Electronic Journal of Differential Equations, Vol. 2008(2008), No. 156, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

N-BODY PROBLEM IN $\mathbb{R}^n$: NECESSARY CONDITIONS FOR A CONSTANT CONFIGURATION MEASURE

K. ZARE

Abstract. A formulation of the N-body problem is presented in which m_i and $r_i \in \mathbb{R}^d$ are the mass and the position vector of the i-th body, $x = (\sqrt{m_1}r_1, \dots, \sqrt{m_N}r_N) \in \mathbb{R}^n$ and $n = dN$ $(d = 1, 2, 3)$. The configuration measure $Z = |x|F$, where F is the Poincare's force function, which plays an important role in this formulation. The orbit plane is a two dimensional linear subspace of $\mathbb{R}^n$ spanned by the position vector x and the velocity vector $\dot{x}$. The N-body motion in $\mathbb{R}^n$ has been decomposed into an orbit in the orbit plane and the instantaneous orientation of the orbit plane. For a solution to stay on a level manifold of Z, it is necessary that the orbit in the orbit plane be elliptic $(h < 0)$, parabolic $(h = 0)$or hyperbolic $(h > 0)$ where h is the total energy. The instantaneous orientation of the orbit plane can be obtained by integration of certain differential equations. These possible solutions include the central configuration solutions in which the orbit plane is fixed in $\mathbb{R}^n$.

1. Introduction

The problem of N bodies may be simply stated as follows. N point masses are attracting each other according to the Newton's laws. Given the initial positions and velocities determine their subsequent motion. The equations of motion are

$$m_i\ddot{r}_i = \frac{\partial F}{\partial r_i}, \quad i = 1, \dots, N \tag{1.1}$$

where m_i and r_i are the masses and the position vectors, a dot over a variable indicates differentiation with respect to time t and

$$F = \sum_{i=1}^{N}\sum_{j>i}^{N} \frac{Gm_im_j}{|r_i - r_j|} \tag{1.2}$$

is the Poincaré's force function or the negative potential. System (1.1) possesses ten first integrals (i.e. the classical integrals). They are: the energy integral

$$h = \frac{1}{2}\sum_{i=1}^{N} m_i\dot{r}_i \cdot \dot{r}_i - F\,, \tag{1.3}$$

2000 *Mathematics Subject Classification.* 70F10.
Key words and phrases. Configuration measure; Saari's conjecture; central configurations; generalized vector product; generalized momentum and eccentricity vectors.

Submitted June 4, 2008. Published November 20, 2008.

the angular momentum integrals

$$c = \sum_{i=1}^{N} m_i r_i \times \dot{r_i}\,, \tag{1.4}$$

and the linear momentum integrals

$$\sum_{i=1}^{N} m_i \dot{r_i} = 0 \quad \text{and} \quad \sum_{i=1}^{N} m_i r_i = 0. \tag{1.5}$$

Note that without loss of generality we have assumed that the origin is at the center of mass.

Some influential works on the subject have been published by Euler [5], Lagrange [8], Jacobi [7], Poincaré [12], Painlevé [11], Sundman [18], Birkholf [1], Chazy [2], Wintner [19], Smale [17], Siegel and Moser [16], Saari[14], etc. There are two publications by Saari that are relevant to the subject of this article. In the first paper, Saari [13] stated the following conjecture (known as the Saari's conjecture):

> If the polar moment of inertia, I, is a constant, then the N-body motion is that of a rotating rigid body.

In the second paper, Saari [15] generalized his original conjecture and stated the following extended conjecture:

> The configuration measure is a constant if and only if the N-body motion is homo-graphic.

Despite the simplicity of its statement, the Saari's conjecture and its generalization have not been proved in the general case. However, in the recent years, significant progress has been made in special cases by Diacu et al. [3, 4], Moeckel [10], and Saari [14], just to mention a few.

In this paper, a new formulation in $\mathbb{R}^n$ is given where $x = (\sqrt{m_1} r_1, \dots, \sqrt{m_N} r_N)$ is in $\mathbb{R}^n$, $r_i \in \mathbb{R}^d$, and $n = dN$ $(d = 1, 2, 3)$. The function $Z = |x|F$ which is equivalent to the configuration measure in the Saari's extended conjecture plays an important role. Using this formulation, the necessary conditions for the configuration measure to be a constant have been obtained.

We begin in Section 2 with the new formulation in $\mathbb{R}^n$ and its similarities to the central force problem. A generalization of vector product is defined in Subsection 2.1. This generalization is necessary to define the generalized angular momentum and eccentricity vectors and to find their properties in Subsection 2.2. As a by product an exact identity for the Golubev's inequality [6] has been found. A new independent variable is introduced in Subsection 2.3. This is necessary to express certain solutions analytically. The orbit plane is defined in Subsection 2.4 and a decomposition of solutions into the orbit plane has been obtained. The necessary (but not sufficient) conditions for a solution to stay on a level manifold of Z are given in Section 3.

2. Formulation in $\mathbb{R}^n$

In this section, we formulate the problem in $\mathbb{R}^n$. We begin with the following theorem.

Theorem 2.1. *The equations of motion* (1.1) *are equivalent to*

$$\ddot{x} = \frac{-Z}{|x|^3}x + \frac{1}{|x|}Z_x, \tag{2.1}$$

where $x = (\sqrt{m_1}\ r_1, \dots, \sqrt{m_N}\ r_N)$ *is an* n*-dimensional vector* ($n = dN$, $d = 1, 2, 3$), $Z = |x|F$, *and*

$$Z_x \cdot x = 0\,. \tag{2.2}$$

Proof. First we note that (1.1) may be written as $\ddot{x} = F_x$. If $Z = |x|F$, then

$$Z_x = \frac{F}{|x|}x + |x|F_x \quad \text{or} \quad F_x = -\frac{F}{|x|^2}x + \frac{1}{|x|}Z_x = -\frac{Z}{|x|^3} + \frac{1}{|x|}Z_x$$

This leads to the equivalence of (1.1) and (2.1). Equation (2.2) follows from the Euler's theorem noting that Z is a homogeneous function of degree zero. □

Remark: The first term for $\ddot{x}$ in (2.1) has been also given in Saari [14] and it was known to Wintner [19]. The second term is new replacing the unspecified function D in the Saari's formulation. This is a necessary step for the analysis that follows.

Theorem 2.2. *If the origin is at the center of mass, the Jacobi function*

$$J(x) = \frac{1}{M}\sum_{i=1}^{N}\sum_{j>i}^{N} m_i m_j |r_i - r_j|^2, \quad M = \sum_{i=1}^{N} m_i \tag{2.3}$$

is an equivalent norm of x.

Proof. Note that

$$\begin{aligned} J(x) &= \frac{1}{2M}\sum_{i=1}^{N}\sum_{j=1}^{N} m_i m_j (r_i - r_j)\cdot(r_i - r_j) \\ &= \frac{1}{2M}\Big[\sum_{i=1}^{N} m_i|r_i|^2 \sum_{j=1}^{N} m_j + \sum_{i=1}^{N} m_i \sum_{j=1}^{N} m_j|r_j|^2 - 2\big(\sum_{i=1}^{N} m_i r_i\big)\big(\sum_{j=1}^{N} m_j r_j\big)\Big] \\ &= |x|^2 \end{aligned}$$

where we have used (1.5). This completes the proof. □

We may now rewrite the classical integrals as functions of x and $\dot{x}$. In particular, the energy integral becomes

$$h = \frac{1}{2}(\dot{x}\cdot\dot{x}) - \frac{Z}{|x|}. \tag{2.4}$$

Definition: A solution $x_c(t)$ of (2.1) is called a central configuration solution if $Z_x(x_c(t)) = 0$.

Since $\dot{Z} = Z_x(x_c(t))\cdot\dot{x}_c(t) = 0$, it follows that on a central configuration solution $Z = \mu$, a constant, and (2.1) reduces to $\ddot{x} = -\mu x/|x|^3$. This is an n-dimensional generalization of the central force problem where a certain value of Z plays the role of the gravitational parameter. In general (2.1) is an n-dimensional central force problem in which the gravitational parameter Z is not a constant and there exists an additional non-central force $Z_x/|x|$. In the next sub-section, we define a vector product of two n-dimensional vectors. This assists us to generalize some of the familiar functions in the central force problem.

2.1. **A vector product for $\mathbb{R}^n$.** The inner product of two n-dimensional vectors is a function defined by $u\cdot v:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$, where $u\cdot v=u_1\cdot v_1+\dots+u_n\cdot v_n$. It follows that $|u|^2=u\cdot u$ where $|u|$ denotes the Euclidean norm of u. This is a direct generalization from $\mathbb{R}^3$. On the contrary the usual vector product for $\mathbb{R}^3$ can not be generalized directly. Here we show that by adding a complementary subspace to $\mathbb{R}^n$,we may define a vector product for $\mathbb{R}^n$ which preserves many properties of the usual vector product.

Let us first consider the usual vector product for $\mathbb{R}^2$, $u\times v:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}$ where $u\times v=(u_1v_2-u_2v_1)e_3$, $u=u_1e_1+u_2e_2$, $v=v_1e_1+v_2e_2$, e_1 and e_2 are an orthonormal bases of $\mathbb{R}^2$, and e_3 is a unit vector normal to the plane containing e_1 and e_2. Note that the vector product is not in $\mathbb{R}^2$, but in a complementary dimension normal to $\mathbb{R}^2$. The addition of this dimension to $\mathbb{R}^2$ forms a higher dimensional space, namely $\mathbb{R}^3$.

To generalize the vector product to $\mathbb{R}^n$ with an orthonormal bases $(e_1,\dots,e_n)$, we introduce a complementary subspace $\mathbb{R}^m$ normal to $\mathbb{R}^n$ with an orthonormal bases denoted by e_{ij} $(1\le i<j\le n)$. The vector product is a function defined by $u\times v:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^m$, $m=n(n-1)/2$, where

$$u\times v=\sum_{i=1}^{n}\sum_{j>i}^{n}(u_iv_j-u_jv_i)e_{ij}. \tag{2.5}$$

The addition of this subspace to $\mathbb{R}^n$ forms a higher dimensional space $\mathbb{R}^k$ with the orthonormal bases $(e_1,\dots,e_n,\dots,e_{ij},\dots)$, where $k=n+m=n(n+1)/2$. Note that in this extended space u and v are in the subspace $\mathbb{R}^n$ and $u\times v$ is in the complementary subspace $\mathbb{R}^m$.

Theorem 2.3. *If u,v,w,z are vectors in $\mathbb{R}^n$, the vector product is defined by* (2.5) *and the inner product is defined in $\mathbb{R}^k$, then*

- (I) $u\times u=0$,
- (II) $u\cdot(u\times v)=0$,
- (III) $v\cdot(u\times v)=0$,
- (IV) $(u\times v)\cdot(w\times z)=(u\cdot w)(v\cdot z)-(u\cdot z)(v\cdot w)$,
- (V) $|u\times v|^2+(u\cdot v)^2=|u|^2|v|^2$,
- (VI) *and assuming $u(t)$ and $v(t)$ are differentiable:* $(u\times v)^{\cdot}=\dot u\times v+u\times\dot v$.

Proof. The proofs for (I)–(III) follow directly from the definition. To prove (IV), we observe that

$$\begin{aligned}(u\times v)\cdot(w\times z)&=\frac12\sum_{i=1}^{n}\sum_{j=1}^{n}(u_iv_j-u_jv_i)(w_iz_j-w_jz_i)\\&=\frac12\Big(\sum_{i=1}^{n}u_iw_i\sum_{j=1}^{n}v_jz_j-\sum_{i=1}^{n}u_iz_i\sum_{j=1}^{n}v_jw_j\\&\quad-\sum_{i=1}^{n}v_iw_i\sum_{j=1}^{n}u_jz_j+\sum_{i=1}^{n}v_iz_i\sum_{j=1}^{n}u_jw_j\Big)\\&=(u\cdot w)(v\cdot z)-(u\cdot z)(v\cdot w).\end{aligned}$$

To prove (V), let $w=u$, and $z=v$ in (IV). The identity (VI) follows directly by differentiating (2.5) with respect to t. $\square$

Remark: For $n = 3$, we have $m = 3$, and the complementary subspace has the same dimension as the original space. In this case we may match the two spaces by letting $e_{12} = e_3$, $e_{23} = e_1$, and $e_{13} = -e_2$. This leads to the usual definition of the vector product in $\mathbb{R}^3$. This matching is not possible for $n > 3$ since $m > n$.

2.2. **Generalized vectors.** We have already shown that (2.1) may be considered as a generalization of the central force problem. With this point of view, it is natural to generalize the vector functions which appear in that problem.

Definition: The generalized angular momentum vector C is defined as

$$C = x \times \dot{x} = \sum_{i=1}^{n}\sum_{j>i}^{n}(x_i\dot{x_j} - x_j\dot{x_i})e_{ij} \tag{2.6}$$

It follows that

$$|C|^2 = C \cdot C = \sum_{i=1}^{n}\sum_{j>i}^{n}(x_i\dot{x_j} - x_j\dot{x_i})^2. \tag{2.7}$$

The difference between this norm and the norm of the angular momentum vector is

$$\phi^2 = |C|^2 - |c|^2 = \sum_{i=1}^{N}\sum_{j=di}^{n}\sum_{k=1}^{d}(x_{d(i-1)+k}\dot{x}_j - x_j\dot{x}_{d(i-1)+k})^2. \tag{2.8}$$

Theorem 2.4. *The rate of change of the generalized angular momentum satisfies*

$$\dot{C} = \frac{1}{|x|}x \times Z_x. \tag{2.9}$$

Proof. Using (VI) in Theorem 2.3,

$$\begin{aligned}
\dot{C} &= x \times \ddot{x} + \dot{x} \times \dot{x} \\
&= x \times \ddot{x} \quad \text{(using (I) in Theorem 2.3)} \\
&= -\frac{Z}{|x|^3}x \times x + \frac{1}{|x|}x \times Z_x \quad \text{(using (2.1))} \\
&= \frac{1}{|x|}x \times Z_x
\end{aligned}$$

were we have used (I) in Theorem 2.3. □

Corollary 2.5. *The generalized angular momentum vector is fixed if and only if the solution is a central configuration*

Proof. From (2.9),

$$\begin{aligned}
|\dot{C}| &= \frac{|x \times Z_x|}{|x|} \\
&= \frac{\sqrt{|x|^2|Z_x|^2 - x \cdot Z_x}}{|x|} \quad \text{(using (V) in Theorem 2.3)} \\
&= |Z_x|,
\end{aligned}$$

where we have used (2.2). □

Corollary 2.6. *The norm of the generalized angular momentum vector is fixed if and only if the solution is on a level manifold of Z.*

Proof. Using (2.6) and (2.9),

$$\begin{aligned}\frac{d}{dt}|C|^2 &= \frac{2}{|x|}(x\times\dot{x})\cdot(x\times Z_x)\\ &= \frac{2}{|x|}[(x\cdot x)(Z_x\cdot\dot{x})-(x\cdot Z_x)(x\cdot\dot{x})] \quad \text{(using (IV) in Theorem 2.3)}\\ &= 2|x|\dot{Z},\end{aligned}$$

where we have used (2.2). □

The generalized angular momentum may be used to obtain the following result.

Corollary 2.7. *The norm of x satisfies*

$$\frac{d^2|x|}{dt^2} = -\frac{Z}{|x|^2}+\frac{|C|^2}{|x|^3}.$$

Proof. From $|x|^2 = x\cdot x$ and differentiating,

$$\begin{aligned}\frac{d^2|x|}{dt^2} &= \frac{1}{|x|}(\dot{x}\cdot\dot{x}+x\cdot\ddot{x}-\frac{(x\cdot\dot{x})^2}{|x|^2})\\ &= \frac{1}{|x|}(\dot{x}\cdot\dot{x}-\frac{Z}{|x|}-\frac{(x\cdot\dot{x})^2}{|x|^2}) \quad \text{(using (2.1) and (2.2))}\\ &= -\frac{Z}{|x|^2}+\frac{|C|^2}{|x|^3},\end{aligned}$$

where we used (V) in Theorem 2.3 and (2.6). □

Remark: A theorem in Saari[14] states that "the equation

$$\frac{d^2|x|}{dt^2} = -\frac{A}{|x|^2}+\frac{B}{|x|^3},$$

where A and B are constants, holds for an N-body solution if and only if Z is a constant. In this case $A = Z$." Note that Corollary 2.7 holds for any N-body solution, and Saari's theorem follows immediately from Corollary 2.6.

Definition: The generalized eccentricity vectors e and f are defined as

$$Ze = (2h+\frac{Z}{|x|})x-(x\cdot\dot{x})\dot{x}, \tag{2.10}$$

$$Zf = \frac{Z(x\cdot\dot{x})}{|C||x|}x+\frac{|C|^2-|x|Z}{|C|}\dot{x}. \tag{2.11}$$

Theorem 2.8. *The generalized vectors satisfy the following:*

(I) $Z^2(1-|e|^2) = -2|C|^2h$
(II) $|f| = |e|$, *and*
(III) $e\cdot f = 0$.

Proof. (I) Using (2.4) and (2.10),

$$Z^2|e|^2 = Z^2+(|\dot{x}|^2-\frac{2Z}{|x|})(|x|^2|\dot{x}|^2-(x\cdot\dot{x})^2) = Z^2+2|C|^2h$$

where we used (2.4), (2.6), and (V) in Theorem 2.3.

(II) Using (2.11),

$$\begin{aligned}
Z^2|f|^2 &= \frac{Z^2}{|C|^2}(|x|^2|\dot{x}|^2-(x\cdot\dot{x})^2)+|C|^2|\dot{x}|^2-\frac{2Z}{|x|}(|x|^2|\dot{x}|^2-(x\cdot\dot{x})^2)\\
&= Z^2+|C|^2(|\dot{x}|^2-2\frac{Z}{|x|}) \quad \text{(using (V) in Theorem 2.3 and (2.6))}\\
&= Z^2+2|C|^2h
\end{aligned}$$

where we used (2.4). This leads to $|f|=|e|$.

(III) Using (2.10) and (2.11),

$$\begin{aligned}
Z^2 e\cdot f &= |C|(x\cdot\dot{x})(-|\dot{x}|^2+2h+\frac{Z}{|x|})+\frac{Z(x\cdot\dot{x})}{|C||x|}(|x|^2|\dot{x}|^2-(x\cdot\dot{x})^2)\\
&= |C|(x\cdot\dot{x})(\frac{-Z}{|x|})+\frac{Z(x\cdot\dot{x})}{|C||x|}|C|^2=0,
\end{aligned}$$

where we used (2.4), (2.6), and (V) in Theorem 2.3. □

Corollary 2.9. *The norm of the generalized eccentricity vector is bounded as follows*

$$\begin{gathered}
0\le|e|\le 1, \quad \text{if } h<0,\\
|e|=1, \quad \text{if } h=0,\\
|e|\ge 1, \quad \text{if } h>0.
\end{gathered}$$

The proof of the above corollary follows immediately from (I) in Theorem 2.8.

Corollary 2.10. *If $h<0$, then $Z^2\ge -2|c|^2h$.*

Proof. Using corollary 2.9,

$$\begin{aligned}
Z^2 &\ge Z^2(1-|e|^2)\\
&= -2|C|^2h \quad \text{(using (I) in Theorem 2.8)}\\
&= -2(|c|^2+\phi^2)h \quad \text{(using (2.8))}\\
&\ge -2|c|^2h.
\end{aligned}$$

□

Remark: The inequality in Corollary 2.10 was first obtained by Golubev [6] using Sundman's inequality, and independently later by Marchal and Saari [9] using the same method, and by Zare [20] and [21] using a theorem for Hamiltonian systems that identifies regions of motion in the configuration space. Combined with the reduction theory in Hamiltonian dynamics, this theorem is applicable to any mechanical system which has integrals linear in the momenta in addition to the energy integral. The theorem has been also applied to the problem of a rigid body with a fixed point by Zare and Levinson [23]. In the three body problem, this inequality leads to the forbidden triangular configurations and to a sufficient condition for no binary exchanges. This topic became very popular in the late seventies and the eighties and many articles on the subject appeared in the literature during that period. I became aware of Golubev's article when Roger Broucke gave me a copy in 1979. To acknowledge the priority, I called it Golubev's inequality in my 1981 paper [22]. For more information on this subject the interested reader may consult the references. The Identity I in Theorem 2.8 is the best possible improvement to this inequality since it is an identity rather than a sharper inequality.

Theorem 2.11. *The rates of change of the generalized eccentricities satisfy*

$$\frac{d}{dt}Ze = \frac{\dot{Z}}{|x|}x - \frac{x \cdot \dot{x}}{|x|}Z_x, \tag{2.12}$$

$$\frac{d}{dt}Zf = \frac{\dot{Z}}{|C|^3}[\frac{x \cdot \dot{x}}{|x|}(|C|^2 - |x|Z)x + |x|^2 Z\dot{x}] + \frac{(|C|^2 - |x|Z)}{|C||x|}Z_x \tag{2.13}$$

Proof. Differentiating (2.10),

$$\begin{aligned} \frac{d}{dt}Ze &= (\frac{\dot{Z}}{|x|} - Z\frac{x \cdot \dot{x}}{|x|^3})x + (2h + \frac{Z}{|x|} - \dot{x} \cdot \dot{x} - x \cdot \ddot{x})\dot{x} - (x \cdot \dot{x})\ddot{x} \\ &= \frac{\dot{Z}}{|x|}x - \frac{x \cdot \dot{x}}{|x|}Z_x \end{aligned}$$

where we used (2.1), (2.2), (2.4). Differentiating (2.11) and using Corollary 2.6,

$$\begin{aligned} \frac{d}{dt}Zf &= -\frac{|x|\dot{Z}}{|C|^3}[\frac{Z(x \cdot \dot{x})}{|x|}x + (|C|^2 - |x|Z)\dot{x}] \\ &\quad + \frac{1}{|C|}[(\frac{\dot{Z}|x|^2(x \cdot \dot{x}) + Z|x|^2(\dot{x} \cdot \dot{x} + x \cdot \ddot{x}) - Z(x \cdot \dot{x})^2}{|x|^3})x \\ &\quad + (|x|\dot{Z})\dot{x} + (|C|^2 - |x|Z)\ddot{x}] \\ &= \frac{\dot{Z}}{|C|^3}[\frac{x \cdot \dot{x}}{|x|}(|C|^2 - |x|Z)x + |x|^2 Z\dot{x}] + \frac{(|C|^2 - |x|Z)}{|C||x|}Z_x \end{aligned}$$

where we used (2.1), (2.2), (2.6), and (V) in Theorem 2.3. □

Corollary 2.12. *The generalized eccentricity vectors are fixed if the solution is a central configuration.*

Proof. On a central configuration solution $Z_x(x_c(t) = 0$, and $\dot{Z}(x_c(t)) = Z_x(x_c(t)) \cdot \dot{x}_c(t) = 0$. Substitution into (2.12) and (2.13) leads to $\dot{e} = 0$ and $\dot{f} = 0$. □

Corollary 2.13. *The norm of the generalized eccentricity vectors is fixed if the solution is on a level manifold of Z.*

Proof. Differentiating (I) in Theorem 2.8),

$$\begin{aligned} \frac{d}{dt}|e|^2 &= \frac{2(1 - |e|^2)}{Z}\dot{Z} + \frac{2h}{Z^2}\frac{d}{dt}|C|^2 \\ &= \frac{-4h\dot{Z}}{Z^3}(|C|^2 - |x|Z), \end{aligned}$$

using (I) in Theorem 2.8 and Corollary 2.6. □

2.3. **A new independent variable.** In this sub-section we introduce a new independent variable s defined as

$$t' = \frac{dt}{ds} = |x|, \tag{2.14}$$

where prime indicates differentiation with respect to s. Using the new independent variable s and the generalized eccentricity e introduced in the previous sub-section we obtain the following result.

Theorem 2.14. *The equations of motion* (2.1) *are equivalent to*

$$x'' - 2hx = -Ze + |x|Z_x. \tag{2.15}$$

Proof. From (2.14),

$$\begin{aligned} x'' &= |x|^2\ddot{x} + (x\cdot\dot{x})\dot{x} \\ &= -\frac{Z}{|x|}x + |x|Z_x + (x\cdot\dot{x})\dot{x} \quad \text{(using (2.1))} \\ &= -\frac{Z}{|x|}x + |x|Z_x + (2h + \frac{Z}{|x|})x - Ze \quad \text{(using (2.10))} \\ &= 2hx - Ze + |x|Z_x \end{aligned}$$

which completes the proof. □

Corollary 2.15. *The norm of x satisfies*

$$|x|'' - 2h|x| = Z. \tag{2.16}$$

Proof. Using $|x|^2 = x.x$ and differentiating,

$$\begin{aligned} |x|'' &= \frac{x'\cdot x' + x\cdot x''}{|x|} - \frac{(x\cdot x')^2}{|x|^3} \\ &= \frac{4h|x|^2 + 2Z|x| - Ze\cdot x}{|x|} - \frac{(x\cdot x')^2}{|x|^3} \quad \text{(using (2.2), (2.4) and (2.15))} \\ &= 2h|x| + Z \end{aligned}$$

where we used (2.10),and (2.14)). □

2.4. **The orbit plane.** The geometric description of the N-body motion by introducing the orbit plane plays an important role in the following section.

Definition: The orbit plane is a two-dimensional linear subspace of $\mathbb{R}^n$ spanned by x and $\dot{x}$ or equivalently by the two orthonormal vectors $\hat{e} = e/|e|$ and $\hat{f} = f/|f|$.

Theorem 2.16. *If $|e|$ is not zero, the N-body motion can be decomposed into the orthonormal bases $(\hat{e},\hat{f})$ in the orbit plane as follows*

$$x(s) = \frac{|C|^2 - |x|Z}{Z|e|}\hat{e} + \frac{|C||x|'}{Z|e|}\hat{f} \tag{2.17}$$

$$x'(s) = -\frac{|x|'}{|e|}\hat{e} + \frac{|C|(2h|x| + Z)}{Z|e|}\hat{f} \tag{2.18}$$

Proof. Solving for x and $\dot{x}$ in the system of equations(2.10) and (2.11) leads to

$$x = \frac{1}{D}[\frac{(|C|^2 - |x|Z)Z}{|C|}e + (x\cdot\dot{x})Zf], \quad \dot{x} = \frac{1}{D}[-\frac{(x\cdot\dot{x})Z^2}{|C||x|}e + (2h + \frac{Z}{|x|})Zf],$$

where

$$D = \frac{1}{C}[2|C|^2h - 2h|x|Z - Z^2 + \frac{Z}{|x|}(|C|^2 + (x\cdot\dot{x})^2)] = \frac{Z^2|e|^2}{|C|},$$

using (2.4), (2.6), (V) in Theorem 2.3 and (I) in Theorem 2.8. Substituting D and using (2.14) leads to (2.17) and (2.18). □

3. Solutions on the level manifolds of Z

The importance of $Z(x)$ in our formulation can not be overemphasized. The function $Z(x)$ is invariant under the change of scale and the rotation. These properties follow since Z is a homogeneous function of degree zero and a function of the mutual distances only. In this section, we obtain the necessary conditions for a solution to stay on a level manifold of Z (i.e. $\dot{Z}(x(t)) = Z_x(x(t)) \cdot \dot{x}(t) = 0$). These solutions include the central configuration solutions (i.e. $Z_x(x_c(t)) = 0$).

Theorem 3.1. *If a solution is on a level manifold of $Z = \mu$, then*
(I) *$|C|$ is fixed*
(II) *$|e|$ is fixed, and*
(III)
$$|x| = \begin{cases} -\frac{\mu}{2h}(1 - |e|\cos(\omega s) & h < 0,\ \omega = \sqrt{-2h} \\ \frac{1}{2}\mu s^2 + |x_0| & h = 0, \\ \frac{\mu}{2h}(|e|\cosh(\omega s) - 1) & h > 0,\ \omega = \sqrt{2h}. \end{cases}$$

Proof. Parts (I) and (II) follow from Corollaries 2.6 and 2.13. To prove (III), let $h < 0$, then the solution of (2.16) with $Z = \mu$ is
$$|x| = -\frac{\mu}{2h}(1 - A\cos(\omega s))$$
where A is a constant of integration. This leads to $|x| = -\mu(1-A)/(2h)$ and $|x|' = 0$ at $s = 0$. Then from (2.10), the initial eccentricity vector is $e = \big(\frac{2h}{\mu} + \frac{1}{|x|}\big)x$. It follows that
$$|e| = (\frac{2h}{\mu} + \frac{1}{|x|})|x| = 1 + \frac{2h}{\mu}|x| = A.$$
The proofs for $h = 0$ and $h > 0$ are similar; we omit them. □

Theorem 3.2. *If a solution is on a level manifold of $Z = \mu$ and $h < 0$, then*
(I) *the orbit in the orbit plane is given by*
$$x = -\frac{\mu}{2h}[(\cos(\omega s) - |e|)\hat{e} + \sqrt{1 - |e|^2}\sin(\omega s)\hat{f}], \tag{3.1}$$
$$x' = \frac{\mu}{\omega}[-\sin(\omega s)\hat{e} + \sqrt{1 - |e|^2}\cos(\omega s)\hat{f}], \tag{3.2}$$
which is an ellipse with the semi-major axis $a = -\mu/(2h)$ and the eccentricity $|e|$. The major-axis and the minor-axis of the ellipse are respectively in $\hat{e}$ and $\hat{f}$ directions;
(II) *the instantaneous orientation of the orbit plane in $\mathbb{R}^n$ can be obtained from*
$$\hat{e}' = -\frac{1}{\omega}\sin(\omega s)Z_x, \tag{3.3}$$
$$\hat{f}' = \frac{1}{\omega\sqrt{1 - |e|^2}}(\cos(\omega s) - |e|)Z_x. \tag{3.4}$$

Proof. Since $Z = \mu$ and $\dot{Z} = 0$, by Theorem 3.1, $|C|$ and $|e|$ are fixed and
$$|x| = -\frac{\mu}{2h}(1 - |e|\cos(\omega s)).$$
Also $|C|^2 = -\mu^2(1 - |e|^2)/(2h)$ by (I) in Theorem 2.8. Substituting for Z, $|x|$, $|x|'$ and $|C|$ in (2.17) and (2.18) leads to (3.1) and (3.2). (3.3) and (3.4) have been obtained by substituting for Z, $\dot{Z}$, $|x|$, $|x|'$ and $|C|$ in (2.12) and (2.13). □

Remark: Similarly it can be shown that if a solution is on a level manifold of Z, the orbit in the orbit plane is a parabola ($h = 0$) or a hyperbola ($h > 0$).

Corollary 3.3. *If a solution is a central configuration and $h < 0$, then*
(I) *the orbit in the orbit plane is an ellipse, and*
(II) *the orbit plane is fixed in $\mathbb{R}^n$.*

Proof. On a central configuration solution, $Z_x(x_c(t)) = 0$ and $Z(x_c(t)) = \mu$ then (I) and (II) follow immediately from Theorem 3.2. □

Remark: Similarly if a solution is a central configuration, then the orbit is a fixed parabola ($h = 0$) or a fixed hyperbola ($h > 0$) in $\mathbb{R}^n$.

Corollary 3.4. *If a solution is on a level manifold of $Z = \mu$, then*
(I) $Z_x \cdot \hat{e} = 0$ *and*
(II) $Z_x \cdot \hat{f} = 0$.

Proof. Using (3.3),

$$-\frac{1}{\omega}\sin(\omega s)(Z_x \cdot \hat{e}) = \hat{e} \cdot \hat{e}' = |\hat{e}||\hat{e}|' = 0,$$

and using (3.4),

$$\frac{1}{\omega\sqrt{1-|e|^2}}(\cos(\omega s) - |e|)(Z_x \cdot \hat{f}) = \hat{f} \cdot \hat{f}' = |\hat{f}||\hat{f}|' = 0.$$

which completes the proof. □

Remark: According to this corollary if a solution is on a level manifold of Z, then Z_x remains normal to the orbit plane.

Necessary conditions for a central configuration solution have been given in the Corollary 3.3. Necessary and sufficient conditions, as expressed in the following theorem, require further restrictions on $\hat{e}$ and $\hat{f}$.

Theorem 3.5. *For $h < 0$, a solution of the N-body problem is a central configuration if and only if it is given by* (3.1) *where $\hat{e}$ and $\hat{f}$ are fixed orthonormal vectors satisfying*
(I) $Z_x(\hat{e}) = 0$;
(II) *if $|e| \neq 1$, then $|\hat{e}_i| = |\hat{f}_i|$;*
(III) *if $|e| \neq 1$, then $\hat{e}_i \cdot \hat{f}_i = 0$;*
(IV) *if $|e| \neq 1$, then $\hat{e}_i$ and $\hat{f}_i$ (for $i = 1, \dots, N$) are planar* ($d = 2$).

Proof. If the solution is a central configuration (i.e. $Z_x(x(s)) = 0$), then by Corollary 3.3, it is given by (3.1) where $\hat{e}$ and $\hat{f}$ are arbitrary fixed orthonormal vectors. We may choose $\hat{e}$ and $\hat{f}$ such that they satisfy conditions (II)-(IV). Condition (I) follows from

$$Z_x(x(0)) = Z_x(-\frac{\mu}{2h}(1-|e|)\hat{e}) = -\frac{2h}{\mu(1-|e|)}Z_x(\hat{e}) = 0,$$

using the fact that Z_x is a homogeneous function of degree -1.

Conversely, if the solution is given by (3.1) with fixed $\hat{e}$ and $\hat{f}$ satisfying conditions (I)-(IV), then

$$r_i = -\frac{\mu}{2h}[(\cos(\omega s) - |e|)\frac{\hat{e}_i}{\sqrt{m_i}} + \sqrt{1-|e|^2}\sin(\omega s)\frac{\hat{f}_i}{\sqrt{m_i}}].$$

This leads to

$$\begin{aligned}|r_i| &= \frac{|\hat{e}_i|}{\sqrt{m_i}}(-\frac{\mu}{2h})(1-|e|\cos(\omega s)) \\ &= \frac{1-|e|\cos(\omega s)}{1-|e|}|r_i(0)|,\end{aligned}$$

using (II) and (III), and

$$\theta = \arctan\frac{\sqrt{1-|e|^2}\sin(\omega s)}{\cos(\omega s)-|e|},$$

where θ is the angle between r_i and $\hat{e}_i$.

Using conditions (II)-(IV), we obtain $\hat{e}_i\cdot\hat{e}_j = \hat{f}_i\cdot\hat{f}_j$, $\hat{e}_i\cdot\hat{f}_j = -\hat{e}_j\cdot\hat{f}_i$, leading to

$$\begin{aligned}|r_i - r_j| &= -\frac{\mu}{2h}(1-|e|\cos(\omega s))\Big|\frac{\hat{e}_i}{\sqrt{m_i}} - \frac{\hat{e}_j}{\sqrt{m_j}}\Big| \\ &= \frac{1-|e|\cos(\omega s)}{1-|e|}|r_i(0)-r_j(0)|.\end{aligned}$$

Note that these relations do not hold if condition (IV) is dropped. It follows that the solutions given by (3.1) and conditions (II)-(IV) are homographic,

$$r_i = \lambda(s)\Omega(\theta)r_i(0), \quad \lambda(s) = \frac{1-|e|\cos(\omega s)}{1-|e|},$$

where $\lambda(s)$ and $\Omega(\theta)$ are respectively the scale and the rotation.

Now we show that the central configuration follows from condition (I). First we write Z_x explicitly in the components form

$$\frac{\partial Z}{\partial r_i} = \sum_{j\neq i}^{N} m_i m_j[-\frac{G|x|}{|r_i-r_j|^3} + \frac{Z}{M|x|^2}](r_i - r_j).$$

Substituting the homo-graphic solution in $Z(x)$ and in $\frac{\partial Z}{\partial r_i}$ leads to $Z(x(s)) = Z(x(0)) = \mu$ and

$$\begin{aligned}\frac{\partial Z}{\partial r_i} &= (\frac{1}{\lambda(s)})\Omega(\theta)\frac{\partial Z}{\partial r_i}(x(0)) \\ &= (\frac{1}{\lambda(s)})\Omega(\theta)\frac{\partial Z}{\partial r_i}(-\frac{\mu}{2h}(1-|e|)\hat{e}) \\ &= (\frac{1}{\lambda(s)})(-\frac{2h}{\mu(1-|e|)})\Omega(\theta)\frac{\partial Z}{\partial r_i}(\hat{e}) = 0,\end{aligned}$$

using condition (I). This proves that the solution is a central configuration.

In the case that $|e| = 1$, there is no rotation ($\theta = 0$), the solution is homothetic and all bodies move in $\hat{e}_i$ directions through the center of mass. The condition (I) is still valid and the solution is a central configuration leading to a total collapse. Conditions (II)-(IV) are irrelevant in this case and the motion is possible for $d = 1, 2, 3$. □

Remark: The proofs for $h > 0$ are similar to the proofs above; so we omit them.

The present formulation has the ability to shed light on the dynamical properties of the N-body problem that are otherwise hidden if the traditional formulations are used. Using this formulation, the necessary conditions for a constant configuration measure are described in terms of the geometry of the orbits in $\mathbb{R}^n$. The orbit must be a conic section in the orbit plane and the instantaneous orientation of the orbit plane is governed by the differential equations (3.3) and (3.4). These orbits include the homographic solutions in which the orbit plane is fixed in $\mathbb{R}^n$. The necessary and sufficient conditions provide a subset of the above set of orbits. The Saari's extended conjecture is true if this subset includes only the solutions in which the orbit plane is fixed. Therefore, it is important to extend the necessary conditions for a constant configuration measure given in this paper to the necessary and sufficient conditions.

Acknowledgments. The author would like to thank the anonymous referee for his/her useful comments and to the editor Prof. J. Dix for his patience during the editorial process.

References

[1] Birkhoff, G. D.; *Dynamical Systems*, Vol. IX Am. Math. Soc., Colloquium Publ. Providence, RI (1927).
[2] Chazy, J.; *Sur l'allure du mouvement dans le probléme des troi corps quand le temps croit indéfiniment*, Analles Scienfitiques de L'Ecole Normale Supérieure. Sér. 3, 39, 29-130 (1922).
[3] Florin Diacu, Ernesto Perez-Chavela, Manuele Santopetre; *Saari's conjecture of the N-body problem in the collinear case*, Trans. Amer. Math. Soc. 357, 4215-4223 (2005).
[4] Florin Diacu, Toshiaki Fujiwara, Ernesto Perez-Chavela, Manuele Santopetre; *Saari's homographic conjecture of the three-body problem*. Transactions of the American Mathematical Society, 360 No. 12, 6447-6473, (2008).
[5] Euler, L.; *De moto rectilineo trium corporum se mutuo attrahentium*, Nov. Comm. Petrop. 11, 144 (1765).
[6] Golubev, V. G.; *Hill stability in the unrestricted three body problem*, Sov. Phys. Dokl. 13, 373-375 (1968).
[7] Jacobi, C. G. J.; *Vorlesungen ́uber Dynamik*, Reimer Publisher, Berlin (1866).
[8] Lagrange, J.; *Œuvres*,(M. J. A. Serret, ed.), Vol. 6. Gauthier-Villars, Paris (1873).
[9] Marchal, C. and Saari, D. G.; *Hill regions for the general three-body problem*, Celest. Mech. 12, 115-129 (1975).
[10] Moeckel, R.; *A computer assisted proof of Saari's conjecture for the planar three body problem*, Trans. Amer. Math. Soc., 357, 3015-3117 (2005).
[11] Painlevé, P.; *Lecones sur la théorie analique des équations différentielles.*, A. Hermann, Paris (1897).
[12] Poincaré, H.; *Les Méthodes Nouvells de la Mécanique Céleste*, Vol. 1,2,3 , Gauthier-villars, Paris (1892-1899).
[13] Saari, D. G.; *On bounded solutions of the n-body problem*, In Giacaglia, G. (ed.) Periodic Orbits, Stability and resonances, D. Riedel, Dordrecht, 76-81 (1970).
[14] Saari, D. G.; *Collisions, Rings, and other Newtonian N-body problems*, Am. Math. Soc., Providence, RI, (2005).
[15] Saari, D. G.; *Reflection on my conjecture and several new ones*, Presentation at Saarifest, Guanajuato, Mexico (2005).
[16] Siegel, G. L. and Moser, J. K.; *Lectures on Celestial Mechanics*, Springer-Verlag, NY (1971).
[17] Smale, S.; *Topology and mechanics*, Invent. Math. 11, 45-64 (1970).
[18] Sundman, K. F.; *Meémoire sur le problème de trois corps*, Acta Math., 36, 105 (1912).
[19] Wintner, A.; *The Analytical Foundations of Celestial Mechanics*, Princeton University Press, Princeton,NJ (1941).
[20] Zare, K.; *The effects of integrals on the totality of solutions of dynamical systems*, Celest.Mech. 14, 73-83 (1976).
[21] Zare, K.; *Bifurcation points in the planar problem of three bodies*, Celest. Mech. 16, 35-38 (1977).
[22] Zare, K.; *Properties of moment of inertia in the problem of three bodies*, Celest. Mech. 24, 345-354 (1981).

[23] Zare,K.; Levinson, D.; *The possible motions of a rigid body with one point fixed*, J. Appl. Mech. 99, 329-332 (1977)

Kal Zare
Department of Mathematics, Texas state University, 601 University Drive, San Marcos, TX 78666, USA
E-mail address: kz11@txstate.edu