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ASYMPTOTIC EXPANSION FORMULAS FOR THE MAXIMUM OF SOLUTIONS TO DIFFUSIVE LOGISTIC EQUATIONS

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Abstract. We consider the nonlinear eigenvalue problems

$$-u''(t) + u(t)^p = \lambda u(t),$$
$$u(t) > 0, \quad t \in I := (0,1), \quad u(0) = u(1) = 0,$$

where $p > 1$ is a constant and $\lambda > 0$ is a parameter. This equation is well known as the original logistic equation of population dynamics when $p = 2$. We establish the precise asymptotic formula for L^∞-norm of the solution u_λ as $\lambda \to \infty$ when $p = 2$ and $p = 5$.

1. Introduction

We consider the nonlinear eigenvalue problem

$$-u''(t) + u^p(t) = \lambda u(t), \quad t \in I := (0,1), \tag{1.1}$$
$$u(t) > 0, \quad t \in I, \tag{1.2}$$
$$u(0) = u(1) = 0, \tag{1.3}$$

where $p > 1$ is a constant and $\lambda > 0$ is a parameter. The equation (1.1)–(1.3) is well known as the original logistic equation of population dynamics when $p = 2$. In this case, the equation (1.1)–(1.3) describes the behavior of a species in I, where u and λ imply the population density and the growth rate, respectively. Therefore, many authors has been investigated the properties of solutions to (1.1)–(1.3) from not only pure mathematical point of view but also an application to the field of biology by computational analysis.

For the existence and uniqueness of the solutions, we know from [1] that for a given $\lambda > \pi^2$, there exists a unique solution $u = u_\lambda \in C^2(I)$ to (1.1)–(1.3). Furthermore, the set of solutions $\{\lambda, u_\lambda\} \subset \mathbf{R}_+ \times C^2(I)$ of (1.1)–(1.3) gives the clear picture of so called bifurcation diagram. Therefore, there are many works which treated the problem (1.1)–(1.3) by bifurcation theory of L^∞-framework.

The purpose of this paper is to study more precisely the asymptotic behavior of $\|u_\lambda\|_\infty$ as $\lambda \to \infty$, which is certainly one of the most important asymptotic properties of u_λ to know the global structure of the bifurcation curve. In spite of the simplicity of the equation, however, a few information of global structure of

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bifurcation diagram in L^∞-framework have been known. Therefore, the problem here is worth considering.

It is known that for general $p > 1$, the following asymptotic formula for $\|u_\lambda\|_\infty$ holds as $\lambda \to \infty$ [5].

$$\|u_\lambda\|_\infty = \lambda^{1/(p-1)}(1 - e^{-\sqrt{(p-1)\lambda}(1+o(1))/2}). \tag{1.4}$$

In particular, if $p = 3$, then the following asymptotic formula has been obtained in [4].

Theorem 1.1 ([4]). *Assume $p = 3$ in (1.1). Then as $\lambda \to \infty$,*

$$\|u_\lambda\|_\infty = \sqrt{\lambda}(1 - 4e^{-\sqrt{\lambda}/\sqrt{2}} - 8e^{-2\sqrt{\lambda}/\sqrt{2}} - 24\sqrt{2}\sqrt{\lambda}e^{-3\sqrt{\lambda}/\sqrt{2}} + o(\sqrt{\lambda}e^{-3\sqrt{\lambda}/\sqrt{2}})). \tag{1.5}$$

The main tool in the proof of Theorem 1.1 is an asymptotic expansion formula for the *complete* elliptic integral of the first kind. However, unfortunately, this approach can not be applied to the general $p > 1$, in particular, to the most important case $p = 2$. We overcome this difficulty by using the asymptotic expansion formula of the first elliptic integral in [2] and obtain the following results.

Theorem 1.2. *Assume $p = 2$ in (1.1). Then as $\lambda \to \infty$,*

$$\|u_\lambda\|_\infty = \lambda\big\{(1 - 6(\sqrt{3}-1)^2 e^{-\sqrt{\lambda}/2} - 12(\sqrt{3}-1)^4 e^{-\sqrt{\lambda}} + O(\sqrt{\lambda}e^{-3\sqrt{\lambda}/2})\big\}. \tag{1.6}$$

We find that almost the same idea, as that to prove Theorem 1.2, can be applied to the case $p = 5$.

Theorem 1.3. *Assume $p = 5$ in (1.1). Then as $\lambda \to \infty$,*

$$\|u_\lambda\|_\infty = \lambda^{1/4}\big\{1 - 6e^{-\sqrt{\lambda}} - 30e^{-2\sqrt{\lambda}} + O(\sqrt{\lambda}e^{-3\sqrt{\lambda}})\big\}. \tag{1.7}$$

The proofs of Theorems 1.2 and 1.3 are quite simple and straightforward. The future direction of this study will be to give the approach which can be applied to general $p > 1$.

2. Proof of Theorem 1.2

In what follows, we assume that $\lambda \gg 1$. We put

$$\xi = \xi_\lambda = 1 - \frac{\|u_\lambda\|_\infty}{\lambda}. \tag{2.1}$$

By (1.4), we know that $\xi_\lambda \to 0$ as $\lambda \to \infty$. To prove Theorem 1.2, we establish the asymptotic formula for ξ_λ. We know from [1] the fundamental properties of u_λ.

$$u_\lambda(t) = u_\lambda(1-t), \quad t \in I, \tag{2.2}$$

$$u'_\lambda(t) > 0, \quad t \in [0, 1/2), \tag{2.3}$$

$$\|u_\lambda\|_\infty = u_\lambda(1/2). \tag{2.4}$$

Multiply (1.1) by $u'_\lambda(t)$ to obtain

$$\{u''_\lambda(t) + \lambda u_\lambda(t) - u_\lambda^2(t)\}u'_\lambda(t) = 0.$$

This implies that

$$\frac{d}{dt}\Big(\frac{1}{2}u'_\lambda(t)^2 + \frac{1}{2}\lambda u_\lambda(t)^2 - \frac{1}{3}u_\lambda(t)^3\Big) = 0.$$

Namely, for $t \in \bar{I}$,

$$\frac{1}{2}u_\lambda'(t)^2 + \frac{1}{2}\lambda u_\lambda(t)^2 - \frac{1}{3}u_\lambda(t)^3 \equiv \text{constant} = \frac{1}{2}\lambda\|u_\lambda\|_\infty^2 - \frac{1}{3}\|u_\lambda\|_\infty^3. \tag{2.5}$$

By this and (2.3), for $0 \le t \le 1/2$,

$$u_\lambda'(t) = \sqrt{S_\lambda(u_\lambda(t))}, \tag{2.6}$$

where

$$S_\lambda(w) = \lambda(\|u_\lambda\|_\infty^2 - w^2) - \frac{2}{3}(\|u_\lambda\|_\infty^3 - w^3). \tag{2.7}$$

By this, we obtain

$$\frac{1}{2} = \int_0^{1/2} dt = \int_0^{1/2} \frac{u_\lambda'(t)}{\sqrt{S_\lambda(u_\lambda(t))}}dt = \frac{1}{\sqrt{\lambda}}\int_0^1 \frac{1}{\sqrt{R_\lambda(s)}}ds. \tag{2.8}$$

Here

$$R_\lambda(s) = 1 - s^2 - \eta_\lambda(1 - s^3), \tag{2.9}$$

$$\eta = \eta_\lambda = \frac{2\|u_\lambda\|_\infty}{3\lambda} = \frac{2}{3}(1-\xi). \tag{2.10}$$

Furthermore, put $\theta = 1 - s$. Then we obtain

$$R_\lambda(s) = U_\lambda(\theta) := \theta\eta(A_\lambda + B_\lambda\theta - \theta^2) = \theta\eta(a_\lambda - \theta)(\theta - c_\lambda), \tag{2.11}$$

where

$$A = A_\lambda = \frac{2-3\eta}{\eta} = \frac{3\xi}{1-\xi} = 3\xi(1 + \xi + O(\xi^2)), \tag{2.12}$$

$$B = B_\lambda = \frac{3\eta - 1}{\eta} = \frac{3(1-2\xi)}{2(1-\xi)} = \frac{3}{2}(1 - \xi - \xi^2 + O(\xi^3)), \tag{2.13}$$

$$a = a_\lambda = \frac{B + \sqrt{B^2 + 4A}}{2} = \frac{3}{2} + \frac{1}{2}\xi - \frac{1}{6}\xi^2 + O(\xi^3), \tag{2.14}$$

$$c = c_\lambda = \frac{B - \sqrt{B^2 + 4A}}{2} = -2\xi - \frac{4}{3}\xi^2 + O(\xi^3). \tag{2.15}$$

We obtain (2.12)–(2.15) by Taylor expansion. By (2.8) and (2.11), we obtain

$$\frac{\sqrt{\lambda}}{2} = \frac{1}{\sqrt{\eta}}\int_0^1 \frac{1}{\sqrt{\theta(a-\theta)(\theta-c)}}d\theta. \tag{2.16}$$

We set

$$\phi = \sin^{-1}\sqrt{\frac{a-c}{a(1-c)}}, \quad k = \sqrt{\frac{a}{a-c}}. \tag{2.17}$$

By [3, pp. 266], we know

$$\int_0^1 \frac{1}{\sqrt{\theta(a-\theta)(\theta-c)}}d\theta = \frac{2}{\sqrt{a-c}}F(\phi,k) = \frac{2}{\sqrt{a-c}}\int_0^\phi \frac{1}{\sqrt{1-k^2\sin^2 t}}dt. \tag{2.18}$$

By this, (2.10) and (2.16), we obtain

$$\frac{\sqrt{\lambda}}{2} = \frac{1}{\sqrt{\eta}}\frac{2}{\sqrt{a-c}}F(\phi,k) = \sqrt{\frac{6}{(1-\xi)}}\sqrt{\frac{1}{a-c}}F(\phi,k). \tag{2.19}$$

Therefore, to prove Theorem 1.2, we calculate $F(\phi,k)$ precisely. By [2, Eq. (1.16)], we know that as $k\to 1-0$ and $\phi\to\pi/2-0$,

$$F(\phi,k)=\frac{\sin\phi}{1-\beta}\big[\big(1+\frac{\zeta^2+\cos^2\phi}{4}\big)\log\frac{4}{\zeta+\cos\phi}-\frac{\zeta^2+\cos^2\phi-\zeta\cos\phi}{4}\big], \quad (2.20)$$

where

$$\zeta=(1-k^2\sin^2\phi)^{1/2},\quad 0<\beta<\frac{3}{8}\zeta^4. \quad (2.21)$$

Lemma 2.1. *For $\lambda\gg 1$,*

$$\begin{aligned}\log(\zeta+\cos\phi) &= \frac12\log\xi+\frac12\log 2-\frac12\log 3+\log(\sqrt3+1) \quad (2.22)\\ &\quad -\frac{\sqrt3}{1+\sqrt3}\big(\frac23+\frac{\sqrt3}{9}\big)\xi+O(\xi^2).\end{aligned}$$

Proof. By (2.15), (2.17) and Taylor expansion,

$$k^2\sin^2\phi=\frac{1}{1-c}=1-2\xi+\frac83\xi^2+O(\xi^3). \quad (2.23)$$

By this, (2.21) and Taylor expansion,

$$\zeta=(1-k^2\sin^2\phi)^{1/2}=\sqrt{2\xi}\big(1-\frac23\xi+O(\xi^2)\big). \quad (2.24)$$

By (2.14), (2.15), (2.17) and (2.23),

$$\sin^2\phi=\frac{a-c}{a}\frac{1}{1-c}=1-\frac23\xi+\frac49\xi^2+O(\xi^3). \quad (2.25)$$

By this and Taylor expansion,

$$\cos\phi=\sqrt{1-\sin^2\phi}=\sqrt{2\xi}\big(\frac{1}{\sqrt3}-\frac{1}{3\sqrt3}\xi+O(\xi^2)\big). \quad (2.26)$$

Then by (2.24) and (2.26),

$$\begin{aligned}\log(\zeta+\cos\phi) &= \frac12\log\xi+\frac12\log 2-\frac12\log 3+\log(\sqrt3+1) \quad (2.27)\\ &\quad -\frac{\sqrt3}{1+\sqrt3}\big(\frac23+\frac{\sqrt3}{9}\big)\xi+O(\xi^2).\end{aligned}$$

Thus the proof is complete. □

Proof of Theorem 1.2. By (2.24) and (2.26), we have

$$\frac{\zeta^2+\cos^2\phi}{4}=\frac23\xi+O(\xi^2). \quad (2.28)$$

Further, by (2.24), (2.25) and (2.26),

$$\sin\phi=1-\frac13\xi+O(\xi^2), \quad (2.29)$$

$$\zeta\cos\phi=\frac{2}{\sqrt3}\xi+O(\xi^2), \quad (2.30)$$

$$\zeta^4=O(\xi^2). \quad (2.31)$$

By (2.14) and (2.15),

$$\frac{1}{\sqrt{a-c}}=\sqrt{\frac23}\Big(1-\frac56\xi+O(\xi^2)\Big). \quad (2.32)$$

Then by (2.20), (2.22) and (2.28)–(2.32),

$$
\begin{aligned}
F(\phi,k) &= (1+O(\xi^2))(1-\frac{1}{3}\xi+O(\xi^2)) \\
&\quad\times\big\{(1+\frac{2}{3}\xi+O(\xi^2))(\log 4-\log(\zeta+\cos\phi))-\frac{4-\sqrt{3}}{6}\xi+O(\xi^2))\big\} \\
&= -\frac{1}{2}\log\xi+D-\frac{1}{6}\xi\log\xi+\big(\frac{1}{6}+\frac{1}{3}D\big)\xi+O(\xi^2\log\xi),
\end{aligned}
\tag{2.33}
$$

where

$$
D=\frac{3}{2}\log 2+\frac{1}{2}\log 3-\log(\sqrt{3}+1)=\frac{1}{2}\log 6(\sqrt{3}-1)^2. \tag{2.34}
$$

By this, (2.19), (2.32) and Lemma 2.1,

$$
\begin{aligned}
\frac{\sqrt{\lambda}}{2} &= \sqrt{6}(1-\xi)^{-1/2}\sqrt{\frac{2}{3}}\Big(1-\frac{5}{6}\xi+O(\xi^2)\Big)F(\phi,k) \\
&= 2(1-\frac{1}{3}\xi+O(\xi^2))F(\phi,k) \\
&= -\log\xi+\log 6(\sqrt{3}-1)^2+\frac{1}{3}\xi+O(\xi^2\log\xi) \\
&= -\log\xi+\log 6(\sqrt{3}-1)^2+\frac{1}{3}\xi+O(\sqrt{\lambda}e^{-\sqrt{\lambda}}).
\end{aligned}
\tag{2.35}
$$

By this and Taylor expansion,

$$
\begin{aligned}
\xi &= e^{-\sqrt{\lambda}/2}\cdot e^{\log 6(\sqrt{3}-1)^2}\cdot e^{\xi/3}\cdot e^{O(\sqrt{\lambda}e^{-\sqrt{\lambda}})} \\
&= 6(\sqrt{3}-1)^2e^{-\sqrt{\lambda}/2}\Big(1+\frac{1}{3}\xi+O(\xi^2)\Big)\Big(1+O(\sqrt{\lambda}e^{-\sqrt{\lambda}})\Big) \\
&= 6(\sqrt{3}-1)^2e^{-\sqrt{\lambda}/2}(1+O(\sqrt{\lambda}e^{-\sqrt{\lambda}}))+2(\sqrt{3}-1)^2\xi e^{-\sqrt{\lambda}/2}.
\end{aligned}
\tag{2.36}
$$

By this, we obtain

$$
\xi\Big(1-2(\sqrt{3}-1)^2e^{-\sqrt{\lambda}/2}\Big)=6(\sqrt{3}-1)^2e^{-\sqrt{\lambda}/2}(1+O(\sqrt{\lambda}e^{-\sqrt{\lambda}})). \tag{2.37}
$$

This implies

$$
\begin{aligned}
\xi &= 6(\sqrt{3}-1)^2e^{-\sqrt{\lambda}/2}\Big(1+O(\sqrt{\lambda}e^{-\sqrt{\lambda}})\Big)\Big(1+2(\sqrt{3}-1)^2e^{-\sqrt{\lambda}/2}+O(e^{-\sqrt{\lambda}})\Big) \\
&= 6(\sqrt{3}-1)^2e^{-\sqrt{\lambda}/2}+12(\sqrt{3}-1)^4e^{-\sqrt{\lambda}}+O(\sqrt{\lambda}e^{-3\sqrt{\lambda}/2}).
\end{aligned}
$$

Thus, the proof is complete. □

3. Proof of Theorem 1.3

In this section, let $p=5$. We put

$$
r=r_\lambda=1-\frac{\|u_\lambda\|_\infty^4}{\lambda}. \tag{3.1}
$$

To prove Theorem 1.3, we calculate r precisely. Let

$$
\sigma=\sigma_\lambda=\frac{1}{3}\frac{\|u_\lambda\|_\infty^4}{\lambda}=\frac{1-r}{3} \tag{3.2}
$$

and

$$
Y_\lambda(s):=1-s^2-\sigma(1-s^6)=(1-s^2)\{1-\sigma(1+s^2+s^4)\}. \tag{3.3}
$$

Then by the same calculation as that to obtain (2.8), we have

$$\frac{1}{2} = \frac{1}{\sqrt{\lambda}}\int_0^1 \frac{1}{\sqrt{Y_\lambda(s)}}ds. \tag{3.4}$$

This along with (3.3) implies

$$\begin{aligned}
\frac{\sqrt{\lambda}}{2} &= \int_0^1 \frac{1}{\sqrt{(1-s^2)\{1-\sigma(1+s^2+s^4)\}}}ds \\
&= \frac{1}{2}\int_0^1 \frac{2sds}{\sqrt{s^2(1-s^2)\{1-\sigma(1+s^2+s^4)\}}}ds \\
&= \frac{1}{2}\int_0^1 \frac{dt}{\sqrt{t(1-t)\{1-\sigma(1+t+t^2)\}}}ds \\
&= \frac{1}{2\sqrt{\sigma}}\int_0^1 \frac{1}{\sqrt{t(1-t)(t-d)(a-t)}}dt,
\end{aligned} \tag{3.5}$$

where

$$a = \frac{-1+\sqrt{1-4\delta}}{2} = 1 + r + \frac{2}{3}r^2 + O(r^3), \tag{3.6}$$

$$d = \frac{-1-\sqrt{1-4\delta}}{2} = -2 - r - \frac{2}{3}r^2 + O(r^3), \tag{3.7}$$

$$\delta = 1 - \frac{1}{\sigma} = -2 - 3r - 3r^2 + O(r^3). \tag{3.8}$$

By (3.5) and [3, p. 290], we know

$$\sqrt{\lambda} = \frac{1}{\sqrt{\sigma}}\frac{2}{\sqrt{a(1-d)}}F\big(\frac{\pi}{2}, k\big), \tag{3.9}$$

where

$$k = \sqrt{\frac{a-d}{a(1-d)}}. \tag{3.10}$$

By (2.20), as $k \to 1-0$,

$$F\left(\frac{\pi}{2}, k\right) = \frac{1}{1-\beta}\big[\big(1+\frac{\zeta^2}{4}\big)\log\frac{4}{\zeta} - \frac{\zeta^2}{4}\big], \tag{3.11}$$

where (by (3.6), (3.7) and (3.10),

$$\zeta^2 = 1 - k^2 = \frac{d(1-a)}{a(1-d)} = \frac{2}{3}r\big(1 - \frac{1}{6}r + O(r^2)\big), \tag{3.12}$$

$$0 < \beta < \frac{3}{8}\zeta^4 = O(r^2). \tag{3.13}$$

By this, (3.11)–(3.13) and Taylor expansion,

$$\begin{aligned}
F\left(\frac{\pi}{2}, k\right) &= (1+O(r^2))\big[(1+\frac{1}{6}r + O(r^2))(\log 4 - \log\zeta) - \frac{1}{6}r + O(r^2)\big] \\
&= -\frac{1}{2}\log r + \frac{1}{2}\log 24 - \frac{1}{12}r\log r + \big(\frac{1}{12}\log 24 - \frac{1}{12}\big)r + O(r^2).
\end{aligned} \tag{3.14}$$

Further, by (3.2), (3.6), (3.7) and Taylor expansion,

$$\frac{1}{\sqrt{\sigma}} = \sqrt{3}\big(1 + \frac{1}{2}r + \frac{3}{8}r^2 + O(r^3)\big), \tag{3.15}$$

$$\frac{1}{\sqrt{a(1-d)}} = \frac{1}{\sqrt{3}}\big(1 - \frac{2}{3}r + O(r^2)\big). \tag{3.16}$$

By (3.9) and (3.14)–(3.16), we obtain

$$\begin{aligned}\sqrt{\lambda} &= \big(1 - \frac{1}{6}r + O(r^2)\big)(-\log r + \log 24 - \frac{1}{6}r\log r + \frac{1}{6}(\log 24 - 1)r + O(r^2)) \\ &= -\log r + \log 24 - \frac{1}{6}r + O(r^2 \log r).\end{aligned} \tag{3.17}$$

By this, we obtain

$$re^{r/6} = 24e^{-\sqrt{\lambda}}e^{O(r^2 \log r)}. \tag{3.18}$$

By this and Taylor expansion, we obtain

$$\begin{aligned}r &= 24\Big(1 - \frac{1}{6}r + O(r^2)\Big)(1 + O(r^2 \log r))e^{-\sqrt{\lambda}} \\ &= 24e^{-\sqrt{\lambda}} - 96e^{-2\sqrt{\lambda}} + o(e^{-2\sqrt{\lambda}}).\end{aligned} \tag{3.19}$$

By substituting this into (3.17) again, we obtain

$$r = 24e^{-\sqrt{\lambda}} - 96e^{-2\sqrt{\lambda}} + O(\sqrt{\lambda}e^{-3\sqrt{\lambda}}).$$

By this and (3.1), we obtain

$$\begin{aligned}\|u_\lambda\|_\infty &= \lambda^{1/4}(1 - 24e^{-\sqrt{\lambda}} + 96e^{-2\sqrt{\lambda}} + O(\sqrt{\lambda}e^{-3\sqrt{\lambda}}))^{1/4} \\ &= \lambda^{1/4}(1 - 6e^{-\sqrt{\lambda}} - 30e^{-2\sqrt{\lambda}} + O(\sqrt{\lambda}e^{-3\sqrt{\lambda}})).\end{aligned}$$

Thus, the proof is complete.

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