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DARCY-TYPE LAW ASSOCIATED TO AN OPTIMAL CONTROL PROBLEM

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Abstract. The aim of this paper is to study the asymptotic behaviour (homogenization) of an optimal control problem in a periodically perforated domain with Dirichlet condition on the boundary of the holes. The optimal control problem considered here is governed by the Stokes system. The holes are assumed to be of the same order as that of the period. The homogenized limit of the Stokes system as well as its adjoint system arising from the optimal control problem is obtained. The convergence of the optimal control and cost functional is obtained on some specific control sets.

1. Introduction

It is now well known that the fluid flow (governed by Stokes or Navier-Stokes equations) in a periodically perforated domain (with a large number of small holes) behaves differently according to the size of the holes (say radius) compared to the period of the distribution. In fact, there is a critical size of the holes, where system behaves like a 'Brinkman-type' flow. Further, when the holes are much smaller, in order, than the critical one, the flow is the standard Stokes or Navier-Stokes. Lastly, when the holes are comparable to the period, then the system tends to a 'Darcy-type' law situation (cf. [1, 2, 3, 17, 12]).

In this article, we consider the Stokes system (it can be carried to non-linear Navier-Stokes system as well) when the holes and period are of the same order with a control acting on the system and an associated cost functional. Some results regarding the critical and subcritical case were proved in [15]. Substantial study had been carried out for optimal control problems under various situation where the state is determined by second order elliptic operators (cf . [6, 7, 8, 14, 10, 5, 11]).

We shall now give some preliminaries and set the environment of the paper.

Let $n \geq 2$ and Ω be a bounded open set in $\mathbb{R}^n$. Let $\{e_1, e_2, \dots, e_n\}$ be the canonical basis of $\mathbb{R}^n$. Let $Y = (-1, 1)^n$ be the reference cell and S be an open subset of Y, such that S is contained in an open ball of radius α $(0 < \alpha < 1)$ centered at the origin. Let $Y' = Y \setminus \bar{S}$ be the fluid part and $\bar{S}$ be the solid (or obstacle) part. If $S + \mathbb{Z}^n = \{x + k \mid x \in S \text{ and } k \in \mathbb{Z}^n\}$, then, for a parameter $\varepsilon > 0$

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tending to zero, we set $S_\varepsilon = \varepsilon(S + \mathbb{Z}^n)$. We define

$$\Omega_\varepsilon = \Omega \cap (\mathbb{R}^n \setminus \bar{S_\varepsilon}).$$

Observe that Ω_ε is bounded and we assume that Ω_ε is connected and its boundary, $\partial\Omega_\varepsilon$, is of Lipschitz type.

In a more general situation, if a_ε denotes the size (say, diameter) of the obstacle S_ε distributed periodically, then observe that under the above setting S_ε is exactly of order ε, *i.e.*, $\lim_{\varepsilon\to 0}(a_\varepsilon/\varepsilon) > 0$. Following the convention of Allaire [3], we define σ_ε as the ratio between the actual size of the obstacles and the critical size:

$$\sigma_\varepsilon = \begin{cases} \varepsilon \left|\log\left(\frac{a_\varepsilon}{\varepsilon}\right)\right|^{1/2} & \text{for } n = 2, \\ \left(\frac{\varepsilon^n}{a_\varepsilon^{n-2}}\right)^{1/2} & \text{for } n \geq 3. \end{cases}$$

Then, in our setting, we have $\sigma_\varepsilon \to 0$.

The inner-product in $(L^2(\Omega))^n$ is given as:

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \int_\Omega \boldsymbol{u}.\boldsymbol{v}\, dx = \sum_{i=1}^n \int_\Omega u_i v_i \, dx, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in (L^2(\Omega))^n.$$

We shall denote the norm in $(L^2(\Omega))^n$ by $\|\cdot\|_2$ and the norm in $(L^2(\Omega_\varepsilon))^n$ by $\|\cdot\|_{2,\Omega_\varepsilon}$. For a function g defined on Ω_ε, we shall denote by $\tilde{g}$ its extension by zero on $\Omega \cap S_\varepsilon$. The symbol C will always denote a generic positive constant independent of ε.

Let a, b be given constants such that $0 < a \leq b$. Let $B = B(y)$ be a $n \times n$ matrix with entries from $L^\infty(Y)$, Y-periodic in y satisfying

$$a|\xi|^2 \leq B(y)\xi.\xi \leq b|\xi|^2 \quad \text{a.e. in } y, \quad \forall \xi = (\xi_i) \in \mathbb{R}^n.$$

In addition, we assume that B is symmetric. The symmetry assumption will not play any role in the homogenization process and is inherited from the optimal control problem.

Let U_ε be a closed convex subset of $(L^2(\Omega_\varepsilon))^n$, called the admissible control set and $\boldsymbol{f} \in (L^2(\Omega))^n$. Let $\nu > 0$ be the cost of the control independent of ε. Given $\boldsymbol{\theta}_\varepsilon \in U_\varepsilon$, let the cost functional J_ε be defined as,

$$J_\varepsilon(\boldsymbol{\theta}_\varepsilon) = \frac{1}{2}\int_{\Omega_\varepsilon} B\left(\frac{x}{\varepsilon}\right)\nabla \boldsymbol{u}_\varepsilon.\nabla \boldsymbol{u}_\varepsilon \, dx + \frac{\nu}{2}\|\boldsymbol{\theta}_\varepsilon\|_{2,\Omega_\varepsilon}^2 \tag{1.1}$$

where $\boldsymbol{u}_\varepsilon = \boldsymbol{u}_\varepsilon(\boldsymbol{\theta}_\varepsilon) \in (H_0^1(\Omega_\varepsilon))^n$ is the state in the unique pair, $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$, of solution of the Stokes equation

$$\begin{gathered} \nabla p_\varepsilon - \Delta \boldsymbol{u}_\varepsilon = \boldsymbol{f} + \boldsymbol{\theta}_\varepsilon \quad \text{in } \Omega_\varepsilon, \\ \operatorname{div}(\boldsymbol{u}_\varepsilon) = 0 \quad \text{in } \Omega_\varepsilon, \\ \boldsymbol{u}_\varepsilon = 0 \quad \text{on } \partial\Omega_\varepsilon. \end{gathered} \tag{1.2}$$

The pressure p_ε is unique up to an additive constant and thus belongs to $L^2(\Omega_\varepsilon)/\mathbb{R}$. It is a classical result from the calculus of variations that there exists a unique $\boldsymbol{\theta}_\varepsilon^* \in U_\varepsilon$ such that

$$J_\varepsilon(\boldsymbol{\theta}_\varepsilon^*) = \min_{\boldsymbol{\theta}_\varepsilon \in U_\varepsilon} J_\varepsilon(\boldsymbol{\theta}_\varepsilon). \tag{1.3}$$

The system (1.1)–(1.2) was considered by Saint Jean Paulin and Zoubairi (cf. [15]) in the abstract framework introduced by Allaire in [2, 3]. They had considered the admissible control sets U_ε to be of obstacle-type. They studied the cases when $\lim_{\varepsilon\to 0}\sigma_\varepsilon > 0$ and $\lim_{\varepsilon\to 0}\sigma_\varepsilon = \infty$. This, precisely, represents the critical and subcritical case, where the holes are much smaller. However, they were unable to conclude concerning the case $\lim_{\varepsilon\to 0}\sigma_\varepsilon = 0$. We remark that the homogenization in the comparable case is always different even in Dirichlet problem of Laplacian.

In this article, we study the asymptotic behaviour of the optimal control problem (1.1)–(1.2), when $\lim_{\varepsilon\to 0}\sigma_\varepsilon = 0$. Our method is based on two-scale convergence. We end this section by recalling the notion of two-scale convergence. We refer to [13, 4, 12, 9] for a detailed study of the same and certain applications.

Definition 1.1. A sequence of functions $\{v_\varepsilon\}$ in $L^2(\Omega)$ is said to *two-scale converge* to a limit $v \in L^2(\Omega \times Y)$ (denoted as $v_\varepsilon \xrightarrow{2s} v$) if

$$\int_\Omega v_\varepsilon \phi\left(x, \frac{x}{\varepsilon}\right) dx \to \int_\Omega \int_Y v(x,y)\phi(x,y)\, dy\, dx, \quad \forall \phi \in L^2[\Omega; C_{\rm per}(Y)].$$

The most interesting property of two-scale convergence is the following compactness result.

Theorem 1.2. *For any bounded sequence v_ε in $L^2(\Omega)$, there exists a subsequence and $v \in L^2(\Omega \times Y)$ such that, v_ε two-scale converges to v along the subsequence.*

The approach of the article is as follows: In the next section, we introduce the adjoint problem associated to the optimal control problem (1.1)–(1.2) and homogenize the state-adjoint system. This yields a Darcy-type result for the adjoint state as well. Finally, we consider special situations where the optimal control problem can be homogenized.

2. Homogenization Process

We begin by stating a lemma on the Poincaré inequality proved by Tartar (cf. [17]) when the size of the obstacles a_ε are exactly of the order of ε.

Lemma 2.1. *There exists a positive constant C, independent of ε, such that*

$$\|v\|_{2,\Omega_\varepsilon} \le C\varepsilon \|\nabla v\|_{2,\Omega_\varepsilon}, \quad \forall v \in H_0^1(\Omega_\varepsilon).$$

Let $\boldsymbol{\theta}_\varepsilon^*$ be the unique optimal control of the system (1.1)–(1.2). Let $(\boldsymbol{u}_\varepsilon^*, p_\varepsilon^*) \in (H_0^1(\Omega_\varepsilon))^n \times L^2(\Omega_\varepsilon)/\mathbb{R}$ be the state and pressure corresponding to $\boldsymbol{\theta}_\varepsilon^*$ given by the system of equations:

$$\begin{gathered} \nabla p_\varepsilon^* - \Delta \boldsymbol{u}_\varepsilon^* = \boldsymbol{f} + \boldsymbol{\theta}_\varepsilon^* \quad \text{in } \Omega_\varepsilon, \\ \operatorname{div}(\boldsymbol{u}_\varepsilon^*) = 0 \quad \text{in } \Omega_\varepsilon, \\ \boldsymbol{u}_\varepsilon^* = 0 \quad \text{on } \partial\Omega_\varepsilon. \end{gathered} \tag{2.1}$$

We introduce the adjoint optimal state associated to the optimal control problem. Let $(\boldsymbol{v}_\varepsilon^*, q_\varepsilon^*) \in (H_0^1(\Omega_\varepsilon))^n \times L^2(\Omega_\varepsilon)/\mathbb{R}$ be the solution of

$$\begin{gathered} \nabla q_\varepsilon^* - \Delta \boldsymbol{v}_\varepsilon^* = -\operatorname{div}\left(B\left(\frac{x}{\varepsilon}\right)\nabla \boldsymbol{u}_\varepsilon^*\right) \quad \text{in } \Omega_\varepsilon, \\ \operatorname{div}(\boldsymbol{v}_\varepsilon^*) = 0 \quad \text{in } \Omega_\varepsilon, \\ \boldsymbol{v}_\varepsilon^* = 0 \quad \text{on } \partial\Omega_\varepsilon. \end{gathered} \tag{2.2}$$

Then the optimality condition, in terms of the adjoint optimal state, is

$$\int_{\Omega_\varepsilon} (\boldsymbol{v}_\varepsilon^* + \nu\boldsymbol{\theta}_\varepsilon^*).(\boldsymbol{\theta}_\varepsilon - \boldsymbol{\theta}_\varepsilon^*)\, dx \ge 0 \quad \forall \boldsymbol{\theta}_\varepsilon \in U_\varepsilon. \tag{2.3}$$

Note that the symmetry hypothesis on B comes in hand to derive the optimality condition (2.3).

Lemma 2.2. *Let* $0 \in U_\varepsilon$ *for all* ε, *then* $\{\varepsilon^{-1}\widetilde{\boldsymbol{\theta}^*_\varepsilon}\}, \{\varepsilon^{-2}\widetilde{\boldsymbol{u}^*_\varepsilon}\}, \{\varepsilon^{-2}\widetilde{\boldsymbol{v}^*_\varepsilon}\}, \{\varepsilon^{-1}\nabla\widetilde{\boldsymbol{u}^*_\varepsilon}\}$ *and* $\{\varepsilon^{-1}\nabla\widetilde{\boldsymbol{v}^*_\varepsilon}\}$ *are bounded in* $(L^2(\Omega))^n$.

Proof. Let $\boldsymbol{w}_\varepsilon$ be the state corresponding to the control $\boldsymbol{\theta}_\varepsilon = 0$ in (1.2). Then it is easy to observe, using Lemma 2.1, that

$$\|\nabla \boldsymbol{w}_\varepsilon\|_{2,\Omega_\varepsilon} \le C\varepsilon \|f\|_2.$$

Since $J_\varepsilon(\boldsymbol{\theta}^*_\varepsilon) \le J_\varepsilon(0)$, we deduce that

$$\|\varepsilon^{-1}\widetilde{\boldsymbol{\theta}^*_\varepsilon}\|_2^2 \le C\|f\|_2^2$$

and similarly, we also deduce that

$$\|\varepsilon^{-1}\nabla\widetilde{\boldsymbol{u}^*_\varepsilon}\|_2^2 \le C\|f\|_2^2.$$

Now using, Lemma 2.1, we get $\|\varepsilon^{-2}\widetilde{\boldsymbol{u}^*_\varepsilon}\|_2^2$ is bounded. Hence, using $\varepsilon^{-2}\boldsymbol{v}^*_\varepsilon$ as a test function in the adjoint state (2.2), we deduce the respective bounds of the adjoint state. □

It follows from the above lemma that the optimal controls $\boldsymbol{\theta}^*_\varepsilon$ converge to zero strongly in $(L^2(\Omega))^n$, $\boldsymbol{u}^*_\varepsilon$ converge to zero strongly in $(H^1_0(\Omega))^n$ and the cost functional $J_\varepsilon(\boldsymbol{\theta}^*_\varepsilon) \to 0$. Our interest is to get information on the further terms on the asymptotic expansion of these quantities. In fact, there exists $\boldsymbol{\theta}^* \in (L^2(\Omega))^n$ such that $\varepsilon^{-1}\widetilde{\boldsymbol{\theta}^*_\varepsilon} \rightharpoonup \boldsymbol{\theta}^*$ weakly in $(L^2(\Omega))^n$. More precisely, our objective is to identify the role of $\boldsymbol{\theta}^*$ in the non-zero homogenized limit of the system (1.1)–(1.2).

We now state a lemma proved in [17, 1].

Lemma 2.3. *There exists a restriction operator* $R_\varepsilon : (H^1_0(\Omega))^n \to (H^1_0(\Omega_\varepsilon))^n$ *such that*

(i) $u \in (H^1_0(\Omega_\varepsilon))^n \Rightarrow R_\varepsilon \widetilde{u} = u$ *in* Ω_ε
(ii) $\operatorname{div}(u) = 0 \Rightarrow \operatorname{div}(R_\varepsilon u) = 0$ *in* Ω_ε
(iii) $\|\nabla(R_\varepsilon u)\|_{2,\Omega_\varepsilon} \le C\left[\frac{1}{\varepsilon}\|u\|_2 + \|\nabla u\|_2\right]$.

The above lemma is used to prove the following result (cf. [17, 3]).

Lemma 2.4. *Let* $0 \in U_\varepsilon$. *Then there exists* P^*_ε *and* Q^*_ε *in* $L^2(\Omega)/\mathbb{R}$ *such that* $P^*_\varepsilon = p^*_\varepsilon$ *and* $Q^*_\varepsilon = q^*_\varepsilon$ *in* Ω_ε, *and both* P^*_ε *and* Q^*_ε *are bounded in* $L^2(\Omega)/\mathbb{R}$.

We deduce from the *a priori* estimates obtained in above lemmas that there exists $\boldsymbol{u}^*_0(x,y), \boldsymbol{v}^*_0(x,y)$ in $(L^2(\Omega\times Y))^n$, $\boldsymbol{\xi}^*_0(x,y), \boldsymbol{\zeta}^*_0(x,y)$ in $(L^2(\Omega\times Y))^{n\times n}$ and $p^*_0(x,y), q^*_0(x,y)$ in $L^2(\Omega\times Y)/\mathbb{R}$ such that, up to a subsequence (cf. Theorem 1.2),

$$\varepsilon^{-2}\widetilde{\boldsymbol{u}^*_\varepsilon} \xrightharpoonup{2s} \boldsymbol{u}^*_0(x,y), \quad \varepsilon^{-2}\widetilde{\boldsymbol{v}^*_\varepsilon} \xrightharpoonup{2s} \boldsymbol{v}^*_0(x,y),$$
$$\varepsilon^{-1}\nabla\widetilde{\boldsymbol{u}^*_\varepsilon} \xrightharpoonup{2s} \boldsymbol{\xi}^*_0(x,y), \quad \varepsilon^{-1}\nabla\widetilde{\boldsymbol{v}^*_\varepsilon} \xrightharpoonup{2s} \boldsymbol{\zeta}^*_0(x,y),$$
$$P^*_\varepsilon \xrightharpoonup{2s} p^*_0(x,y), \quad Q^*_\varepsilon \xrightharpoonup{2s} q^*_0(x,y).$$

Let $\boldsymbol{u}^*(x) = \frac{1}{|Y|}\int_Y \boldsymbol{u}^*_0(x,y)\,dy$ and $\boldsymbol{v}^*(x) = \frac{1}{|Y|}\int_Y \boldsymbol{v}^*_0(x,y)\,dy$. It is a known fact from the two-scale convergence theory that, for the same subsequence,

$$\varepsilon^{-2}\widetilde{\boldsymbol{u}^*_\varepsilon} \rightharpoonup \boldsymbol{u}^* \text{ weakly in } (L^2(\Omega))^n,$$
$$\varepsilon^{-2}\widetilde{\boldsymbol{v}^*_\varepsilon} \rightharpoonup \boldsymbol{v}^* \text{ weakly in } (L^2(\Omega))^n.$$

The extension of the pressures are not the trivial extension by zero and it was observed in [17, 3] that, for the same subsequence, P_ε^*, Q_ε^* converges strongly in $L^2(\Omega)/\mathbb{R}$ and, in fact, we get those limits as

$$P_\varepsilon^* \to \frac{1}{|Y|}\int_Y p_0^*(x,y)\,dy \quad \text{strongly in } L^2(\Omega)/\mathbb{R},$$
$$Q_\varepsilon^* \to \frac{1}{|Y|}\int_Y q_0^*(x,y)\,dy \quad \text{strongly in } L^2(\Omega)/\mathbb{R}.$$

Remark 2.5. Given $\operatorname{div}(\boldsymbol{u}_\varepsilon^*) = 0$ in Ω_ε implies that $\operatorname{div}(\widetilde{\boldsymbol{u}_\varepsilon^*}) = 0$ in Ω, and hence $\operatorname{div}(\boldsymbol{u}^*) = 0$ in Ω and $\boldsymbol{u}^* \cdot n = 0$ in $\partial\Omega$, where n is the unit outward normal. Similarly, $\operatorname{div}(\boldsymbol{v}^*) = 0$ in Ω and $\boldsymbol{v}^* \cdot n = 0$ in $\partial\Omega$ (cf. [16, 1]).

We shall now define some cell problems which will be used in the sequel to identify the limit problem. For $1 \le i \le n$, let the function $(\boldsymbol{\mu}_i, \rho_i) \in (H^1_{\text{per}}(Y'))^n \times L^2_{\text{per}}(Y')/\mathbb{R}$ be the solution of the cell problem

$$\begin{gathered} \nabla_y \rho_i - \Delta_y \boldsymbol{\mu}_i = e_i \quad \text{in } Y' \\ \operatorname{div}_y(\boldsymbol{\mu}_i) = 0 \quad \text{in } Y' \\ \boldsymbol{\mu_i} = 0 \quad \text{on } \partial Y' \setminus \partial Y \\ \boldsymbol{\mu}_i \text{ and } \rho_i \text{ are } Y\text{-periodic.} \end{gathered} \tag{2.4}$$

Let $(\boldsymbol{\chi}_i, \lambda_i) \in (H^1_{\text{per}}(Y'))^n \times L^2_{\text{per}}(Y')/\mathbb{R}$ be the solution of the adjoint cell problem

$$\begin{gathered} \nabla_y \lambda_i - \Delta_y \boldsymbol{\chi}_i = -\operatorname{div}_y\left(B(y)\nabla_y \boldsymbol{\mu}_i\right) \quad \text{in } Y' \\ \operatorname{div}_y(\boldsymbol{\chi}_i) = 0 \quad \text{in } Y' \\ \boldsymbol{\chi_i} = 0 \quad \text{on } \partial Y' \setminus \partial Y \\ \boldsymbol{\chi}_i \text{ and } \lambda_i \text{ are } Y\text{-periodic.} \end{gathered} \tag{2.5}$$

We extend $\boldsymbol{\mu}_i$ and $\boldsymbol{\chi}_i$ by zero to $Y \setminus Y'$ and use the same notation for the extension. Let M and N be the $n \times n$ matrices defined as follows:

$$Me_i = \frac{1}{|Y|}\int_Y \boldsymbol{\mu}_i(y)\,dy,$$
$$Ne_i = \frac{1}{|Y|}\int_Y \boldsymbol{\chi}_i(y)\,dy.$$

The matrix M is standard in the homogenization of Stokes system in perforated domain and it is known that M is both symmetric and positive definite (cf. [16]). We have to establish similar results for N.

Lemma 2.6. *If B is symmetric, then the matrix N is symmetric.*

Proof. Using $\boldsymbol{\chi}_j$ as a test function in (2.4), we get

$$\int_Y \nabla_y \boldsymbol{\mu}_i \cdot \nabla_y \boldsymbol{\chi}_j\,dy = \int_Y e_i.\boldsymbol{\chi}_j\,dy = |Y|\langle e_i, Ne_j\rangle.$$

Now, using $\boldsymbol{\mu}_i$ as a test function in (2.5) corresponding to the index j, we get

$$\int_Y \nabla_y \boldsymbol{\mu}_i.\nabla_y \boldsymbol{\chi}_j\,dy = \int_Y B(y)\nabla_y \boldsymbol{\mu}_j.\nabla_y \boldsymbol{\mu}_i\,dy.$$

Thus,

$$|Y|\langle e_i, Ne_j\rangle = \int_Y B(y)\nabla_y \boldsymbol{\mu}_j.\nabla_y \boldsymbol{\mu}_i\,dy.$$

Similarly, interchanging the role of i and j in the above argument and using the fact that scalar product commutes, we deduce

$$|Y|\langle e_j, Ne_i\rangle = \int_Y B(y)\nabla_y \boldsymbol{\mu}_i.\nabla_y \boldsymbol{\mu}_j \, dy.$$

Hence, if B is symmetric, then N is symmetric. □

Lemma 2.7. *If B is positive definite, then N is positive definite.*

Proof. Let $\xi \in \mathbb{R}^n$. Define $(\boldsymbol{\alpha}_\xi, \gamma_1) \in (H^1_{\text{per}}(Y'))^n \times L^2_{\text{per}}(Y')/\mathbb{R}$ as the solution of the problem

$$\begin{gathered}\nabla_y \gamma_1 - \Delta_y \boldsymbol{\alpha}_\xi = \xi \quad \text{in } Y' \\ \operatorname{div}_y(\boldsymbol{\alpha}_\xi) = 0 \quad \text{in } Y' \\ \boldsymbol{\alpha}_{\boldsymbol{\xi}} = 0 \quad \text{on } \partial Y' \setminus \partial Y \\ \boldsymbol{\alpha}_\xi \text{ and } \gamma_1 \text{ are } Y\text{-periodic.}\end{gathered} \tag{2.6}$$

Also, let $(\boldsymbol{\beta}_\xi, \gamma_2) \in (H^1_{\text{per}}(Y'))^n \times L^2_{\text{per}}(Y')/\mathbb{R}$ be the solution of the corresponding adjoint problem

$$\begin{gathered}\nabla_y \gamma_2 - \Delta_y \boldsymbol{\beta}_\xi = -\operatorname{div}_y\left(B(y)\nabla_y \boldsymbol{\alpha}_\xi\right) \quad \text{in } Y' \\ \operatorname{div}_y(\boldsymbol{\beta}_\xi) = 0 \quad \text{in } Y' \\ \boldsymbol{\beta}_{\boldsymbol{\xi}} = 0 \quad \text{on } \partial Y' \setminus \partial Y \\ \boldsymbol{\beta}_\xi \text{ and } \gamma_2 \text{ are } Y\text{-periodic.}\end{gathered} \tag{2.7}$$

We extend $\boldsymbol{\alpha}_\xi$ and $\boldsymbol{\beta}_\xi$ by zero to $Y \setminus Y'$ and use the same notation for the extension. Observe that $\boldsymbol{\alpha}_\xi = \sum_{i=1}^n \xi_i \boldsymbol{\mu}_i$ and $\boldsymbol{\beta}_\xi = \sum_{i=1}^n \xi_i \boldsymbol{\chi}_i$. Therefore, $\langle N\xi, \xi\rangle = \frac{1}{|Y|}\int_Y \boldsymbol{\beta}_\xi(y).\xi \, dy$. Using $\boldsymbol{\beta}_\xi$ as a test function in (2.6) and $\boldsymbol{\alpha}_\xi$ as a test function in (2.7), we deduce,

$$\begin{aligned}\langle N\xi, \xi\rangle &= \frac{1}{|Y|}\int_Y \boldsymbol{\beta}_\xi(y).\xi \, dy \\ &= \frac{1}{|Y|}\int_Y \nabla_y \boldsymbol{\beta}_\xi(y).\nabla_y \boldsymbol{\alpha}_\xi(y) \, dy \\ &= \frac{1}{|Y|}\int_Y B(y)\nabla_y \boldsymbol{\alpha}_\xi(y).\nabla_y \boldsymbol{\alpha}_\xi(y) \, dy \\ &\geq \frac{a}{|Y|}\|\nabla_y \boldsymbol{\alpha}_\xi(y)\|_2^2 \quad \geq 0.\end{aligned}$$

Thus, we have shown that N is positive. It now remains to show the positive definiteness of N. Suppose that $\langle N\xi, \xi\rangle = 0$. Hence, $\|\nabla_y \boldsymbol{\alpha}_\xi(y)\|_2^2 = 0$ and consequently $\boldsymbol{\alpha}_\xi(y) = 0$. This implies that $\nabla_y \gamma_1 = \xi$, but since γ_1 is Y-periodic, we have $\xi = 0$. □

We now provide the homogenization theorem for the state and adjoint-state equations. Before doing so, we pause to remark that div_y will denote the divergence w.r.t the y variable and a 'div' without subscript will denote the divergence w.r.t the x variable.

Theorem 2.8. *Let $0 \in U_\varepsilon$. If $(\boldsymbol{u}_\varepsilon^*, p_\varepsilon^*)$ and $(\boldsymbol{v}_\varepsilon^*, q_\varepsilon^*)$ are the solution of* (2.1) *and* (2.2), *respectively, then there exists p^* and q^* in $L^2(\Omega)/\mathbb{R}$ such that*

$$\begin{gathered}\boldsymbol{u}^* = M(\boldsymbol{f} - \nabla p^*) \quad \text{in } \Omega, \\ \operatorname{div}(\boldsymbol{u}^*) = 0 \quad \text{in } \Omega, \\ \boldsymbol{u}^* \cdot n = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{2.8}$$

and

$$\begin{gathered}\boldsymbol{v}^* = N^t(\boldsymbol{f} - \nabla p^*) - M\nabla q^* \quad \text{in } \Omega, \\ \operatorname{div}(\boldsymbol{v}^*) = 0 \quad \text{in } \Omega, \\ \boldsymbol{v}^* \cdot n = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{2.9}$$

where N^t denotes the transpose of N. Further, the following convergence hold for the entire sequence:

$$\begin{gathered}\varepsilon^{-2}\widetilde{\boldsymbol{u}_\varepsilon^*} \rightharpoonup \boldsymbol{u}^* \quad \text{weakly in } (L^2(\Omega))^n, \\ \varepsilon^{-2}\widetilde{\boldsymbol{v}_\varepsilon^*} \rightharpoonup \boldsymbol{v}^* \quad \text{weakly in } (L^2(\Omega))^n, \\ P_\varepsilon^* \to p^* \quad \text{strongly in } L^2(\Omega)/\mathbb{R}, \\ Q_\varepsilon^* \to q^* \quad \text{strongly in } L^2(\Omega)/\mathbb{R} \end{gathered} \tag{2.10}$$

and

$$\varepsilon^{-2}\int_\Omega B\left(\frac{x}{\varepsilon}\right)\nabla\widetilde{\boldsymbol{u}_\varepsilon^*}\cdot\nabla\widetilde{\boldsymbol{u}_\varepsilon^*}\,dx \to \int_\Omega N\left(M^{-1}\boldsymbol{u}^*\right)\cdot M^{-1}\boldsymbol{u}^*\,dx \tag{2.11}$$

Proof. Let $\Phi(x,y) \in [\mathcal{D}(\Omega; C^\infty_{\rm per}(Y))]^{n\times n}$ be such that $\Phi(\cdot, y) = 0$ for all $y \in Y \setminus Y'$. Then integration by parts will yield,

$$\int_\Omega \nabla\widetilde{\boldsymbol{u}_\varepsilon^*}\cdot\Phi\left(x, \frac{x}{\varepsilon}\right)\,dx = -\int_\Omega \widetilde{\boldsymbol{u}_\varepsilon^*}\Big[\operatorname{div}\Phi\left(x,\frac{x}{\varepsilon}\right) + \frac{1}{\varepsilon}\operatorname{div}_y\Phi(x,\frac{x}{\varepsilon})\Big]\,dx.$$

Multiplying by ε^{-1} on both sides of the equality and then passing to the limit, as $\varepsilon \to 0$, we get,

$$\int_\Omega\int_Y \boldsymbol{\xi}_0^*(x,y)\cdot\Phi(x,y)\,dx\,dy = -\int_\Omega\int_Y \boldsymbol{u}_0^*(x,y)\operatorname{div}_y\Phi(x,y)\,dx\,dy.$$

Since Φ is arbitrary, we have $\boldsymbol{\xi}_0^*(x,y) = \nabla_y\boldsymbol{u}_0^*(x,y)$. A similar argument for the adjoint-state $\boldsymbol{v}_\varepsilon^*$ will yield $\boldsymbol{\zeta}_0^*(x,y) = \nabla_y\boldsymbol{v}_0^*(x,y)$.

Let $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2 \in [\mathcal{D}(\Omega; \mathcal{D}(Y'))]^{n\times n}$ be such that $\operatorname{div}_y(\boldsymbol{\phi}_2) = 0$. Using $\varepsilon\boldsymbol{\phi}_1(x, \frac{x}{\varepsilon}) + \boldsymbol{\phi}_2(x, \frac{x}{\varepsilon})$ as a two-scale test function in (2.1), we get,

$$\begin{aligned}&-\int_\Omega\int_Y p_0^*(x,y)\Big[\operatorname{div}(\boldsymbol{\phi}_2(x,y)) + \operatorname{div}_y(\boldsymbol{\phi}_1(x,y))\Big]\,dx\,dy \\ &+\int_\Omega\int_Y \boldsymbol{\xi}_0^*(x,y)\nabla_y\boldsymbol{\phi}_2(x,y)\,dx\,dy \\ &=\int_\Omega\int_Y \boldsymbol{f}\boldsymbol{\phi}_2(x,y)\,dx\,dy.\end{aligned}$$

By putting, $\boldsymbol{\phi}_2 \equiv 0$, we get

$$-\int_\Omega\int_Y p_0^*(x,y)\operatorname{div}_y(\boldsymbol{\phi}_1(x,y))\,dx\,dy = 0.$$

Hence, $\nabla_y p_0^*(x,y) = 0$ a.e. and thus there exists a $p^* \in L^2(\Omega)/\mathbb{R}$ such that $p_0^*(x,y) = p^*(x)$ in $\Omega \times Y$.

By putting, $\boldsymbol{\phi}_1 \equiv 0$, we get

$$\begin{aligned} &- \int_\Omega \int_Y p^*(x) \operatorname{div}(\boldsymbol{\phi}_2(x,y))\, dx\, dy \\ &+ \int_\Omega \int_Y \nabla_y \boldsymbol{u}_0^*(x,y) \nabla_y \boldsymbol{\phi}_2(x,y)\, dx\, dy \\ &= \int_\Omega \int_Y \boldsymbol{f} \boldsymbol{\phi}_2(x,y)\, dx\, dy. \end{aligned}$$

Since $\boldsymbol{\phi}_2$ is such that $\operatorname{div}_y(\boldsymbol{\phi}_2) = 0$, there exists $p_1^*(x,y) \in L^2(\Omega \times Y)/\mathbb{R}$ such that

$$-\nabla_y p_1^*(x,y) - \Delta_y \boldsymbol{u}_0^*(x,y) = \boldsymbol{f} - \nabla p^* \text{ in } \Omega \times Y.$$

By using the cell problem (2.4) and from the uniqueness of solution for the Stokes system, we derive

$$\frac{\partial p_1^*}{\partial y_i} = \nabla_y \rho_i \cdot (\boldsymbol{f} - \nabla p^*) \text{ in } \Omega \times Y.$$

and $\boldsymbol{u}^* = M^t(\boldsymbol{f} - \nabla p^*)$ in Ω, where M^t is the transpose of M. But since M is symmetric, we have

$$\boldsymbol{u}^* = M(\boldsymbol{f} - \nabla p^*) \text{ in } \Omega,$$

This combined with the facts mentioned in Remark 2.5 gives (2.8).

Now, using $\varepsilon\boldsymbol{\phi}_1(x, \frac{x}{\varepsilon}) + \boldsymbol{\phi}_2(x, \frac{x}{\varepsilon})$ as a two-scale test function in (2.2) and following a similar analysis as before, we deduce that there exists a $q^* \in L^2(\Omega)/\mathbb{R}$ such that $q_0^*(x,y) = q^*(x)$ in $\Omega \times Y$ and there exists $q_1^*(x,y) \in L^2(\Omega \times Y)/\mathbb{R}$ such that

$$-\nabla_y q_1^*(x,y) - \Delta_y \boldsymbol{v}_0^*(x,y) = -\operatorname{div}_y \left(B(y)\nabla_y \boldsymbol{u}_0^*(x,y)\right) - \nabla q^* \text{ in } \Omega \times Y.$$

By using the cell problem (2.5) and from the uniqueness of solution for the Stokes system, we derive

$$\frac{\partial q_1^*}{\partial y_i} = \nabla_y \lambda_i \cdot (\boldsymbol{f} - \nabla p^*) - \nabla_y \rho_i \cdot \nabla q^* \text{ in } \Omega \times Y.$$

and

$$\boldsymbol{v}^* = N^t(\boldsymbol{f} - \nabla p^*) - M^t \nabla q^* \text{ in } \Omega,$$

where M^t is the transpose of M. But since M is symmetric, we have,

$$\boldsymbol{v}^* = N^t(\boldsymbol{f} - \nabla p^*) - M \nabla q^* \text{ in } \Omega\,.$$

This combined with the facts mentioned in Remark 2.5 gives (2.9). The uniqueness of p^* and q^*, up to additive constants, can be obtained by solving the Neumann problem hidden in (2.8) and (2.9), respectively. Hence the uniqueness of $\boldsymbol{u}^*$ and $\boldsymbol{v}^*$, and we deduce the convergence (2.10) for the entire sequence.

It now remains to prove (2.11). Using $\varepsilon^{-2}\boldsymbol{v}_\varepsilon^*$ as a test function in (2.1) and $\varepsilon^{-2}\boldsymbol{u}_\varepsilon^*$ as a test function in (2.2), we deduce that

$$\varepsilon^{-2} \int_{\Omega_\varepsilon} B\left(\frac{x}{\varepsilon}\right) \nabla \boldsymbol{u}_\varepsilon^* \cdot \nabla \boldsymbol{u}_\varepsilon^*\, dx = \int_{\Omega_\varepsilon} (\boldsymbol{f} + \boldsymbol{\theta}_\varepsilon^*).(\varepsilon^{-2}\boldsymbol{v}_\varepsilon^*)\, dx.$$

Now, passing to the limit on the right-hand side of the above equality, we get

$$\varepsilon^{-2} \int_{\Omega_\varepsilon} B\left(\frac{x}{\varepsilon}\right) \nabla \boldsymbol{u}_\varepsilon^* \cdot \nabla \boldsymbol{u}_\varepsilon^*\, dx \to \int_\Omega \boldsymbol{f}.\boldsymbol{v}^*\, dx.$$

We deduce from the equation of $\boldsymbol{u}^*$ and $\boldsymbol{v}^*$ in (2.8) and (2.9), respectively, that $\boldsymbol{f} = M^{-1}\boldsymbol{u}^* + \nabla p^*$ and $\boldsymbol{v}^* = N^t M^{-1}\boldsymbol{u}^* - M\nabla q^*$. Thus,

$$\begin{aligned}\int_\Omega \boldsymbol{f}\boldsymbol{v}^*\,dx &= \int_\Omega M^{-1}\boldsymbol{u}^*\cdot\boldsymbol{v}^*\,dx - \int_\Omega p^*\operatorname{div}(\boldsymbol{v}^*)\,dx + \int_{\partial\Omega} p^*(\boldsymbol{v}^*.n)\,d\sigma\\ &= \int_\Omega M^{-1}\boldsymbol{u}^*\cdot N^t(M^{-1}\boldsymbol{u}^*)\,dx - \langle M^{-1}\boldsymbol{u}^*, M\nabla q^*\rangle\\ &= \int_\Omega M^{-1}\boldsymbol{u}^*\cdot N^t(M^{-1}\boldsymbol{u}^*)\,dx + \int_\Omega q^*\operatorname{div}(\boldsymbol{u}^*)\,dx + \int_{\partial\Omega} q^*(\boldsymbol{u}^*.n)\,d\sigma\\ &= \int_\Omega N(M^{-1}\boldsymbol{u}^*)\cdot M^{-1}\boldsymbol{u}^*\,dx.\end{aligned}$$

Thus, we have shown (2.11). □

We do not have the symmetry hypothesis on B for the above theorem. This hypothesis will only affect the symmetry property of N in the above proof.

Remark 2.9. The limit pressure terms p^* and q^*, in fact, are in $H^1(\Omega)/\mathbb{R}$ since they solve the Neumann problem hidden in (2.8) and (2.9), respectively. Moreover, in particular, if B is the identity matrix, then (2.11) gives back the usual energy convergence. □

Remark 2.10. In the above theorem, we have concluded regarding the limit behaviour of the state-adjoint system. Equation (2.8) is called the Darcy law and, along with convergences (2.10), is well-known in the literature. However, the above theorem is original in the conclusion of (2.9) and the convergence (2.11) which generalises the notion of energy convergence. It is an easy exercise to note that when B is the identity matrix, then $M = N$, $\boldsymbol{u}^*_\varepsilon = \boldsymbol{v}^*_\varepsilon$ and $\boldsymbol{u}^* = \boldsymbol{v}^*$. Also,

$$\|\varepsilon^{-1}\nabla\boldsymbol{u}^*_\varepsilon\|^2_{2,\Omega_\varepsilon} \to \int_\Omega \boldsymbol{u}^*\cdot M^{-1}\boldsymbol{u}^*\,dx.$$

□

Recall that $\boldsymbol{\theta}^*$ is the weak limit of a subsequence of $\varepsilon^{-1}\widetilde{\boldsymbol{\theta}^*_\varepsilon}$ in $(L^2(\Omega))^n$. We have no means of concluding that $\boldsymbol{\theta}^*$ is the optimal control of an appropriate limit optimal control problem. Moreover, we have no result on the convergence of $\{\varepsilon^{-2}J_\varepsilon(\boldsymbol{\theta}^*_\varepsilon)\}$, in the general case when $0 \in U_\varepsilon$. However, we do note that $\varepsilon^{-1}J_\varepsilon(\boldsymbol{\theta}^*_\varepsilon) \to 0$ and

$$\liminf_{\varepsilon\to 0}\varepsilon^{-2}J_\varepsilon(\boldsymbol{\theta}^*_\varepsilon) \geq \frac{1}{2}\int_\Omega N\left(M^{-1}\boldsymbol{u}^*\right)\cdot M^{-1}\boldsymbol{u}^*\,dx + \frac{\nu}{2}\|\boldsymbol{\theta}^*\|_2^2.$$

The main difficulty in concluding regarding the optimal control problem is the disappearance of $\boldsymbol{\theta}^*$ in the limit state equation (2.8). We give two situations in which the convergence of the optimal controls will be obtained due to extra regularity available on the optimal controls.

Theorem 2.11 (unconstrained case)**.** *Let $U_\varepsilon = (L^2(\Omega_\varepsilon))^n$ and $\boldsymbol{\theta}^*_\varepsilon$ be the minimiser of the optimal control problem* (1.1)–(1.2) *and $(\boldsymbol{u}^*_\varepsilon, p^*_\varepsilon)$ be the corresponding state and pressure. Then* (2.8), (2.9), (2.10) *and* (2.11) *holds and, in addition, we have for the entire sequence,*

$$\begin{gathered}\varepsilon^{-1}\widetilde{\boldsymbol{\theta}^*_\varepsilon} \to 0 \quad \textit{strongly in } (L^2(\Omega))^n,\\ \varepsilon^{-2}\widetilde{\boldsymbol{\theta}^*_\varepsilon} \rightharpoonup \frac{-\boldsymbol{v}^*}{\nu} \quad \textit{weakly in } (L^2(\Omega))^n,\end{gathered}$$

and

$$\varepsilon^{-2} J_\varepsilon(\boldsymbol{\theta}^*_\varepsilon) \to \frac{1}{2}\int_\Omega N\left(M^{-1}\boldsymbol{u}^*\right)\cdot M^{-1}\boldsymbol{u}^*\,dx.$$

Proof. Since $0 \in U_\varepsilon$, by Theorem 2.8, (2.8), (2.9), (2.10) and (2.11) holds. It follows from (2.3) that $\boldsymbol{\theta}^*_\varepsilon$ is the projection of $\frac{-\boldsymbol{v}^*_\varepsilon}{\nu}$ in U_ε. Therefore, $\boldsymbol{\theta}^*_\varepsilon = \frac{-\boldsymbol{v}^*_\varepsilon}{\nu}$. Hence, from the bounds on $\boldsymbol{v}^*_\varepsilon$, we deduce that $\boldsymbol{\theta^*} = 0$. Thus, using Lemma 2.2 (also the arguments below it), we get for a subsequence,

$$\varepsilon^{-1}\widetilde{\boldsymbol{\theta}^*_\varepsilon} \rightharpoonup 0 \quad \text{weakly in } (L^2(\Omega))^n,$$
$$\varepsilon^{-2}\widetilde{\boldsymbol{\theta}^*_\varepsilon} \rightharpoonup \frac{-\boldsymbol{v}^*}{\nu} \quad \text{weakly in } (L^2(\Omega))^n.$$

In fact, the first convergence is strong, since $\varepsilon^{-1}\widetilde{\boldsymbol{v}^*_\varepsilon}$ converge to zero strongly in $(L^2(\Omega))^n$. Moreover, from the uniqueness of (2.8) and (2.9), the convergences hold for the entire sequence. The convergence of $\varepsilon^{-2}J_\varepsilon(\boldsymbol{\theta}^*_\varepsilon)$ follows from (2.11) and the strong convergence of $\varepsilon^{-1}\widetilde{\boldsymbol{\theta}^*_\varepsilon}$. □

Theorem 2.12 (Constrained case)**.** *Let U_ε be the positive cone of $(L^2(\Omega_\varepsilon))^n$,* i.e., $U_\varepsilon = \left\{\boldsymbol{\theta} \in (L^2(\Omega_\varepsilon))^n \mid \boldsymbol{\theta} \geq 0 \text{ a.e. in } \Omega_\varepsilon\right\}$ *and $\boldsymbol{\theta}^*_\varepsilon$ be the minimiser of the optimal control problem* (1.1)–(1.2) *and $(\boldsymbol{u}^*_\varepsilon, p^*_\varepsilon)$ be the corresponding state and pressure. Then* (2.8)*,* (2.9)*,* (2.10) *and* (2.11) *holds and, in addition, we have for the entire sequence,*

$$\varepsilon^{-1}\widetilde{\boldsymbol{\theta}^*_\varepsilon} \to 0 \quad \text{strongly in } (L^2(\Omega))^n,$$
$$\varepsilon^{-2}\widetilde{\boldsymbol{\theta}^*_\varepsilon} \rightharpoonup \frac{(\boldsymbol{v}^*)^-}{\nu} \quad \text{weakly in } (L^2(\Omega))^n, \varepsilon^{-2}J_\varepsilon(\boldsymbol{\theta}^*_\varepsilon) \to \frac{1}{2}\int_\Omega N\left(M^{-1}\boldsymbol{u}^*\right)\cdot M^{-1}\boldsymbol{u}^*\,dx$$

where $(\boldsymbol{v}^*)^-$ is the negative part of $\boldsymbol{v}^*$.

Proof. The proof is similar to that of Theorem 2.11, except that we now note that $\boldsymbol{\theta}^*_\varepsilon = \frac{(\boldsymbol{v}^*_\varepsilon)^-}{\nu}$ where $(\boldsymbol{v}^*_\varepsilon)^-$ is the negative part of $\boldsymbol{v}^*_\varepsilon$. □

Remark 2.13. When $n = 2$ or 3, the results of §2 easily carry forward to the situation when the state variable in the cost functional is determined by the Navier-Stokes equations:

$$\begin{gathered}\nabla p^*_\varepsilon + \boldsymbol{u}^*_\varepsilon.\nabla\boldsymbol{u}^*_\varepsilon - \Delta\boldsymbol{u}^*_\varepsilon = \boldsymbol{f} + \boldsymbol{\theta}^*_\varepsilon \quad \text{in } \Omega_\varepsilon,\\ \operatorname{div}(\boldsymbol{u}^*_\varepsilon) = 0 \quad \text{in } \Omega_\varepsilon,\\ \boldsymbol{u}^*_\varepsilon = 0 \quad \text{on } \partial\Omega_\varepsilon.\end{gathered} \tag{2.12}$$

The existence of solution for the above system is known when $n = 2$ or 3. The *a priori* bounds obtained in Lemma 2.2 remain valid, since

$$\int_{\Omega_\varepsilon}[(\boldsymbol{u}^*_\varepsilon.\nabla)\boldsymbol{u}^*_\varepsilon].\boldsymbol{u}^*_\varepsilon\,dx = -\frac{1}{2}\sum_{i=1}^{3}\int_{\Omega_\varepsilon}(\nabla.\boldsymbol{u}^*_\varepsilon)(u^*_\varepsilon)^2_i\,dx = 0.$$

The result of Theorem 2.8 also remains valid, since

$$\int_\Omega [(\tilde{\boldsymbol{u}^*_\varepsilon}.\nabla)\tilde{\boldsymbol{u}^*_\varepsilon}].\left(\varepsilon\boldsymbol{\phi}_1\left(x,\frac{x}{\varepsilon}\right) + \boldsymbol{\phi}_2\left(x,\frac{x}{\varepsilon}\right)\right)\,dx \to 0$$

as $\varepsilon \to 0$. However, we note that the matrices M and N are the same and are defined as in (2.4) and (2.5), respectively. In other words, we do not have the non-linear terms in the definition of the cell problems (2.4) and (2.5).

Conclusion. We have studied the asymptotic behaviour of the optimal control problem (1.1)–(1.2) when the holes are large, a situation left open in [15]. We have employed the two-scale method to achieve our result. We are successful in computing the limit of the state-adjoint pair (2.1)–(2.2), in the general case when $0 \in U_\varepsilon$. However, we are unable to conclude anything about the optimal control problem in this generality. Nevertheless, the optimal control problem is completely settled in the unconstrained case and the positive cone case. It would be interesting to see other non-trivial admissible control sets in which the problem could be settled.

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