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CRITICAL NEUMANN PROBLEM FOR NONLINEAR ELLIPTIC SYSTEMS IN EXTERIOR DOMAINS

SHENGBING DENG, JIANFU YANG

Abstract. In this paper, we investigate the Neumann problem for a critical elliptic system in exterior domains. Assuming that the coefficient $Q(x)$ is a positive smooth function and λ, $\mu \geq 0$ are parameters, we examine the common effect of the mean curvature of the boundary $\partial\Omega$ and the shape of the graph of the coefficient $Q(x)$ on the existence of the least energy solutions.

1. Introduction

In this paper, we are concerned with the following Neumann problem for elliptic systems

$$\begin{gathered}
-\Delta u + \lambda u = \frac{2\alpha}{\alpha+\beta} Q(x)|u|^{\alpha-2}u|v|^{\beta} \quad \text{in } \Omega^c, \\
-\Delta v + \mu v = \frac{2\beta}{\alpha+\beta} Q(x)|u|^{\alpha}|v|^{\beta-2}v \quad \text{in}\Omega^c, \\
\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \\
u, v > 0 \quad \text{in } \Omega^c,
\end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain and $\Omega^c = \mathbb{R}^N \setminus \Omega$, we assume that Ω^c has no bounded components. $\lambda, \mu \geq 0$ are parameters, $\alpha, \beta > 1$ and $\alpha + \beta = 2^*$, where 2^* denotes the critical Sobolev exponent, that is, $2^* = \frac{2N}{N-2}$ for $N \geq 3$. ν is the unit inner normal at the boundary $\partial\Omega$. The coefficient $Q(x)$ is Hölder continuous on Ω^c and $Q(x) > 0$ for all x $\in \Omega^c$.

In[4], critical semilinear elliptic problems for one equation with Dirichlet boundary conditions was solved by variational methods. Although the (PS) does not hold globally, it was found in [4] that the condition is valid locally. Critical point theory then can be used locally to find critical points of associated functionals. The critical Neumann problem was considered in [16] using the same idea as [4]. Later on, the critical Neumann has been extensively studied. Various existence results concerning the graph of coefficients, topology of domains etc., can be found in [1, 2, 6, 7, 9, 10, 13, 14, 16] and references therein. Particularly, the Neumann

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problem in exterior domains

$$\begin{gathered} -\Delta u+\lambda u=Q(x)|u|^{2^*-2}u \quad \text{in} \quad \Omega^c, \\ \frac{\partial u}{\partial \nu}=0 \quad \text{on} \quad \partial\Omega, \\ u>0 \quad \text{in} \quad \Omega^c, \end{gathered} \tag{1.2}$$

was considered in [7]. Existence results for (1.2) were obtained by showing

$$S(\Omega^c,Q,\lambda)=\inf\big\{\int_{\Omega^c}(|\nabla u|^2+\lambda u^2)\,dx, u\in H^1(\Omega^c), \int_{\Omega^c}Q(x)|u|^{2^*}\,dx=1\big\},$$

is achieved. The effect of the graph of Q and the geometry of the domain was taken into account on the existence of solutions of (1.2).

For the system (1.1), it was considered in [3] the existence of solutions for subcritical nonlinearities. In the critical case, problem (1.1) in bounded domains was investigated in [8], where the effect of the shape of $Q(x)$ was considered in the existence of least energy solutions. Inspired of [7] and [8], in this paper, we consider the existence of solutions of problem (1.1) in exterior domains. The problem is both critical and setting on unbounded domains. The loss of compactness is caused by noncompact groups of translations and dilations. In applying variational methods, it is necessary to figure out energy levels so that the (PS) condition holds. These energy levels are not only affected by noncompact groups of translations and dilations but also the shape of the coefficient Q. Solutions of problem (1.1) will be found as a minimizer of the variational problem

$$S_{\lambda,\mu}(\Omega^c,Q)=\inf_{u,v\in H^1(\Omega^c)\setminus\{0\}}\frac{\int_{\Omega^c}(|\nabla u|^2+|\nabla v|^2+\lambda u^2+\mu v^2)\,dx}{\big(\int_{\Omega^c}Q(x)|u|^\alpha|v|^\beta\,dx\big)^{2/(\alpha+\beta)}}, \tag{1.3}$$

which is a weak solution of (1.1) up to a multiple of a constant. It was proved in [3] that every weak solution of problem (1.1) is classical. As we will see, problem (1.3) is closely related to the problem

$$S_{\alpha,\beta}=\inf_{u,v\in D_0^{1,2}(\Omega^c)\setminus\{0\}}\frac{\int_{\Omega^c}(|\nabla u|^2+|\nabla v|^2)\,dx}{\big(\int_{\Omega^c}|u|^\alpha|v|^\beta\,dx\big)^{2/(\alpha+\beta)}}. \tag{1.4}$$

We may verify as [3] that

$$S_{\alpha,\beta}=\big[\big(\frac{\alpha}{\beta}\big)^{\beta/(\alpha+\beta)}+\big(\frac{\beta}{\alpha}\big)^{\alpha/(\alpha+\beta)}\big]S:=A_{\alpha,\beta}S,$$

where S is the best Sobolev constant defined by

$$S:=\inf_{u\in D_0^{1,2}(\Omega)\setminus\{0\}}\frac{\int_\Omega|\nabla u|^2\,dx}{\big(\int_\Omega|u|^{2^*}\,dx\big)^{2/2^*}},$$

which is achieved if and only if $\Omega=\mathbb{R}^N$ by the function

$$U(x)=\big[\frac{N(N-2)}{N(N-2)+|x|^2}\big]^{(N-2)/2}.$$

The function U satisfies

$$-\Delta U=U^{2^*-1}, \quad \text{in } \mathbb{R}^N$$

and

$$\int_{\mathbb{R}^N}|\nabla U|^2\,dx=\int_{\mathbb{R}^N}|U|^{2^*}\,dx=S^{\frac{N}{2}}.$$

Denote

$$Q_m = \max_{\partial\Omega} Q(x), \quad Q_M = \max_{\Omega^c} Q(x), \quad Q_\infty = \lim_{|x|\to\infty} Q(x).$$

Suppose Q_m, Q_M and Q_∞ are positive, we set

$$S_\infty = \min\Big\{ \frac{S_{\alpha,\beta}}{2^{2/N} Q_m^{\frac{N-2}{N}}}, \frac{S_{\alpha,\beta}}{Q_M^{\frac{N-2}{N}}}, \frac{S_{\alpha,\beta}}{Q_\infty^{\frac{N-2}{N}}} \Big\}.$$

Our main result is as follows.

Theorem 1.1. *If $S_{\lambda,\mu}(\Omega^c, Q) < S_\infty$ for $\lambda, \mu \geq 0$, then $S_{\lambda,\mu}(\Omega^c, Q)$ is achieved.*

In section2, we show a variant of second concentration lemma, and then prove Theorem 1.1. In the rest of the paper, we will verify the condition

$$S_{\lambda,\mu}(\Omega^c, Q) < S_\infty. \tag{1.5}$$

In the case $Q_M \leq 2^{\frac{2}{N-2}} Q_m$, we assume that

(Q1) There exists a point y $\in \partial\Omega$ such that $Q_m = Q(y)$ and $H(y) < 0$ and for x near y,

$$|Q(x) - Q(y)| = o(|x-y|) \tag{1.6}$$

where $H(y)$ denotes the mean curvature of $\partial\Omega$ at $y \in \partial\Omega$ with respect to the inner normal to $\partial\Omega$ at y.

In the case $Q_M > 2^{\frac{2}{N-2}} Q_m$, we assume

(Q2) $Q_M = Q(y)$ for some $y \in \Omega^c$ and for x near y, there holds

$$|Q(y) - Q(x)| = o(|x-y|^{N-2}). \tag{1.7}$$

If there is $x \in \Omega^c$ such that $Q(x) \geq Q_\infty$, then

$$S_\infty = \min\Big\{ \frac{S_{\alpha,\beta}}{2^{2/N} Q_m^{\frac{N-2}{N}}}, \frac{S_{\alpha,\beta}}{Q_M^{\frac{N-2}{N}}} \Big\}.$$

If $Q(x) < Q_\infty$ for all $x \in \Omega^c$, we suppose

(Q3) There exists some cone $K \subset \mathbb{R}^N$ on which the convergence $Q(x) \to Q_\infty$ holds and there exist $\bar{z} \in \partial B_1(0), \delta > 0$, $C > 0$ such that for $R \to \infty$,

$$0 < Q_\infty - Q(R\bar{y}) \leq \frac{C}{R^p}, \quad p > \frac{N^2}{2} \tag{1.8}$$

for every $\bar{y} \in \partial B_1(0) \cap B_\delta(\bar{z})$.

Under conditions (Q1)-(Q3), S_∞ is well defined.

Theorem 1.2. *The condition $S_{\lambda,\mu}(\Omega^c, Q) < S_\infty$ holds if one of the following conditions holds.*

(i) $Q_M \leq 2^{\frac{2}{N-2}} Q_m$ *and* (Q1) *holds;*
(ii) $Q_M > 2^{\frac{2}{N-2}} Q_m$ *and* (Q2) *holds;*
(iii) $Q_\infty > 2^{2/(N-2)} Q_m$, $Q(x) < Q_\infty$ *for all* $x \in \Omega^c$ *and* (Q3) *holds.*

The above theorem will be proved in sections 3, 4 and 5.

2. Proof of Theorem 1.1

Let

$$J_{\lambda,\mu}(u,v)=\int_{\Omega^c}(|\nabla u|^2+|\nabla v|^2+\lambda u^2+\mu v^2)\,dx$$

be a functional defined on $E:=H^1(\Omega^c)\times H^1(\Omega^c)$. Then

$$S_{\lambda,\mu}(\Omega^c,Q)=\inf\big\{J_{\lambda,\mu}(u,v):(u,v)\in E,\int_{\Omega^c}Q(x)|u|^\alpha|v|^\beta\,dx=1\big\}. \tag{2.1}$$

The following Brézis-Lieb type lemma is proved in [8].

Lemma 2.1. *Let $u_n\rightharpoonup u$ and $v_n\rightharpoonup v$ in $H^1(\Omega^c)$. Then*

$$\lim_{n\to\infty}\int_{\Omega^c}|u_n|^\alpha|v_n|^\beta\,dx=\lim_{n\to\infty}\int_{\Omega^c}|u_n-u|^\alpha|v_n-v|^\beta\,dx+\int_{\Omega^c}|u|^\alpha|v|^\beta\,dx. \tag{2.2}$$

Denote by $B_1(0)$ the unit ball in $\mathbb{R}^N$. We have the following results, see [8].

Lemma 2.2. *Let $\tilde{B}=B_1(0)\cap\{x_N>h(x')\}$ and $h(x')$ be a C^1 function defined on $\{x'\in\mathbb{R}^{N-1}:|x'|<1\}$ with h, ∇h vanishing at 0. For every u, $v\in H^1(B_1(0))$ with $\operatorname{supp}u,\operatorname{supp}v\subset\tilde{B}$, we have*

(A) *If $h\equiv 0$, then*

$$\int_{\tilde{B}}(|\nabla u|^2+|\nabla v|^2)\,dx\geq 2^{-2/N}S_{\alpha,\beta}\Big(\int_{\tilde{B}}|u|^\alpha|v|^\beta\Big)^{2/2^*}.$$

(B) *For every $\varepsilon>0$ there exists a $\delta>0$ depending only on ε such that if $|\nabla h|\leq\delta$, then*

$$\int_{\tilde{B}}(|\nabla u|^2+|\nabla v|^2)\,dx\geq\Big(\frac{S_{\alpha,\beta}}{2^{2/N}}-\varepsilon\Big)\Big(\int_{\tilde{B}}|u|^\alpha|v|^\beta\Big)^{2/2^*}.$$

To show the compactness of a (PS) sequence, we need a concentration - compactness lemma. In [12], it gave a remarkably characterization of non-compactness of the injection of $W^{1,q}(\Omega)$ into $L^{q^*}(\Omega)$ for $1\leq q<n$ and $q^*=\frac{qn}{n-q}$. The proof of the following results are essentially in spirit of [12], see also [11].

Lemma 2.3. *Let $u_n\rightharpoonup u$ and $v_n\rightharpoonup v$ in $H^1(\Omega^c)$. Suppose that $(|\nabla u_n|^2+|\nabla v_n|^2)\rightharpoonup\mu$, $|u_n|^\alpha|v_n|^\beta\rightharpoonup\nu$ in the sense of measure, and denote*

$$\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\Omega^c\cap\{|x|\geq R\}}(|\nabla u_n|^2+|\nabla v_n|^2)\,dx=\mu_\infty,$$
$$\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\Omega^c\cap\{|x|\geq R\}}|u_n|^\alpha|v_n|^\beta\,dx=\nu_\infty.$$

Then there exist an at most countable index set J and sequences $\{x_j\}\subset\Omega^c\cup\partial\Omega,\{\mu_j\},\{\nu_j\}\subset(0,\infty),j\in J$,such that

$$\nu=|u|^\alpha|v|^\beta+\sum_{j\in J}\nu_j\delta_{x_j}+\nu_\infty\delta_\infty, \tag{2.3}$$

$$\mu\geq|\nabla u|^2+|\nabla v|^2+\sum_{j\in J}\mu_j\delta_{x_j}+\mu_\infty\delta_\infty, \tag{2.4}$$

and

$$S_{\alpha,\beta}\nu_\infty^{2/2^*} \le \mu_\infty, \tag{2.5}$$

$$S_{\alpha,\beta}\nu_j^{2/2^*} \le \mu_j, \quad \text{if } x_j \in \Omega^c, \tag{2.6}$$

$$\frac{S_{\alpha,\beta}}{2^{2/N}}\nu_j^{2/2^*} \le \mu_j, \quad \text{if } x_j \in \partial\Omega^c, \tag{2.7}$$

where δ_x denotes the Dirac-mass of mass 1 concentrated at x.

Proof. We consider first the case $u = v = 0$. Since μ is a finite measure, the set $F := \{x \in \overline{\Omega^c} | \mu(\{x\}) > 0\}$ is at most countable. We can therefore write $F = \{x_j\}_{j\in J}$, $\mu_j := \mu(x_j)$, $j \in J$ so that

$$\mu \ge \sum_{j\in J} \mu_j\delta_{x_j} + \mu_\infty\delta_\infty.$$

If $x_j \in \Omega^c$, for any $\xi \in C_0^\infty(\Omega^c) \cap L^\infty(\Omega^c)$, we have

$$\begin{aligned} \int_{\Omega^c} |\xi|^{2^*}\, d\nu &= \lim_{n\to\infty} \int_{\Omega^c} |\xi|^{2^*} |u_n|^\alpha |v_n|^\beta\, dx \\ &\le \lim_{n\to\infty} S_{\alpha,\beta}^{-2^*/2}\Big(\int_{\Omega^c} |\nabla(\xi u_n)|^2 + |\nabla(\xi v_n)|^2\, dx\Big)^{2^*/2}. \end{aligned} \tag{2.8}$$

Since $u_n \to 0$, $v_n \to 0$ in $L^2_{\rm loc}(\Omega^c)$, we deduce

$$\int_{\Omega^c} |\xi|^{2^*}\, d\nu \le S_{\alpha,\beta}^{-2^*/2}\Big(\int_{\Omega^c} |\xi|^2\, d\mu\Big)^{2^*/2}. \tag{2.9}$$

By approximation, for any Borel set $E \in \Omega^c$ we have

$$\nu(E) \le S_{\alpha,\beta}^{-2^*/2}\mu(E)^{2^*/2} \tag{2.10}$$

as well as particularly, (2.5) and (2.6) hold. Because (2.10) implies $\nu \ll \mu$, we have for $E \in \Omega^c$ being Borel set that

$$\nu(E) = \int_E D_\mu\nu d\mu, \tag{2.11}$$

where

$$D_\mu\nu(x) = \lim_{r\to 0} \frac{\nu(B_r(x))}{\mu(B_r(x))}, \tag{2.12}$$

this limit exists for μ-a.e. $x \in \mathbb{R}^N$. From (2.10), we have

$$D_\mu\nu = 0, \quad \mu - \text{a.e. } x \in \Omega^c \setminus F. \tag{2.13}$$

Define $\nu_j = D_\mu\nu(x_j)\mu_j$, we see from (2.10)-(2.13) that (2.3) holds for the case $u = v = 0$.

If $x_j \in \partial\Omega^c$, by Lemma 2.2,

$$\begin{aligned} &\int_{\Omega^c\cap B_\varepsilon(x_j)\cap\{x_N>h(x')\}} (|\nabla u|^2 + |\nabla v|^2)\, dx \\ &\ge \Big(\frac{S_{\alpha,\beta}}{2^{2/N}} - \varepsilon\Big)\Big(\int_{\Omega^c\cap B_\varepsilon(x_j)\cap\{x_N>h(x')\}} |u|^\alpha|v|^\beta\Big)^{2/2^*} \end{aligned}$$

implying similarly that

$$\frac{S_{\alpha,\beta}}{2^{2/N}}\nu_j^{2/2^*} \le \mu_j.$$

Next, in the general case, let $\hat{u}_n = u_n - u$ and $\hat{v}_n = v_n - v$. We may apply above results to $\hat{u}_n$ and $\hat{v}_n$. Moreover, in terms of Lemma 2.1,

$$|\nabla \hat{u}_n|^2 + |\nabla \hat{v}_n|^2 \rightharpoonup \mu + |\nabla u|^2 + |\nabla v|^2,$$
$$|\nabla \hat{u}_n|^\alpha |\nabla \hat{v}_n|^\beta \rightharpoonup \nu + |\nabla u|^\alpha |\nabla v|^\beta$$

in the sense of measures. The proof is complete. □

Proof of Theorem 1.1. Let $\{u_n, v_n\}$ be a minimizing sequence for $S_{\lambda,\mu}(\Omega^c, Q)$; that is,

$$\int_{\Omega^c} (|\nabla u_n|^2 + |\nabla v_n|^2 + \lambda u_n^2 + \mu v_n^2)\,dx \to S_{\lambda,\mu}(\Omega^c, Q), \quad \int_{\Omega^c} Q(x)|u_n|^\alpha |v_n|^\beta\,dx = 1.$$

We may assume that $u_n \rightharpoonup u$, $v_n \rightharpoonup v$ in $H^1(\Omega^c)$. By Lemma 2.3,

$$|\nabla u_n|^2 + |\nabla v_n|^2 \rightharpoonup \mu \geq |\nabla u|^2 + |\nabla v|^2 + \sum_{j\in J} \mu_j \delta_{x_j} + \mu_\infty \delta_\infty,$$
$$Q(x)|u_n|^\alpha |v_n|^\beta \rightharpoonup \nu = Q(x)|u|^\alpha |v|^\beta + \sum_{j\in J} \nu_j Q(x_j)\delta_{x_j} + \nu_\infty Q_\infty \delta_\infty$$

and

$$1 = \int_{\Omega^c} Q(x)|u|^\alpha |v|^\beta\,dx + \sum_{j\in J} \nu_j Q(x_j) + \nu_\infty Q_\infty. \tag{2.14}$$

Therefore, using (2.5)-(2.7) we obtain

$$\begin{aligned}
&S_{\lambda,\mu}(\Omega^c, Q)\\
&\geq \int_{\Omega^c} (|\nabla u|^2 + |\nabla v|^2 + \lambda u^2 + \mu v^2)\,dx + \sum_{j\in J} \mu_j + \mu_\infty\\
&\geq S_{\lambda,\mu}(\Omega^c, Q)\Big(\int_{\Omega^c} Q(x)|u|^\alpha|v|^\beta\,dx\Big)^{2/(\alpha+\beta)} + \sum_{x_j\in\Omega^c} S_{\alpha,\beta}\nu_j^{2/(\alpha+\beta)}\\
&\quad + \sum_{x_j\in\partial\Omega} \frac{S_{\alpha,\beta}}{2^{2/N}}\nu_j^{2/(\alpha+\beta)} + S_{\alpha,\beta}\nu_\infty^{2/(\alpha+\beta)}\\
&= S_{\lambda,\mu}(\Omega^c, Q)\Big(\int_{\Omega^c} Q(x)|u|^\alpha|v|^\beta\,dx\Big)^{(N-2)/N}\\
&\quad + \sum_{x_j\in\Omega^c} \frac{S_{\alpha,\beta}}{Q(x_j)^{(N-2)/N}}\,(\nu_j Q(x_j))^{(N-2)/N}\\
&\quad + \sum_{x_j\in\partial\Omega} \frac{S_{\alpha,\beta}}{2^{2/N}Q(x_j)^{(N-2)/N}}(\nu_j Q(x_j))^{(N-2)/N} + \frac{S_{\alpha,\beta}}{Q_\infty^{(N-2)/2}}(\nu_\infty Q_\infty)^{(N-2)/N}\\
&\geq S_{\lambda,\mu}(\Omega^c, Q)\Big(\int_{\Omega^c} Q(x)|u|^\alpha|v|^\beta\,dx\Big)^{(N-2)/N} + \sum_{x_j\in\Omega^c} \frac{S_{\alpha,\beta}}{Q_M^{(N-2/N)}}(\nu_j Q(x_j))^{(N-2)/N}\\
&\quad + \sum_{x_j\in\partial\Omega} \frac{S_{\alpha,\beta}}{2^{2/N}Q_m^{(N-2)/N}}(\nu_j Q(x_j))^{(N-2)/N} + \frac{S_{\alpha,\beta}}{Q_\infty^{(N-2)/2}}(\nu_\infty Q_\infty)^{(N-2)/N}.
\end{aligned} \tag{2.15}$$

Since $S_{\lambda,\mu}(\Omega^c,Q) < S_\infty$, we deduce that $\nu_j = 0$ for all $j \in J \cup \{\infty\}$. Indeed, otherwise, we infer from (2.14) and (2.15) that

$$\begin{aligned} &S_{\lambda,\mu}(\Omega^c,Q)\\ &> S_{\lambda,\mu}(\Omega^c,Q)\Big(\int_{\Omega^c} Q(x)|u|^\alpha|v|^\beta\,dx + \nu_j Q(x_j) + \nu_j Q(x_j) + \nu_\infty Q_\infty\Big)^{(N-2)/N}\\ &= S_{\lambda,\mu}(\Omega^c,Q), \end{aligned}$$

which is a contradiction. Hence, $\int_{\Omega^c} Q(x)|u|^\alpha|v|^\beta\,dx = 1$, and then

$$\int_\Omega (|\nabla u|^2 + |\nabla v|^2 + \lambda u^2 + \mu v^2)\,dx \le S_{\lambda,\mu}(\Omega^c,Q).$$

The assertion follows. □

To verify condition (1.5), we need some preliminaries. Set

$$S(\Omega^c,\lambda) = \inf_{u\in H^1(\Omega^c)\setminus\{0\}} \frac{\int_{\Omega^c}(|\nabla u|^2 + \lambda u^2)\,dx}{(\int_{\Omega^c}|u|^{2^*}\,dx)^{2/2^*}},$$

The following result was proved in [14].

Lemma 2.4. *Assume Ω is a smooth bounded domain in $\mathbb{R}^N$ such that Ω^c has no bounded components. Then we have*

- (i) *$S(\Omega^c,\lambda)$ is nondecreasing in λ;*
- (ii) *$0 < S(\Omega^c,\lambda) \le (1/2)^{2/N} S$ for all $\lambda \ge 0$;*
- (iii) *If the mean curvature of $\partial\Omega$ is negative at some point, then for all $\lambda \ge 0$, $S(\Omega^c,\lambda) < (1/2)^{2/N} S$;*
- (iv) *If $\lambda \ge 0$ and $S(\Omega^c,\lambda) < (1/2)^{2/N} S$, then $S(\Omega^c,\lambda)$ is achieved.*

Let

$$U_{\varepsilon,y}(x) = \varepsilon^{-\frac{N-2}{2}} U\big(\frac{x-y}{\varepsilon}\big),$$

for $y \in \mathbb{R}^N$, $\varepsilon > 0$, and denote $U_\varepsilon = U_{\varepsilon,0}$.

Corollary 2.5. *Assume Ω is a smooth bounded domain in $\mathbb{R}^N$ such that Ω^c has no bounded components. Then there hold*

- (i) *$S_{\lambda,\mu}(\Omega^c,1)$ is nondecreasing in λ,μ;*
- (ii) *$0 < S_{\lambda,\mu}(\Omega^c,1) \le (1/2)^{2/N} S_{\alpha,\beta}$ for all $\lambda,\mu \ge 0$;*
- (iii) *If the mean curvature of $\partial\Omega$ is negative at some point, then for all $\lambda,\mu \ge 0$, $S_{\lambda,\mu}(\Omega^c,1) < (1/2)^{2/N} S_{\alpha,\beta}$.*

Proof. (i) is obvious. Now we show (ii) and (iii) only. Since $S_{0,0}(\Omega^c,1) = S_{\alpha,\beta} > 0$, $S_{\lambda,\mu}(\Omega^c,1) > 0$ follows from (i). Now we show that $S_{\lambda,\mu}(\Omega^c,1) \le (1/2)^{2/N} S_{\alpha,\beta}$ for all $\lambda,\mu \ge 0$. Let y be a point on $\partial\Omega$. Let Φ be the diffeomorphism from a small ball $B_\delta(0)$ centered at the origin to a neighborhood ω of y so that $\Phi : B_\delta^+(0) \to \bar\omega \cap \Omega^c$ and $\Phi : B_\delta(0) \cap \{y_N = 0\} \to \bar\omega \cap \partial\Omega^c$. We denote $\Psi := \Phi^{-1}$. Take a radial cut-off function η such that $\eta(r) \equiv 1$ for $r \le \frac{\delta}{2}$, $\eta(r) = 0$ for $r \ge \delta$ and $0 \le \eta \le 1$. Define

$$\tilde U_\varepsilon(x) = \begin{cases} (\eta U_\varepsilon)(\Psi(x)) & \text{if } x \in \bar\omega \cap \Omega^c;\\ 0 & \text{if } x \in \Omega^c \setminus \bar\omega. \end{cases}$$

It has shown in [14] that

$$\frac{\int_{\Omega^c}[|\nabla \tilde{U}_\varepsilon(x)|^2+\lambda \tilde{U}_\varepsilon(x)^2]\,dx}{\{\int_{\Omega^c}|\tilde{U}_\varepsilon(x)|^{2^*}\,dx\}^{2/2^*}}=\frac{S}{2^{2/N}}+A_N H(y)\beta_1(\varepsilon)+O(\beta_2(\varepsilon)), \tag{2.16}$$

where $A_N>0$ is a constant and $H(y)$ denotes the mean curvature of $\partial\Omega$ at y, when viewed from inside, and

$$\beta_1(\tau)=\begin{cases}\tau\log\frac{(}{1}\tau) & \text{if } N=3;\\ \tau & \text{if } N\geq 4,\end{cases}\quad \beta_2(\tau)=\begin{cases}\tau & \text{if } N=3;\\ \tau^2\log(\frac{1}{\tau}) & \text{if } N=4;\\ \tau^2 & \text{if } N\geq 5.\end{cases} \tag{2.17}$$

Choosing s and t such that

$$\frac{2\alpha}{\alpha+\beta}s^{-(\alpha-2)}t^{-\beta}=1\quad\text{and}\quad \frac{2\beta}{\alpha+\beta}s^{-\alpha}t^{-(\beta-2)}=1; \tag{2.18}$$

that is, $\frac{s^2}{t^2}=\frac{\beta}{\alpha}$, we have

$$\frac{s^2+t^2}{(s^\alpha t^\beta)^{2/2^*}}=A_{\alpha,\beta}. \tag{2.19}$$

By (2.16),

$$\begin{aligned}S_{\lambda,\mu}(\Omega^c,1)&\leq\frac{J_{\lambda,\mu}(s\tilde{U}_\epsilon(x),t\tilde{U}_\epsilon(x))}{\big(\int_{\Omega^c}|s\tilde{U}_\epsilon(x)|^\alpha|t\tilde{U}_\epsilon(x)|^\beta\,dx\big)^{2/2^*}}\\ &\leq\frac{s^2+t^2}{(s^\alpha t^\beta)^{2/2^*}}\frac{\int_{\Omega^c}[|\nabla\tilde{U}_\epsilon(x)|^2+\max\{\lambda,\mu\}\tilde{U}_\epsilon(x)^2]\,dx}{\big(\int_{\Omega^c}\tilde{U}_\epsilon(x)^{2^*}\big)^{2/2^*}}\\ &\leq A_{\alpha,\beta}\frac{\int_{\Omega^c}[|\nabla\tilde{U}_\epsilon(x)|^2+(\lambda+\mu)\tilde{U}_\epsilon(x)^2]\,dx}{\big(\int_{\Omega^c}\tilde{U}_\epsilon(x)^{2^*}\big)^{2/2^*}}\\ &=\frac{S_{\alpha,\beta}}{2^{2/N}}+B_N H(y)\beta_1(\varepsilon)+O(\beta_2(\varepsilon)),\end{aligned} \tag{2.20}$$

where $B_N>0$ is a constant. Let $\epsilon\to 0$ in (2.20), we obtain (ii). Equation (2.20) also implies (iii) since we may assume $H(y)<0$. □

Lemma 2.6. *For every $\lambda,\mu\geq 0$ we have*

$$\frac{S_{\lambda,\mu}(\Omega^c,1)}{Q_M^{\frac{N-2}{N}}}\leq S_{\lambda,\mu}(\Omega^c,Q)\leq S_\infty.$$

If $Q(x)<Q_\infty$ for every $x\in\Omega^c$, Q_M on the left side should be replaced by Q_∞.

Proof. The first inequality is obvious. To show the second inequality, first, since $Q_\infty=\lim_{|x|\to\infty}Q(x)$, for any $\varepsilon>0$ there exists $R>0$ such that $\Omega\subset B_R(0)$, and $Q(x)\geq Q_\infty-\varepsilon$ for $x\in\mathbb{R}^N\setminus B_R(0)$. Hence,

$$S_{\lambda,\mu}(\Omega^c,Q)\leq\frac{\int_{\Omega^c}(|\nabla u|^2+|\nabla v|^2+\lambda u^2+\mu v^2)\,dx}{(Q_\infty-\varepsilon)^{\frac{N-2}{N}}\big(\int_{\mathbb{R}^N\setminus B_R(0)}|u|^\alpha|v|^\beta\,dx\big)^{2/2^*}}$$

for all $u,v\in H^1(\Omega^c)$. Taking infimum over $u,v\in H_0^1(\mathbb{R}^N\setminus B_R(0))$ and noting that $S_{\alpha,\beta}$ is independent of Ω^c, since $\epsilon>0$ is arbitrary, we obtain from the above inequality that

$$S_{\lambda,\mu}(\Omega^c,Q)\leq\frac{S_{\alpha,\beta}}{Q_\infty^{\frac{N-2}{N}}}.$$

Next, if $Q(y) = Q_M$ for some $y \in \Omega^c$, using $(sU_{\varepsilon,y}, tU_{\varepsilon,y})$ as test function in the expression of $S_{\lambda,\mu}(\Omega^c, Q)$, where s and t satisfy (2.16), we obtain

$$S_{\lambda,\mu}(\Omega^c, Q) \le \frac{S_{\alpha,\beta}}{Q_M^{\frac{N-2}{N}}}.$$

Finally, if $y \in \partial\Omega$ is such that $Q_m = Q(y)$, we deduce as (2.20) that

$$S_{\lambda,\mu}(\Omega^c, Q) \le \frac{S_{\alpha,\beta}}{2^{2/N} Q_m^{\frac{N-2}{N}}}.$$

The proof is complete. □

3. Case $Q_M \le 2^{2/(N-2)} Q_m$

We will prove Theorem 1.2.

Proposition 3.1. *Assume $Q_M \le 2^{2/(N-2)} Q_m$ and (Q1). Then there holds*

$$S_{\lambda,\mu}(\Omega^c, Q) < \frac{S_{\alpha,\beta}}{2^{2/N} Q_m^{(N-2)/N}}. \tag{3.1}$$

Proof. If $N \ge 5$, let s, $t > 0$ be chosen as (2.18). Then

$$\frac{J_{\lambda,\mu}(sU_{\varepsilon,y}, tU_{\varepsilon,y})}{\left(\int_{\Omega^c} Q(x)|sU_{\varepsilon,y}|^\alpha |tU_{\varepsilon,y}|^\beta\, dx\right)^{2/2^*}} \le A_{\alpha,\beta} \frac{\int_{\Omega^c} [|\nabla U_{\varepsilon,y}|^2 + (\lambda+\mu) U_{\varepsilon,y}^2]\, dx}{\left(\int_{\Omega^c} Q(x) U_{\varepsilon,y}^{2^*}\right)^{2/2^*}}. \tag{3.2}$$

By the assumption (Q1),

$$\int_{\Omega^c} Q(x) U_{\varepsilon,y}^{2^*}\, dx = Q_m \int_{\Omega^c} U_{\varepsilon,y}^{2^*}\, dx + o(\varepsilon). \tag{3.3}$$

Corollary 2.5, (3.2) and (3.3) yield

$$\begin{aligned} S_{\lambda,\mu}(\Omega^c, Q) &\le \frac{J_{\lambda,\mu}(sU_{\varepsilon,y}, tU_{\varepsilon,y})}{\left(\int_{\Omega^c} Q(x)|sU_{\varepsilon,y}|^\alpha |tU_{\varepsilon,y}|^\beta\, dx\right)^{2/2^*}} \\ &< \frac{SA_{\alpha,\beta}}{2^{2/N} Q_m^{(N-2)/N}} = \frac{S_{\alpha,\beta}}{2^{2/N} Q_m^{(N-2)/N}}. \end{aligned}$$

If $N = 3, 4$, we replace $U_{\varepsilon,y}$ by $U_{\varepsilon,y}\phi_R$, where $\phi_R \in C^1(\mathbb{R}^N)$, $\phi_R(x) = 1$ for $x \in B_R(0)$, $\phi_R(x) = 0$ for $x \in \mathbb{R}^N \backslash B_{R+1}(0)$, and $0 \le \phi_R(x) \le 1$ on $\mathbb{R}^N$. Then, (3.1) can be proved in the same way. □

4. Case $Q_M > 2^{2/(N-2)} Q_m$

In this section, we show (ii) of Theorem 1.2.

Proposition 4.1. *Suppose $Q_M > 2^{2/(N-2)} Q_m$ and (Q2), then there exists $\Lambda > 0$ such that*

$$S_{\lambda,\mu}(\Omega^c, Q) < \frac{S_{\alpha,\beta}}{Q_M^{\frac{N-2}{N}}},$$

for all $0 \le \lambda,\ \mu < \Lambda$.

Proof. First we consider the case $N \geq 5$. For any $\delta > 0$, using (1.7) we have

$$\int_{\Omega^c} Q(x)|sU_{\varepsilon,y}|^\alpha |tU_{\varepsilon,y}|^\beta\,dx = \int_{\Omega^c} s^\alpha t^\beta Q_M U_{\varepsilon,y}^{2^*}\,dx + \int_{\Omega^c} s^\alpha t^\beta (Q(x)-Q_M)U_{\varepsilon,y}^{2^*}\,dx$$

Since

$$\int_{\Omega^c} s^\alpha t^\beta Q_M U_{\varepsilon,y}^{2^*}\,dx = Q_M \int_{\mathbb{R}^N} s^\alpha t^\beta U_{\varepsilon,y}^{2^*}\,dx - Q_M \int_{\Omega} s^\alpha t^\beta U_{\varepsilon,y}^{2^*}\,dx$$

and

$$\begin{aligned}
&\int_{\Omega^c} s^\alpha t^\beta (Q(x)-Q_M)U_{\varepsilon,y}^{2^*}\,dx \\
&= \int_{\Omega^c\cap B_\delta(y)} s^\alpha t^\beta (Q(x)-Q_M)U_{\varepsilon,y}^{2^*}\,dx + \int_{\Omega^c\setminus B_\delta(y)} s^\alpha t^\beta (Q(x)-Q_M)U_{\varepsilon,y}^{2^*}\,dx \\
&= \int_{\Omega^c\cap B_\delta(y)} s^\alpha t^\beta o(|x-y|^{N-2})U_{\varepsilon,y}^{2^*}\,dx + o(\varepsilon^N),
\end{aligned}$$

we have

$$\int_{\Omega^c} Q(x)|sU_{\varepsilon,y}|^\alpha |tU_{\varepsilon,y}|^\beta\,dx = s^\alpha t^\beta Q_M K_2 + o(\varepsilon^{N-2}), \tag{4.1}$$

where $K_2 = \int_{\mathbb{R}^N} U^{2^*}\,dx$. Since $y \in \Omega^c$, there exists a constant $C_1 > 0$ such that

$$\int_{\Omega^c} |\nabla U_{\varepsilon,y}|^2\,dx = \int_{\mathbb{R}^N} |\nabla U_{\varepsilon,y}|^2\,dx - \int_{\Omega} |\nabla U_{\varepsilon,y}|^2\,dx \leq K_1 - C_1\varepsilon^{N-2},$$

where $K_1 = \int_{\mathbb{R}^N} |\nabla U|^2\,dx$, and $\frac{K_1}{(K_2)^{2/2^*}} = S$. Hence,

$$\begin{aligned}
&\frac{\int_{\Omega^c}[|\nabla(sU_{\varepsilon,y})|^2 + |\nabla(tU_{\varepsilon,y})|^2 + \lambda(sU_{\varepsilon,y})^2 + \mu(tU_{\varepsilon,y})^2]\,dx}{\big(\int_{\Omega^c} s^\alpha t^\beta Q(x)U_{\varepsilon,y}^{2^*}\big)^{2/2^*}} \\
&\leq \frac{s^2+t^2}{(s^\alpha t^\beta)^{2/2^*}} \frac{\int_{\Omega^c}[|\nabla U_{\varepsilon,y}|^2 + \max\{\lambda,\mu\}U_{\varepsilon,y}^2]\,dx}{(\int_{\Omega^c} Q(x)U_{\varepsilon,y}^{2^*})^{2/2^*}} \\
&\leq \frac{s^2+t^2}{(s^\alpha t^\beta)^{2/2^*}} \frac{K_1 - C_1\varepsilon^{N-2} + K_3\max\{\lambda,\mu\}\varepsilon^2}{(Q_M K_2 + o(\varepsilon^{N-2}))^{2/2^*}} \\
&= A_{\alpha,\beta}\big(K_1 - C_1\varepsilon^{N-2} + K_3\max\{\lambda,\mu\}\varepsilon^2\big)\big\{(Q_M K_2)^{-(N-2)/N} \\
&\quad - \frac{N-2}{N}(Q_M K_2)^{-(2N+2)/N} o(\varepsilon^{N-2})\big\} \\
&< \frac{A_{\alpha,\beta}S}{Q_M^{(N-2)/N}} = \frac{S_{\alpha,\beta}}{Q_M^{(N-2)/N}}
\end{aligned}$$

for $\varepsilon > 0$, λ and $\mu \geq 0$ sufficiently small, where K_3 is a constant independent of ε. If $N = 3, 4$, we replace $U_{\varepsilon,y}$ by $U_{\varepsilon,y}\phi_R$, where ϕ_R is a C^1 function such that $\phi_R = 1$ if $x \in B_R(y)$, $\phi_R = 0$ if $x \in \mathbb{R}^N \subset B_{R+1}(y)$ with $R > 0$ large, and we may proceed in the same way. □

5. Case $S_\infty = \frac{S_{\alpha,\beta}}{Q_\infty^{(N-2)/N}}$

In this section, we show (iii) of Theorem 1.2.

Proposition 5.1. *Suppose* (1.8) *holds, then there exists* $\Lambda > 0$ *such that for* $0 \le \lambda$, $\mu < \Lambda$ *there holds*

$$S_{\lambda,\mu}(\Omega^c, Q) < \frac{S_{\alpha,\beta}}{Q_\infty^{(N-2)/N}}.$$

Proof. Let $R > 0$ be such that $\Omega \subset B_{R/2}(0)$ and K_δ denote the cone $K_\delta = \{\tau\bar{y}; \bar{y} \in \partial B_1(0) \cap B_\delta(\bar{z}), \tau > 0\}$, then for s and t satisfying (2.18) we have

$$\begin{aligned}
&\int_{\Omega^c} Q(x)|sU_{\varepsilon,R\bar{z}}|^\alpha |tU_{\varepsilon,R\bar{z}}|^\beta \,dx \\
&= s^\alpha t^\beta \Big\{ \int_{B_{R/2}(0)\setminus\Omega} Q(x) U_{\varepsilon,R\bar{z}}^{2^*} \,dx + \int_{\mathbb{R}^N\setminus B_{R/2}(0)} (Q(x) - Q_\infty) U_{\varepsilon,R\bar{z}}^{2^*} \,dx \\
&\quad + \int_{\mathbb{R}^N\setminus B_{R/2}(0)} Q_\infty U_{\varepsilon,R\bar{z}}^{2^*} \,dx \Big\} \\
&\ge s^\alpha t^\beta \Big\{ \int_{\mathbb{R}^N\setminus(B_{R/2}(0)\cup K_\delta)} (Q(x) - Q_\infty) U_{\varepsilon,R\bar{z}}^{2^*} \,dx \\
&\quad + \int_{K_\delta\setminus B_{R/2}(0)} (Q(x) - Q_\infty) U_{\varepsilon,R\bar{z}}^{2^*} \,dx + \int_{\mathbb{R}^N} Q_\infty U_{\varepsilon,R\bar{z}}^{2^*} \,dx \\
&\quad - \int_{B_{R/2}(0)} Q_\infty U_{\varepsilon,R\bar{z}}^{2^*} \,dx \Big\} \\
&= s^\alpha t^\beta (I_1 + I_2 + Q_\infty K_2 + I_3).
\end{aligned} \tag{5.1}$$

Since $|x - R\bar{z}| > \delta R$ if $x \in B_{2R}(0) \setminus (B_{R/2}(0)) \cup K_\delta)$, and $|x - R\bar{z}| \ge |x|/2$ if $x \in \mathbb{R}^N \setminus B_{2R}(0)$,

$$\begin{aligned}
I_1 &\ge -Q_\infty \int_{\mathbb{R}^N\setminus(B_{R/2}(0)\cup K_\delta)} \frac{\varepsilon^N C_N^{2^*}}{(N(N-2)\varepsilon^2 + |x - R\bar{z}|^2)^N} \,dx \\
&\ge -Q_\infty \varepsilon^N C_N^{2^*} \int_{R/2}^{2R} \frac{r^{N-1}}{(\delta R)^{2N}} \,dr - Q_\infty \varepsilon^N C_N^{2^*} \int_{2R}^\infty \frac{r^{N-1}}{(\frac{r}{2})^{2N}} \,dr \\
&= -C \frac{\varepsilon^N}{R^N}.
\end{aligned} \tag{5.2}$$

Next, by assumption (1.8),

$$I_2 = \int_{K_\delta\setminus B_{R/2}(0)} (Q(x) - Q_\infty) U_{\varepsilon,R\bar{z}}^{2^*} \,dx \ge -\frac{C2^p}{R^p} \int_{\mathbb{R}^N} U_{\varepsilon,R\bar{z}}^{2^*} \,dx = -\frac{CK_2}{R^p}. \tag{5.3}$$

Finally, by the fact that $|x - R\bar{z}| \ge \frac{R}{2}$ for $x \in B_{R/2}(0)$,

$$I_3 = \int_{B_{R/2}(0)} Q_\infty U_{\varepsilon,R\bar{z}}^{2^*} \,dx \ge -\frac{Q_\infty C_N^{2^*} 2^{2N}}{R^{2N}} \int_0^{\frac{R}{2}} r^{N-1} \,dr = -\frac{CK_2}{R^N} \tag{5.4}$$

Consequently,

$$\int_{\Omega^c} Q(x)|sU_{\varepsilon,R\bar{z}}|^\alpha |tU_{\varepsilon,R\bar{z}}|^\beta \,dx \ge s^\alpha t^\beta \Big(K_2 Q_\infty - C\frac{\varepsilon^N}{R^p} - \frac{CK_2}{R^p} \Big). \tag{5.5}$$

On the other hand,

$$\begin{aligned}\int_{\Omega^c} |\nabla U_{\varepsilon,R\bar{z}}|^2\,dx &= \int_{\mathbb{R}^N} |\nabla U_{\varepsilon,R\bar{z}}|^2\,dx - \int_{\Omega} |\nabla U_{\varepsilon,R\bar{z}}|^2\,dx \\ &\le K_1 - C_N^2(N-2)\varepsilon^{N-2}\int_{\Omega} \frac{|x-R\bar{z}|^2}{(\varepsilon^2 N(N-2)+|x-R\bar{z}|^2)^N}\,dx \\ &\le K_1 - \frac{C\varepsilon^{N-2}}{R^{2N-2}}.\end{aligned} \tag{5.6}$$

Therefore,

$$\frac{J_{\lambda,\mu}(sU_{\varepsilon,R\bar{z}}, tU_{\varepsilon,R\bar{z}})}{(\int_{\Omega^c} Q(x)|sU_{\varepsilon,R\bar{z}}|^\alpha |tU_{\varepsilon,R\bar{z}}|^\beta\,dx)^{2/2^*}} \le A_{\alpha,\beta}\frac{K_1 - C\frac{\varepsilon^{N-2}}{R^{2N-2}} + \max\{\lambda,\mu\}C\varepsilon^2}{(K_2Q_\infty - C\frac{\varepsilon^N}{R^N} - \frac{CK_2}{R^p})^{(N-2)/N}}. \tag{5.7}$$

Hence, there exist constants $A>0$, $B>0$, $C>0$ and $D>0$ such that

$$\begin{aligned}&\frac{J_{\lambda,\mu}(sU_{\varepsilon,R\bar{z}}, tU_{\varepsilon,R\bar{z}})}{(\int_{\Omega^c} Q(x)|sU_{\varepsilon,R\bar{z}}|^\alpha |tU_{\varepsilon,R\bar{z}}|^\beta\,dx)^{2/2^*}} \\ &\le \frac{SA_{\alpha,\beta}}{Q_\infty^{(N-2)/N}} - \frac{A\varepsilon^{N-2}}{R^{2N-2}} + \max\{\lambda,\mu\}B\varepsilon^2 + \frac{C\varepsilon^N}{R^N}\frac{D}{R^p}.\end{aligned}$$

If $\lambda=\mu=0$, we choose $\varepsilon=\varepsilon(R)$ such that

$$\frac{A\varepsilon^{N-2}}{2R^{2N-2}} = \frac{C\varepsilon^N}{R^N}, \quad \text{i.e., } \frac{1}{\varepsilon^2} = \frac{2CR^{N-2}}{A}.$$

Then, we choose $R>0$ so that

$$\frac{A\varepsilon^{N-2}}{2R^{2N-2}} = \frac{A}{2R^{2N-2}}\Big(\frac{A}{2CR^{N-2}}\Big)^{(N-2)/2} > \frac{D}{R^p};$$

that is,

$$\frac{A^{N/2}}{2^{N/2}R^{N^2/2}} > \frac{D}{R^p}\,,$$

which is possible if $p > \frac{N^2}{2}$. Hence for this choice of R we have

$$\frac{J_{\lambda,\mu}(sU_{\varepsilon,R\bar{z}}, tU_{\varepsilon,R\bar{z}})}{(\int_{\Omega^c} Q(x)|sU_{\varepsilon,R\bar{z}}|^\alpha |tU_{\varepsilon,R\bar{z}}|^\beta\,dx)^{2/2^*}} < \frac{S_{\alpha,\beta}}{Q_\infty^{(N-2)/N}}.$$

If $\lambda,\mu>0$. we can similarly choose R such that the above inequality holds. The proof is complete. □

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Shengbing Deng
Department of Mathematics, Jiangxi Normal University, Nanchang, Jiangxi 330022, China
E-mail address: shbdeng@yahoo.com.cn

Jianfu Yang
Department of Mathematics, Jiangxi Normal University, Nanchang, Jiangxi 330022, China
E-mail address: jfyang_2000@yahoo.com