\
&\leq a_2^{p+1}\int_{\mathbb{R}^N} e^{-\mu|x|}e^{-(p+1)|x-Re|}dx \\
&= a_2^{p+1}\int_{\mathbb{R}^N} e^{-\mu|x+Re|}e^{-(p+1)|x|}dx \\
&\leq a_2^{p+1}e^{-\mu R}\int_{\mathbb{R}^N} e^{-((p+1)-\mu)|x|}dx.
\end{aligned}
$$

Because of the assumption $\mu \in (2, p+1)$, we have

$$
\int_{\mathbb{R}^N} e^{-\mu|x|}\omega(x-Re)^{p+1}dx \leq C_3 e^{-\mu R} \quad \text{for } R \geq 1.
$$

The proof is complete. $\square$

Proof of Proposition 3.3. Let $\omega_\alpha(x) = \omega(x-\alpha e)$, $\alpha \in [0,+\infty)$, where ω is the ground state solution of the limit problem (3.2). We also remark that for all $s>0$, $t>0$,

$$
(s+t)^{p+1} - s^{p+1} - t^{p+1} - (p+1)s^p t \geq 0; \tag{3.5}
$$

and for any $s_0>0$ and $r_0>0$ there exists $C_4(s_0,r_0)>0$ such that for all $s\in[s_0,r_0]$, $t\in[0,r_0]$,

$$
(s+t)^{p+1} - s^{p+1} - t^{p+1} - (p+1)s^p t \geq C_4(s_0,r_0)t^2. \tag{3.6}
$$

(1) From (3.5), (A2), and Lemma 3.4 (III), we have

$$
\begin{aligned}
J(t\omega_R) &\leq \frac{1}{2}t^2\|\omega_R\|^2 - \frac{\lambda}{p+1}\int_{\mathbb{R}^N} a(x)(t\omega_R)^{p+1}dx \\
&\leq \frac{1}{2}t^2\|\omega_R\|^2 - \frac{\lambda}{p+1}t^{p+1}\int_{\mathbb{R}^N}(1-Ce^{-\mu|x|})\omega_R^{p+1}dx \\
&\leq \frac{1}{2}t^2\|\omega_R\|^2 - \frac{\lambda}{p+1}t^{p+1}\Big(\|\omega_R\|_{L^{p+1}}^{p+1} - CC_3e^{-\mu R}\Big).
\end{aligned}
$$

Choosing $t_0>0$ large enough, we have (1).

(2) Since J is continuous in $H^1(\mathbb{R}^N)$, there exists $\underline{t}>0$ such that for $t<\underline{t}$, $J(t\omega_R) < S^\infty$ for all $R\geq 0$, and from (1), $J(t\omega_R)\to-\infty$ as $t\to\infty$ uniformly in $R\geq 1$, then there exists $\bar{t}>0$ such that $\sup_{t\geq 0} J(t\omega_R) = \sup_{0\leq t\leq \bar{t}} J(t\omega_R)$. Then we only need to verify the inequality $\sup_{\underline{t}\leq t\leq\bar{t}} J(t\omega_R) < S^\infty$ for all R large enough. Straightforward computations give us

$$
\begin{aligned}
&J(t\omega_R) \\
&= I^\infty(tw_R) \\
&\quad - \frac{\lambda}{p+1}\int_{\mathbb{R}^N} a(x)\left[(u(\lambda)+t\omega_R)^{p+1} - u(\lambda)^{p+1} - (t\omega_R)^{p+1} - (p+1)u(\lambda)^p t\omega_R\right]dx \\
&\quad + \frac{\lambda}{p+1}\int_{\mathbb{R}^N}((t\omega_R)^{p+1} - a(x)(t\omega_R)^{p+1})dx \\
&\leq S^\infty - \frac{\lambda}{p+1}\Lambda_1 + \frac{\lambda}{p+1}\Lambda_2,
\end{aligned}
$$

where

$$\Lambda_1 = \int_{\mathbb{R}^N} a(x)\big[(u(\lambda)+t\omega_R)^{p+1} - u(\lambda)^{p+1} - (t\omega_R)^{p+1} - (p+1)u(\lambda)^p t\omega_R\big]dx$$

and

$$\Lambda_2 = \int_{\mathbb{R}^N} ((t\omega_R)^{p+1} - a(x)(t\omega_R)^{p+1})dx.$$

We need to show that there exists a constant $R_0 \geq 1$ such that

$$-\Lambda_1 + \Lambda_2 < 0 \quad \text{for all } t \in [\underline{t}, \bar{t}]. \tag{3.7}$$

Now we estimate Λ_1 and Λ_2.

$$\Lambda_1 \geq \int_{|x|\leq 1} a(x)\big[(u(\lambda)+t\omega_R)^{p+1} - u(\lambda)^{p+1} - (t\omega_R)^{p+1} - (p+1)u(\lambda)^p t\omega_R\big]dx.$$

Setting $s_0 = \min_{|x|\leq 1} u(\lambda)(x)$, $r_0 = \max\{\max_{x\in\mathbb{R}^N} u(\lambda)(x),\ \bar{t}\max_{x\in\mathbb{R}^N}\omega(x)\} > 0$, $\underline{a} = \inf_{|x|\leq 1} a(x)$, by (3.6), we obtain

$$\Lambda_1 \geq \underline{a}\int_{|x|\leq 1} C_4(s_0,r_0)(t\omega_R)^2 dx \geq \underline{a}C_4(s_0,r_0)\underline{t}^2 \int_{|x|\leq 1}\omega_R^2 dx \quad \text{for all } t \in [\underline{t}, \bar{t}].$$

From Lemma 3.4 (I), we have for $A = \underline{a}C_1C_4(s_0,r_0)\underline{t}^2$,

$$\Lambda_1 \geq AR^{-(N-1)}e^{-2R}. \tag{3.8}$$

Next from (A2), we have for any $R \geq 1$,

$$\begin{aligned}\Lambda_2 &= \int_{\mathbb{R}^N} ((t\omega_R)^{p+1} - a(x)(t\omega_R)^{p+1})dx \\ &\leq \int_{\mathbb{R}^N} Ce^{-\mu|x|}(t\omega_R)^{p+1}dx \\ &\leq \bar{t}^{p+1}\int_{\mathbb{R}^N} Ce^{-\mu|x|}\omega_R^{p+1}dx.\end{aligned}$$

From (II) and (III) of Lemma 3.4, we have for $B = \bar{t}^{p+1}CC_3$,

$$\Lambda_2 \leq Be^{-\mu R}. \tag{3.9}$$

and we choose $R_0 \geq 1$ so that

$$Be^{-\mu R_0} < AR_0^{-(N-1)}e^{-2R_0}. \tag{3.10}$$

Thus from (3.8)-(3.10), we obtain (3.7). $\square$

Theorem 3.5. *Assume* (A1)-(A2) *hold. Then for $\lambda \in (0,\lambda^*)$, there is a second positive solution $U(\lambda)$ of* (1.1).

Proof. We only need to prove that problem (3.1) has a positive solution. Set $\gamma = \inf_{g\in\Gamma}\max_{t\in[0,1]} J(g(t))$, where $\Gamma = \{g \in C([0,1], H^1(\mathbb{R}^N)) : g(0) = 0, g(1) = t_0\omega_{R_0}\}$. By Lemma 3.2 and Proposition 3.3, we deduce that $0 < \alpha \leq \gamma < S^\infty$. The Mountain Pass Lemma insures the existence of a (PS)-sequence $\{v_n\}$ for J at level γ. By Proposition 3.1, we deduce that

$$\gamma = \lim_{n\to\infty} J(v_n) = J(v_\lambda) + \sum_{i=1}^m I^\infty(v^i),$$

for some v_λ, v^i satisfying $J'(v_\lambda) = 0$ and $I^{\infty\prime}(v^i) = 0$ for $1 \leq i \leq m$.

By the strong maximum principle, we need to prove only that $v_\lambda \not\equiv 0$. We have $\gamma = J(v_\lambda) \geq \alpha > 0$ if $m = 0$, and $S^\infty > \gamma \geq J(v_\lambda) + S^\infty$ if $m \geq 1$. This implies $v_\lambda \not\equiv 0$. $\square$

4. The case $\lambda = \lambda^*$: properties and bifurcation of positive solutions

First, we show that the existence of positive solutions of (1.1) for $\lambda = \lambda^*$.

Proposition 4.1. *The set of minimal positive solutions $\{u(\lambda) : \lambda \in (0, \lambda^*)\}$ of (1.1) is uniformly bounded in λ in $L^\infty(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$.*

Proof. From Theorem 2.4,

$$\int_{\mathbb{R}^N} (|\nabla u(\lambda)|^2 + u(\lambda)^2) dx \geq \mu_\lambda(u(\lambda)) \int_{\mathbb{R}^N} pa(x)u(\lambda)^{p+1} dx.$$

Also we have

$$\int_{\mathbb{R}^N} (|\nabla u(\lambda)|^2 + u(\lambda)^2) dx = \lambda \int_{\mathbb{R}^N} \big(a(x)u(\lambda)^{p+1} + h(x)u(\lambda)\big)\, dx.$$

By the Hölder's and Young's inequalities we deduce

$$\Big(1 - \frac{\lambda}{p\mu_\lambda(u(\lambda))} - \frac{\epsilon}{2}\lambda\Big)\|u(\lambda)\|^2 \leq \frac{\lambda}{2\epsilon}\|h\|_{H^{-1}}$$

for all $\epsilon > 0$. Taking ϵ small enough so that

$$\Big(1 - \frac{\lambda}{p\mu_\lambda(u(\lambda))} - \frac{\epsilon}{2}\lambda\Big) > 0$$

and hence we have $\|u(\lambda)\| \leq C$ which shows that $u(\lambda)$ is uniformly bounded in $H^1(\mathbb{R}^N)$.

By Theorem 2.2 the solution $u(\lambda)$ is strictly increasing with respect to λ, we may suppose that $u(\lambda) \to u(\lambda^*)$ weakly in $H^1(\mathbb{R}^N)$ as $\lambda \to \lambda^*$, and hence $u(\lambda^*)$ is a solution of (1.1) with $\lambda = \lambda^*$.

¿From Theorem 2.3 we can deduce that $\|u(\lambda^*)\|_{L^\infty} \leq C$. Then by Theorem 2.2, the solution $u(\lambda)$ is strictly increasing with respect to λ and Theorem 4.6, the uniqueness of $u(\lambda^*)$ (the proof will be given later), we conclude that $\|u(\lambda)\|_{L^\infty} \leq \|u(\lambda^*)\|_{L^\infty} \leq C$. $\square$

Secondly, we show that problem (1.1) has exactly two distinct positive solutions $u(\lambda)$, $U(\lambda)$ for $\lambda \in (0, \lambda^*)$.

Denote by $Q = \{(\lambda, u) : u \text{ solves } (1.1)_\lambda\}$, the set of positive solutions of (1.1). For each $(\lambda, u) \in Q$, let's recall that $\mu_\lambda(u)$ denote the number defined by

$$\mu_\lambda(u) = \inf\Big\{\int_{\mathbb{R}^N} (|\nabla v|^2 + v^2) dx : v \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} pa(x)u^{p-1}v^2 dx = 1\Big\},$$

which is the smallest eigenvalue of the following problem:

$$\begin{gathered} -\Delta v + v = \mu_\lambda(u)pa(x)u^{p-1}v \quad \text{in } \mathbb{R}^N, \\ v \in H^1(\mathbb{R}^N), v > 0 \quad \text{in } \mathbb{R}^N. \end{gathered} \tag{4.1}$$

Proposition 4.2. *Let $(\lambda, u) \in Q$, $0 < \lambda < \lambda^*$. Then*

(i) *$\mu_\lambda(u) > \lambda$ if and only if $u = u(\lambda)$;*
(ii) *$\mu_\lambda(U(\lambda)) < \lambda$, where $U(\lambda)$ is the second positive solution of 1.1.*

Proof. (i) Let $\psi \geq 0$, $\psi \in H^1(\mathbb{R}^N)$. Then from the convexity of u^p, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \nabla(u - u(\lambda)) \cdot \nabla\psi dx + \int_{\mathbb{R}^N} (u - u(\lambda))\psi dx \\
&= \lambda \int_{\mathbb{R}^N} a(x)(u^p - u(\lambda)^p)\psi dx \\
&= \lambda \int_{\mathbb{R}^N} a(x) \int_{u(\lambda)}^{u} pt^{p-1}\psi dt\, dx \\
&\leq \lambda \int_{\mathbb{R}^N} pa(x)u^{p-1}(u - u(\lambda))\psi dx.
\end{aligned} \tag{4.2}
$$

Set $\psi = (u - u(\lambda))^+$. If $\psi \not\equiv 0$, then by (4.2) and the definition of $\mu_\lambda(u)$,

$$
\begin{aligned}
\int_{\mathbb{R}^N} (|\nabla\psi|^2 + |\psi|^2)dx &\leq \lambda \int_{\mathbb{R}^N} pa(x)u^{p-1}\psi^2 dx \\
&< \mu_\lambda(u) \int_{\mathbb{R}^N} pa(x)u^{p-1}\psi^2 dx \\
&\leq \int_{\mathbb{R}^N} (|\nabla\psi|^2 + |\psi|^2)dx,
\end{aligned}
$$

which is impossible. Hence $\psi \equiv 0$, i.e. $u = u(\lambda)$ in $\mathbb{R}^N$. On the other hand, by Theorem 2.4, $\mu_\lambda(u(\lambda)) > \lambda$. This completes the proof of (i).

(ii) By (i), we get that $\mu_\lambda(U(\lambda)) \leq \lambda$ for $\lambda \in (0, \lambda^*)$. We claim that $\mu_\lambda(U(\lambda)) = \lambda$ can not occur. We proceed by contradiction. Set $w = U(\lambda) - u(\lambda)$; we have

$$
-\Delta w + w = \lambda a(x)(U(\lambda)^p - (U(\lambda) - w)^p), \quad w > 0 \quad \text{in } \mathbb{R}^N. \tag{4.3}
$$

By $\mu_\lambda(U(\lambda)) = \lambda$, we have that the problem

$$
-\Delta\phi + \phi = \lambda pa(x)U(\lambda)^{p-1}\phi, \quad \phi \in H^1(\mathbb{R}^N) \tag{4.4}
$$

possesses a positive solution ϕ_1.

Multiplying (4.3) by ϕ_1 and (4.4) by w, integrating and subtracting we deduce that

$$
\begin{aligned}
0 &= \int_{\mathbb{R}^N} \lambda a(x)[U(\lambda)^p - (U(\lambda) - w)^p - pU(\lambda)^{p-1}w]\phi_1 dx \\
&= -\frac{1}{2} \int_{\mathbb{R}^N} \lambda a(x)p(p-1)\xi_\lambda^{p-2}w^2\phi_1 dx,
\end{aligned}
$$

where $\xi_\lambda \in (u(\lambda), U(\lambda))$. Thus $w \equiv 0$, that is $U(\lambda) = u(\lambda)$ for $\lambda \in (0, \lambda^*)$. This is a contradiction. Hence, we have that $\mu_\lambda(U(\lambda)) < \lambda$ for $\lambda \in (0, \lambda^*)$. $\square$

Remark 4.3. Since $\mu_\lambda(U(\lambda)) < \lambda$, one may employ a similar argument to the one used for $u(\lambda)$ to show that $U(\lambda)$ is strictly decreasing in λ, $\lambda \in (0, \lambda^*)$.

Remark 4.4. To the authors' knowledge, it is still unknown about the existence of solutions for the case of $\lambda > \lambda^*$.

When $\lambda = \lambda^*$, we have the existence of positive solution $u(\lambda^*)$ of $(1.1)_{\lambda^*}$ in Proposition 4.1. Now we show that $u(\lambda^*)$ is the unique positive solution of $(1.1)_{\lambda^*}$.

Lemma 4.5. *For any $g(x) \in H^{-1}(\mathbb{R}^N)$, the problem*

$$
-\Delta w + w = \lambda pa(x)u(\lambda)^{p-1}w + g(x), w \in H^1(\mathbb{R}^N) \tag{4.5}
$$

has a solution w, where $u(\lambda)$ is the minimal solution of (1.1) *for $\lambda \in (0, \lambda^*)$.*

Proof. Consider the functional

$$\Phi(w) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla w|^2 + w^2\right) dx - \frac{1}{2}\int_{\mathbb{R}^N} \lambda p a(x) u(\lambda)^{p-1} w^2 dx - \int_{\mathbb{R}^N} g(x) w dx.$$

From the definition of μ_λ, $\mu_\lambda > \lambda$, the Hölder's and Young's inequality we have

$$\begin{aligned}
\Phi(w) &= \frac{1}{2}\|w\|^2 - \frac{1}{2}\int_{\mathbb{R}^N} \lambda p a(x) u(\lambda)^{p-1} w^2 dx - \int_{\mathbb{R}^N} g(x) w dx \\
&\geq \frac{1}{2}\|w\|^2 - \frac{\lambda}{2\mu_\lambda}\|w\|^2 - \int_{\mathbb{R}^N} g(x) w dx \\
&\geq (\frac{1}{2} - \frac{\lambda}{2\mu_\lambda})\|w\|^2 - \frac{\varepsilon}{2}\|w\|^2 - \frac{C_\varepsilon}{2}\|g\|^2_{H^{-1}} \\
&= (\frac{1}{2} - \frac{\lambda}{2\mu_\lambda} - \frac{\varepsilon}{2})\|w\|^2 - \frac{C_\varepsilon}{2}\|g\|^2_{H^{-1}} \\
&\geq -C\|g\|^2_{H^{-1}}
\end{aligned} \tag{4.6}$$

if we choose ϵ small.

Let $\{w_n\} \subset H^1(\mathbb{R}^N)$ be the minimizing sequence of the variational problem $d = \inf\{\Phi(w) : w \in H^1(\mathbb{R}^N)\}$. From (4.6),

$$(\frac{1}{2} - \frac{\lambda}{2\mu_\lambda} - \frac{\varepsilon}{2})\|w_n\|^2 \leq \Phi(w_n) + \frac{C_\varepsilon}{2}\|g\|^2_{H^{-1}} = d + \frac{C_\varepsilon}{2}\|g\|^2_{H^{-1}} + o(1)$$

as $n \to \infty$. Since $\mu_\lambda > \lambda$, we deduce that $\{w_n\}$ is bounded in $H^1(\mathbb{R}^N)$ if we choose ε small. So we may suppose that

$$\begin{gathered}
w_n \rightharpoonup w \quad \text{weakly in } H^1(\mathbb{R}^N), \\
w_n \to w \quad \text{almost everywhere in } \mathbb{R}^N, \\
w_n \to w \quad \text{strongly in } L^s_{\rm loc}(\mathbb{R}^N) \quad \text{for } 2 \leq s < \frac{2N}{N-2}.
\end{gathered}$$

We now prove that

$$\int_{\mathbb{R}^N} \lambda p a(x) u(\lambda)^{p-1} \left(w_n - w\right)^2 dx \to 0 \quad \text{as } n \to \infty. \tag{4.7}$$

By Theorem 2.3 and (A1), we have $pa(x)u(\lambda)^{p-1} \to 0$ as $|x| \to \infty$, for any $\varepsilon > 0$ there exists $R > 0$ such that for $x \in \mathbb{R}^N$, $|x| \geq R$, $|pa(x)u(\lambda)^{p-1}| < \varepsilon$. Consequently, there exists a constant $C > 0$ such that $|pa(x)u(\lambda)^{p-1}| \leq C$ for $x \in \mathbb{R}^N$. Then

$$\begin{aligned}
&\int_{\mathbb{R}^N} p a(x) u(\lambda)^{p-1} \left(w_n - w\right)^2 dx \\
&= \int_{B_R} p a(x) u(\lambda)^{p-1} \left(w_n - w\right)^2 dx + \int_{\mathbb{R}^N \setminus B_R} p a(x) u(\lambda)^{p-1} \left(w_n - w\right)^2 dx \\
&\leq C\int_{B_R} \left(w_n - w\right)^2 dx + \varepsilon \int_{\mathbb{R}^N \setminus B_R} \left(w_n - w\right)^2 dx.
\end{aligned}$$

Since $w_n \to w$ strongly in $L^s_{\rm loc}(\mathbb{R}^N)$ for $2 \leq s < 2N/N-2$, $\{w_n\}$ is a bounded sequence in $H^1(\mathbb{R}^N)$, taking $n \to \infty$, then $R \to \infty$, and finally $\varepsilon \to 0$, we deduce our claim.

From (4.7) we deduce that $\int_{\mathbb{R}^N} pa(x)u(\lambda)^{p-1}w_n^2dx \to \int_{\mathbb{R}^N} pa(x)u(\lambda)^{p-1}w^2dx$ and $\int_{\mathbb{R}^N} g(x)w_ndx \to \int_{\mathbb{R}^N} g(x)wdx$ as $n \to \infty$. Thus by Fatou's lemma,

$$\begin{aligned}\Phi(w) &= \frac{1}{2}\|w\|^2 - \frac{1}{2}\int_{\mathbb{R}^N} \lambda pa(x)u(\lambda)^{p-1}w^2dx - \int_{\mathbb{R}^N} g(x)wdx\\ &\le \frac{1}{2}\liminf_{n\to\infty}\|w_n\|^2 - \frac{1}{2}\lim_{n\to\infty}\int_{\mathbb{R}^N} \lambda pa(x)u(\lambda)^{p-1}w_n^2dx - \lim_{n\to\infty}\int_{\mathbb{R}^N} g(x)w_ndx\\ &= \liminf_{n\to\infty}\Phi(w_n)\\ &= d = \inf_{w\in H^1(\mathbb{R}^N)}\Phi(w)\end{aligned}$$

and hence $\Phi(w) = d$, which implies that w is a solution of (4.5). □

Theorem 4.6. *Let $u(\lambda^*)$ be a solution of* (1.1) *with $\lambda = \lambda^*$. Then $\mu_{\lambda^*}(u(\lambda^*)) = \lambda^*$. Moreover, $u(\lambda^*)$ is the unique positive solution of* (1.1) *with $\lambda = \lambda^*$.*

Proof. Define $F : \mathbb{R} \times H^1(\mathbb{R}^N) \to H^{-1}(\mathbb{R}^N)$ by

$$F(\lambda, u) = \Delta u - u + \lambda(a(x)u^p + h(x)).$$

Since $\mu_\lambda(u(\lambda)) > \lambda$ for $\lambda \in (0, \lambda^*)$, it follows that $\mu_{\lambda^*}(u(\lambda^*)) \ge \lambda^*$. If $\mu_{\lambda^*}(u(\lambda^*)) > \lambda^*$, the equation $F_u(\lambda^*, u(\lambda^*))\phi = 0$ has no nontrivial solution. From Lemma 4.5, F_u maps $\mathbb{R} \times H^1(\mathbb{R}^N)$ onto $H^{-1}(\mathbb{R}^N)$. Applying the implicit function theorem to F we can find a neighborhood $(\lambda^* - \delta, \lambda^* + \delta)$ of λ^* such that (1.1) possesses a solution $u(\lambda)$ if $\lambda \in (\lambda^* - \delta, \lambda^* + \delta)$. This is contradictory to the definition of λ^*.

Next, we are going to prove that $u(\lambda^*)$ is unique. In fact, suppose (1.1) with $\lambda = \lambda^*$ has another solution $U(\lambda^*)$. Set $w = U(\lambda^*) - u(\lambda^*)$; we have

$$-\Delta w + w = \lambda^* a(x)((w + u(\lambda^*))^p - u(\lambda^*)^p),\ w > 0 \quad \text{in } \mathbb{R}^N. \tag{4.8}$$

By $\mu_{\lambda^*}(u(\lambda^*)) = \lambda^*$, we have that the problem

$$-\Delta\phi + \phi = \lambda^* pa(x)u(\lambda^*)^{p-1}\phi, \phi \in H^1(\mathbb{R}^N) \tag{4.9}$$

possesses a positive solution ϕ_1.

Multiplying (4.8) by ϕ_1 and (4.9) by w, integrating and subtracting we deduce that

$$\begin{aligned}0 &= \int_{\mathbb{R}^N} \lambda^* a(x)[(w + u(\lambda^*))^p - u(\lambda^*)^p - pu(\lambda^*)^{p-1}w]\phi_1dx\\ &= \frac{1}{2}\int_{\mathbb{R}^N} \lambda^* p(p-1)\xi_{\lambda^*}^{p-2}w^2\phi_1dx,\end{aligned}$$

where $\xi_{\lambda^*} \in (u(\lambda^*), u(\lambda^*) + w)$. Thus $w \equiv 0$. □

Theorem 4.7. *$(\lambda^*, u(\lambda^*))$ is a bifurcation point in $C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$.*

We prove that $(\lambda^*, u(\lambda^*))$ is a bifurcation point in $C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$ by using an idea in [16]. To this end, we need the following bifurcation theorem.

Bifurcation Theorem [10, Theorem 3.2]**.** Let X and Y be Banach spaces. Let $(\bar\lambda, \bar x) \in \mathbb{R} \times X$ and let G be a twice continuously differentiable mapping of an open neighborhood of $(\bar\lambda, \bar x)$ into Y. Let $N(G_x(\bar\lambda, \bar x)) = \operatorname{span}\{x_0\}$ be one dimensional and *codim*$R(G_x(\bar\lambda, \bar x)) = 1$. Let $G_\lambda(\bar\lambda, \bar x) \notin R(G_x(\bar\lambda, \bar x))$. If Z is the complement of *span*$\{x_0\}$ in X, then the solutions of $G(\lambda, x) = G(\bar\lambda, \bar x)$ near $(\bar\lambda, \bar x)$ form a curve $(\lambda(s), x(s)) = (\bar\lambda + \tau(s), \bar x + sx_0 + z(s))$, where $s \to (\tau(s), z(s)) \in \mathbb{R} \times Z$ is a twice continuously differentiable function near $s = 0$ and $\tau(0) = \tau'(0) = z(0) = z'(0) = 0$.

We define $G : \mathbb{R} \times C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N) \to C^\alpha(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ by

$$G(\lambda, u) = \Delta u - u + \lambda(a(x)u^p + h(x)),$$

where $C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$ and $C^\alpha(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ are endowed with the natural norms. Then they become Banach spaces. We show that the Bifurcation Theorem [10] applies at $(\lambda^*, u(\lambda^*))$. Indeed, from Remark 2.5 and Theorem 4.6, problem (4.1) has a solution $\phi_1 > 0$ in $\mathbb{R}^N$. $\phi_1 > 0 \in C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$ if $h \in C^\alpha(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$. Thus $G_u(\lambda^*, u(\lambda^*))\phi = 0$, $\phi \in C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$, has a solution $\phi_1 > 0$. This implies that $\dim N(G_u(\lambda^*, u(\lambda^*))) = \dim \operatorname{span}\{\phi_1\} = 1$ and $\operatorname{codim} R(G_u(\lambda^*, u(\lambda^*))) = 1$ by the Fredholm alternative.

It remains to check that $G_\lambda(\lambda^*, u(\lambda^*)) \notin R(G_u(\lambda^*, u(\lambda^*)))$. By contradictory, it would imply the existence of $v(x) \not\equiv 0$ such that

$$\Delta v - v + \lambda^* pa(x)u(\lambda^*)^{p-1}v = -(a(x)u(\lambda^*)^p + h(x)), v \in C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N).$$

From $G_u(\lambda^*, u(\lambda^*))\phi_1 = 0$ we conclude that $\int_{\mathbb{R}^N}(a(x)u(\lambda^*)^p + h(x))\phi_1 dx = 0$. This is impossible because $h(x) \geq 0$, $h(x) \not\equiv 0$ and $\phi_1(x) > 0$ in $\mathbb{R}^N$. Applying the Bifurcation Theorem [10] we conclude that $(\lambda^*, u(\lambda^*))$ is the bifurcation point near which, the solutions of $(1.1)_{\lambda^*}$ form a curve $(\lambda^* + \tau(s), u(\lambda^*) + s\phi_1 + z(s))$ with s near $s = 0$ and $\tau(0) = \tau'(0) = 0$, $z(0) = z'(0) = 0$.

We claim that $\tau''(0) < 0$ which implies that the bifurcation curve turns strictly to the left in (λ, u) plane. Since $\lambda = \lambda^* + \tau(s)$, $u = u(\lambda^*) + s\phi_1 + z(s)$ in

$$-\Delta u + u - \lambda(a(x)u^p + h(x)) = 0, u > 0, u \in C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N). \tag{4.10}$$

Differentiate (4.10) in s twice we have

$$\begin{aligned}
&-\Delta u_{ss} + u_{ss} - \lambda_{ss}(a(x)u^p + h(x)) - \lambda_s(pa(x)u^{p-1}u_s) - \lambda_s(pa(x)u^{p-1}u_s)\\
&-\lambda(p(p-1)a(x)u^{p-2}u_s^2 + pa(x)u^{p-1}u_{ss}) = 0.
\end{aligned}$$

Set here $s = 0$ and use that $\tau'(0) = 0$, $u_s = \phi_1(x)$ and $u = u(\lambda^*)$ as $s = 0$ we obtain

$$\begin{aligned}
&-\Delta u_{ss} + u_{ss} - \tau''(0)(a(x)u(\lambda^*)^p + h(x))\\
&-\lambda^*(p(p-1)a(x)u(\lambda^*)^{p-2}\phi_1(x)^2 + pa(x)u(\lambda^*)^{p-1}u_{ss}) = 0.
\end{aligned} \tag{4.11}$$

Multiplying $G_u(\lambda^*, u(\lambda^*))\phi_1 = 0$ by u_{ss}, and (4.11) by ϕ_1, integrating and subtracting the results we obtain

$$\lambda^* p(p-1)\int_{\mathbb{R}^N} a(x)u(\lambda^*)^{p-2}\phi_1(x)^3 dx + \tau''(0)\int_{\mathbb{R}^N}(a(x)u(\lambda^*)^p + h(x))\phi_1(x)dx = 0$$

which immediately gives $\tau''(0) < 0$. Thus

$$\begin{aligned}
u(\lambda) \to u(\lambda^*) \quad &\text{in } C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N) \quad \text{as } \lambda \to \lambda^*,\\
U(\lambda) \to u(\lambda^*) \quad &\text{in } C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N) \quad \text{as } \lambda \to \lambda^*.
\end{aligned}$$

Proposition 4.8. *The solution set Q is unbounded in $C^{2,\alpha}(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$.*

Proof. By Proposition 4.1, the set of minimal positive solutions $\{u(\lambda)\}$ of (1.1) is uniformly bounded in $H^1(\mathbb{R}^N)$. We show that $\{U(\lambda) : \lambda \in (0, \lambda^*)\}$ is unbounded in $H^1(\mathbb{R}^N)$. First, we show that for any $\delta > 0$, $\{U(\lambda) : \lambda \in [\delta, \lambda^*)\}$ is bounded in

$H^1(\mathbb{R}^N)$. Since $U(\lambda)$ is decreasing in λ, therefore, we have

$$\begin{aligned}\|U(\lambda)\|^2 &= \lambda \int_{\mathbb{R}^N} \big(a(x)U(\lambda)^{p+1} + h(x)U(\lambda)\big)\,dx \\ &\le \lambda^* \int_{\mathbb{R}^N} \big(U(\delta)^{p+1} + h(x)U(\delta)\big)\,dx \\ &\le C_1\end{aligned}$$

for some constant $C_1 > 0$.

By Adams [2], Gilbarg-Trudinger [14, Theorem 8.32, 9.16], and the Sobolev embedding theorem, for all $0 < \alpha < 1$,

$$\begin{aligned}\|U(\lambda)\|_{C^{1,\alpha}} &\le C\|U(\lambda)\|_{W^{2,q_\alpha}} \\ &\le C\left(\|U(\lambda)\|_{L^{q_\alpha}} + \|\lambda\left(a(x)U(\lambda)^p + h(x)\right)\|_{L^{q_\alpha}}\right) \\ &\le C\left(\|U(\delta)\|_{L^{q_\alpha}} + \|U(\delta)^p + h\|_{L^{q_\alpha}}\right) \\ &\le C_2\end{aligned}$$

for some constant $C_2 > 0$.

Then we need only to show that $\{U(\lambda) : \lambda \in (0,\delta)\}$ is unbounded in $H^1(\mathbb{R}^N)$. Since $U(\lambda) = u(\lambda) + v_\lambda$, we claim that $\{v_\lambda : \lambda > 0\}$ is unbounded in $H^1(\mathbb{R}^N)$. If not, then

$$\|v_\lambda\| \le M \tag{4.12}$$

for all $\lambda \in (0,\delta]$. Choose $\lambda_n \downarrow 0$ and let v_{λ_n} be the corresponding solutions of $(3.1)_{\lambda_n}$. Then v_{λ_n} satisfies

$$\begin{aligned}\int_{\mathbb{R}^N} \left(|\nabla v_{\lambda_n}|^2 + v_{\lambda_n}^2\right) dx &= \lambda_n \int_{\mathbb{R}^N} a(x)\left(U(\lambda_n)^p - u(\lambda_n)^p\right) v_{\lambda_n}\,dx \\ &\le C\lambda_n \|U(\lambda_n)\|^p \|v_{\lambda_n}\| \\ &\le \overline{C}\lambda_n\end{aligned}$$

for some constant $\overline{C}$, independent of v_{λ_n}, where we have used (4.12) and the boundedness of $\{u(\lambda_n)\}$ in $H^1(\mathbb{R}^N)$. Hence, we have $lim_{n\to\infty}\|v_{\lambda_n}\|^2 = 0$. It implies that

$$\lim_{n\to\infty} \|v_{\lambda_n}\|_{L^2} = 0. \tag{4.13}$$

On the other hand, we notice that $U(\lambda) = u(\lambda) + v_\lambda$ is decreasing and $u(\lambda)$ is increasing in λ. Therefore, $v_{\lambda_n} \ge v_\delta$ for all n. Then we obtain

$$\|v_{\lambda_n}\|_{L^2} \ge \|v_\delta\|_{L^2} > 0 \quad \text{for all } n, \tag{4.14}$$

which contradicts (4.13).

Next we claim that $\{U(\lambda) : \lambda \in (0,\delta)\}$ is unbounded in $C^{2,\alpha}(\mathbb{R}^N)$. Otherwise, we suppose $\|U(\lambda_n)\|_{C^{2,\alpha}} \le M$, for all $\lambda_n \in (0,\delta)$, $\lambda_n \downarrow 0$. The compact Sobolev embedding theorem and a diagonal process allow us to extract a subsequence of $\{U(\lambda_n)\}$, still denoted by $\{U(\lambda_n)\}$, such that $U(\lambda_n) \to U$ in $C^2(\mathbb{R}^N)$. Then U would satisfy $-\Delta U + U = 0$ in $\mathbb{R}^N$, which implies $U = 0$.

However, by Remark 4.4 that $m_n := \text{meas}\{x \in \mathbb{R}^N : U(\lambda_n) > U(\delta)\} > 0$. If $m_n \to 0$, then by the Lebesgue's dominated convergence theorem it results that

$$\begin{aligned} \|U(\lambda_n)\|^2 &= \lambda_n \int_{\mathbb{R}^N} (a(x)U(\lambda_n)^p + h(x))U(\lambda_n)dx \\ &= \lambda_n \int_{U(\lambda_n)\le U(\delta)} + \lambda_n \int_{U(\lambda_n)>U(\delta)} (a(x)U(\lambda_n)^p + h(x))U(\lambda_n)dx \\ &\le \lambda_n \Big(\int_{U\le U(\delta)} (a(x)U^p + h(x))U dx + Cm_n \Big) \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Thus $\|v_{\lambda_n}\|^2 \to 0$, which contradicts (4.14). Therefore, we may assume that there exists $\alpha > 0$ independent of λ_n such that $\text{meas}\{x \in \mathbb{R}^N : U(\lambda_n) > U(\delta)\} \ge \alpha > 0$ for all λ_n, which would imply $\text{meas}\{x \in \mathbb{R}^N : U > U(\delta)\} \ge \alpha > 0$. But $U \equiv 0$, a contradiction. This completes the proof. □

The proof of Theorem 1.1 follows from Theorems 2.2, 2.3, 2.4(3), 3.5, Proposition 4.1, Theorems 4.6, 4.7, and Proposition 4.8.

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