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APPROXIMATE CONTROLLABILITY OF NEUTRAL STOCHASTIC INTEGRODIFFERENTIAL SYSTEMS IN HILBERT SPACES

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Abstract. In this paper sufficient conditions are established for the controllability of a class of neutral stochastic integrodifferential equations with nonlocal conditions in abstract space. The Nussbaum fixed point theorem is used to obtain the controllability results, which extends the linear system to the stochastic settings with the help of compact semigroup. An example is provided to illustrate the theory.

1. Introduction

Controllability is one of the fundamental concepts in mathematical control theory and plays an important role in both deterministic and stochastic control systems. It is well known that controllability of deterministic systems are widely used in many fields of science and technology. The controllability of nonlinear deterministic systems represented by evolution equations in abstract spaces has been extensively studied by several authors [2, 3]. Stochastic control theory is a stochastic generalization of classical control theory.

However, in many cases, the accurate analysis, design and assessment of systems subjected to realistic environments must take into account the potential of random loads and randomness in the system properties. Randomness is intrinsic to the mathematical formulation of many phenomena such as fluctuations in the stock market, or noise in communication networks. Mathematical modelling of such systems often leads to differential equations with random parameters. The use of deterministic equations that ignore the randomness of the parameter or replace them by their mean values can result in gross errors. All such problems are mathematically modelled and described by various stochastic systems described by stochastic differential equations, stochastic delay equations and in some cases stochastic integrodifferential equations which are mathematical models for phenomena with irregular fluctuations.

The problem of controllability of the linear stochastic system of the form

$$dx(t) = [Ax(t) + Bu(t)]dt + \tilde{\sigma}(t)dw(t)$$

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$$x(0) = x_0, \quad t \in I = [0, T]$$

in Hilbert spaces has been studied by Dubov and Mordukhovich [10], Mahmudov [12].

The problem of controllability of nonlinear stochastic system in infinite dimensional spaces have been studied by many authors. Sirbu and Tessitore [16] studied null controllability of an infinite dimensional stochastic differential equations with state and control dependent noise using Riccati equation approach. Mahmudov [13] investigated the sufficient conditions for approximate controllability of nonlinear systems in Hilbert spaces by using the Nussbaum fixed point theorem. Dauer and Mahmudov [9], Mahmudov [14] studied controllability of semilinear stochastic system by using the Banach fixed point technique. Sunahara et al [17] introduced the concept of stochastic ϵ-controllability and controllability with probability and established sufficient conditions for stochastic controllability of a class of nonlinear systems. Sufficient conditions for stochastic ϵ-controllability have been established by Klamka and Socha [11] using a stochastic Lyapunov-like approach.

Balachandran et al [6] discussed the controllability of neutral functional integrodifferential systems in Banach spaces by using semigroup theory and the Nussbaum fixed point theorem. Recently, Balachandran and Karthikeyan [4], Balachandran et al [5] derived sufficient conditions for the controllability of stochastic integrodifferential systems in finite dimensional spaces. This paper is different from previous works in which dependence of the nonlinear map contain integrodifferential term with nonlocal condition. Here we are interested to establish a set of sufficient conditions for the approximate controllability of the following nonlinear neutral stochastic integrodifferential systems with non-local condition

$$\begin{aligned} &d[x(t) - q(t,x)] \\ &= [Ax(t) + Bu(t) + f(t, x(t)) + \int_0^t g(t, s, x(s))ds]dt + \sigma(t, x(t))dw(t) \qquad (1.1) \\ &\qquad x(0) + h(x) = x_0, \quad t \in I = [0, T]. \end{aligned}$$

in a Hilbert space H by using the Nussbaum fixed point theorem. Here $(\Omega, \mathcal{F}, P)$ is a probability space with a normal filtration

$$\{\mathcal{F}_t = \sigma(w(s) : s \le t),\ 0 \le t \le T\}$$

generated by w; H, E, U are three separable Hilbert spaces, and w is a Q-Wiener process on $(\Omega, \mathcal{F}, P)$, with the covariance operator $Q \in \mathcal{L}(E)$. We assume that there exists a complete orthonormal system $\{e_k\}$ in E, a bounded sequence of non-negative real numbers λ_k such that $Qe_k = \lambda_k e_k$ and a sequence of real independent Brownian motions such that $w(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\beta_k(t)e_k$. Let $L_2^0 = L_2(Q^{\frac{1}{2}}E, H)$ be the space of all Hilbert-Schmidt operators. The space L_2^0 is a separable Hilbert space, equipped with the norm $\|\Psi\|_Q^2 = \mathrm{tr}[\Psi Q \Psi^*]$. $L_2^{\mathcal{F}}(I, H)$ is the space of all $\mathcal{F}_t$-adapted, H-valued measurable square integrable processes on $I \times \Omega$. $\mathcal{C}(I, L_2(\Omega, F, P, H))$ is the Banach space of continuous maps from I into $L_2(\Omega, F, P, H)$ satisfying the condition that $\sup_{t\in I} \mathbf{E}\|x(t)\|^2 < \infty$. $\mathcal{C}(I, L_2)$ is the closed subspace of $\mathcal{C}(I, L_2(\Omega, F, P, H))$ consisting of measurable and $\mathcal{F}_t$-adapted processes $x(t)$ with norm $\|x\|_*^2 = \sup_{t\in I} \mathbf{E}\|x(t)\|^2$.

Concerning the operators A, B, f, q, g, σ, h we assume the following hypotheses:

- (H1) The operator A generates a compact semigroup $S(\cdot)$ and B is a bounded linear operator from a Hilbert space U into H.

(H2) The functions $f : I \times H \to H$, $q : I \times H \to H$, $g : I \times I \times H \to H$, $\sigma : I \times H \to L_2^0$ and $h : C(I,H) \to H$, satisfy the Lipschitz condition and there exist constants $L_1, L_2, L_3, l > 0$ for $x_1, x_2 \in H$ and $0 \le s < t \le T$ such that

$$\begin{gathered}
\|f(t,x_1) - f(t,x_2)\|^2 + \|\sigma(t,x_1) - \sigma(t,x_2)\|_Q^2 \le L_1\|x_1 - x_2\|^2 \\
\|g(t,s,x_1(s)) - g(t,s,x_2(s))\|^2 \le L_2\|x_1 - x_2\|^2 \\
\|q(t,x_1) - q(t,x_2)\|^2 \le L_3\|x_1 - x_2\|^2 \\
\|h(x_1) - h(x_2)\|^2 \le l\|x_1 - x_2\|^2
\end{gathered}$$

(H3) The functions f, q, g, h and σ are continuous and there exist constants $L_4, L_5, L_6, l_1 > 0$ for $x \in H$ and $0 \le t \le T$ such that

$$\begin{gathered}
\|f(t,x)\|^2 + \|\sigma(t,x)\|_Q^2 \le L_4 \\
\|g(t,s,x(s))\|^2 \le L_5 \\
\|q(t,x)\|^2 \le L_6 \\
\|h(x)\|^2 \le l_1
\end{gathered}$$

It is clear that under these conditions the system (1.1) admits a mild solution $x(\cdot) \in \mathcal{C}(I, L_2)$ for any $x_0 \in H$, $u(\cdot) \in L_2^{\mathcal{F}}(I,U)$ in the following form (see [8]).

$$\begin{aligned}
x(t) = S(t)[x_0 - h(x) - q(0,x(0))] + q(t,x(t)) + \int_0^t AS(t-s)q(s,x(s))ds \\
+ \int_0^t S(t-s)Bu(s)ds + \int_0^t S(t-s)f(s,x(s))ds \\
+ \int_0^t S(t-s)\sigma(s,x(s))dw(s) + \int_0^t S(t-s)\Big[\int_0^s g(s,\tau,x(\tau))d\tau\Big]ds.
\end{aligned} \tag{1.2}$$

To study the approximate controllability of the system (1.2), we consider the approximate controllability of its corresponding linear part

$$\begin{gathered}
d[x(t) - q(t)] = \Big[Ax(t) + Bu(t) + f(t) + \int_0^t g(t,s)ds\Big]dt + \tilde{\sigma}(t)dw(t) \\
x(0) + h(x) = x_0, \quad t \in I = [0,T].
\end{gathered} \tag{1.3}$$

where $\tilde{\sigma} \in L_2^0$ and assume the approximate controllability of the system (1.3)

We need the Nussbaum fixed-point theorem (see [15]) to establish our results.

Theorem 1.1. *Suppose that Y is a closed, bounded convex subset of a Banach space $\mathcal{H}$. Suppose that $\mathbb{P}_1, \mathbb{P}_2$ are continuous mappings from Y into $\mathcal{H}$ such that*

(i) $(\mathbb{P}_1 + \mathbb{P}_2)Y \subset Y$,
(ii) $\|\mathbb{P}_1 x - \mathbb{P}_2 y\| \le k\|x - y\|$ *for all $x, y \in Y$ where $0 \le k < 1$ is a constant,*
(iii) $\overline{\mathbb{P}}_2[Y]$ *is compact.*

Then the operator $\mathbb{P}_1 + \mathbb{P}_2$ has a fixed point in Y.

2. Controllability Results for Linear Systems

In this section we find an optimal control for solving the stochastic linear regulator problem in terms of stochastic controllability operator which drives a point $x_0 \in H$ to a small neighbourhood of an arbitrary point $b \in L_2(\mathcal{F}_T, H)$. Further, we study the relation between controllability operator Γ_0^T and its stochastic analogue Π_0^T.

Define the linear regulator problem: minimize

$$J(u) = \mathbf{E}\|x_\alpha(T) - b\|^2 + \alpha \mathbf{E}\int_0^T \|u(t)\|^2 dt \tag{2.1}$$

over all $u(\cdot) \in L_2^{\mathcal{F}}(I, U)$, where the solution $x(\cdot)$ of (1.3) is given by

$$\begin{aligned} x(t) &= S(t)[x_0 - h(x) - q(0)] + q(t) + \int_0^t AS(t-s)q(s)ds \\ &\quad + \int_0^t S(t-s)Bu(s)ds + \int_0^t S(t-s)f(s)ds \\ &\quad + \int_0^t S(t-s)\tilde{\sigma}(s)dw(s) + \int_0^t S(t-s)\Big[\int_0^s g(s,\tau)d\tau\Big]ds, \end{aligned} \tag{2.2}$$

Here $b \in L_2(\mathcal{F}_T, H)$ and $\alpha > 0$ are parameters and $f(\cdot) \in L_2^{\mathcal{F}}(I, H)$, $q(\cdot) \in L_2^{\mathcal{F}}(I, H)$, $g(\cdot) \in L_2^{\mathcal{F}}(I, I, H)$, $\tilde{\sigma}(\cdot) \in L_2^{\mathcal{F}}(I, L_0^2)$ and $h(\cdot) \in C(I, H)$.

It is convenient to introduce the relevant operators and the basic controllability condition

(i) The operator $L_0^T \in \mathcal{L}(L_2^{\mathcal{F}}(I, H), L_2(\Omega, \mathcal{F}_T, H))$ is defined by

$$L_0^T u = \int_0^T S(T-s)Bu(s)ds.$$

Clearly the adjoint $(L_0^T)^* : L_2(\Omega, \mathcal{F}_T, H) \to L_2^{\mathcal{F}}(I, H)$ is defined by

$$[(L_0^T)^* z](t) = B^* S^*(T-t)\mathbf{E}\{z \mid \mathcal{F}_t\}.$$

(ii) The controllability operator Π_0^T associated with (1.3) is defined by

$$\Pi_0^T\{\cdot\} = L_0^T (L_0^T)^*\{\cdot\} = \int_0^T S(T-t)BB^*S^*(T-t)\mathbf{E}\{\cdot \mid \mathcal{F}_t\}dt.$$

which belongs to $\mathcal{L}(L_2(\Omega, \mathcal{F}_T, H), L_2(\Omega, \mathcal{F}_T, H))$ and the controllability operator $\Gamma_s^T \in \mathcal{L}(H, H)$ is

$$\Gamma_s^T = \int_s^T S(T-t)BB^*S^*(T-t)dt, \quad 0 \le s < t.$$

(iii) The resolvent operator

$$\mathcal{R}(\alpha, \Gamma_0^T) := (\alpha I + \Gamma_0^T)^{-1}, \quad \mathcal{R}(\alpha, \Pi_0^T) := (\alpha I + \Pi_0^T)^{-1}.$$

(AC) $\alpha\mathcal{R}(\alpha, \Pi_0^T) := (\alpha I + \Pi_0^T)^{-1} \to 0$ as $\alpha \to 0^+$ in the strong topology.

It is known that the assumption (AC) holds if and only if the linear stochastic system(1.3) is approximately controllable on $[0, T]$ (see [12]). The following lemmas whose proof can be found in [13] and lemma 2.2 give a formula for a control which steers the system (2.2) from a point $x_0 \in H$ to a small neighbourhood of an arbitrary point $b \in L_2(\mathcal{F}_T, H)$.

Lemma 2.1. *(a) For arbitrary $z \in L_2(\mathcal{F}_T, H)$ there exists $k_z(\cdot) \in L_2^{\mathcal{F}}(I, L_2^0)$ such that*

$$\mathbf{E}\{z|\mathcal{F}_t\} = \mathbf{E}z + \int_0^t k_z(s)dw(s), \tag{2.3}$$

$$\Pi_0^T z = \Gamma_0^T \mathbf{E} z + \int_0^t \Gamma_s^T k_z(s) dw(s), \tag{2.4}$$

$$R(\alpha, \Pi_0^T) z = R(\alpha, \Gamma_0^T) \mathbf{E} z + \int_0^T R(\alpha, \Gamma_s^T) k_z(s) dw(s), \tag{2.5}$$

$$\begin{aligned} &\Pi_0^t S^*(T-t) R(\alpha, \Pi_0^T) z \\ &= \Gamma_0^t S^*(T-t) R(\alpha, \Gamma_0^T) \mathbf{E} z + \int_0^t \Gamma_r^t S^*(T-t) R(\alpha, \Gamma_r^T) k_z(r) dw(r). \end{aligned} \tag{2.6}$$

(b) If $f : I \times H \to H$, $q : I \times H \to H$, $g : I \times I \times H \to H$, satisfies the condition (H2) and $x(\cdot) \in L_2^F(I, H)$, then there exist $k_f(\cdot, x(s)) \in L_2^F(I, L_2^0)$, $k_q(\cdot, x(s)) \in L_2^F(I, L_2^0)$ and $k_g(\cdot, \cdot, x(s)) \in L_2^F(I, I, L_2^0)$ such that

$$\begin{aligned} &\mathbf{E}\{\int_0^T S(T-s) f(s, x(s)) ds | \mathcal{F}_t\} \\ &= \mathbf{E} \int_0^T S(T-s) f(s, x(s)) ds + \int_0^T k_f(s, x(s)) dw(s) \end{aligned} \tag{2.7}$$

$$\begin{aligned} &\mathbf{E}\{\int_0^T AS(T-s) q(s, x(s)) ds | \mathcal{F}_t\} \\ &= \mathbf{E} \int_0^T AS(T-s) q(s, x(s)) ds + \int_0^T k_q(s, x(s)) dw(s) \end{aligned} \tag{2.8}$$

$$\begin{aligned} &\mathbf{E}\{\int_0^T S(T-s)\Big[\int_0^s g(s, \tau, x(\tau)) d\tau\Big] ds | \mathcal{F}_t\} \\ &= \mathbf{E} \int_0^T S(T-s)\Big[\int_0^s g(s, \tau, x(\tau)) d\tau\Big] ds + \int_0^T \Big[\int_0^s k_g(s, \tau, x(\tau)) d\tau\Big] dw(s) \end{aligned} \tag{2.9}$$

and for all $x(\cdot), y(\cdot) \in L_2^F(I, H)$

$$\mathbf{E} \int_0^T \|k_f(s, x(s)) - k_f(s, y(s))\|^2 ds \le K^2 T L_1 \Big(\mathbf{E} \int_0^T \|x(s) - y(s)\|^2 ds\Big) \tag{2.10}$$

$$\mathbf{E} \int_0^T \|k_f(s, x(s))\|^2 ds \le K^2 T^2 L_4 \tag{2.11}$$

$$\mathbf{E} \int_0^T \|k_q(s, x(s)) - k_q(s, y(s))\|^2 ds \le K^2 l_0^2 T L_3 \Big(\mathbf{E} \int_0^T \|x(s) - y(s)\|^2 ds\Big) \tag{2.12}$$

$$\mathbf{E} \int_0^T \|k_q(s, x(s))\|^2 ds \le K^2 l_0^2 T^2 L_6, \tag{2.13}$$

$$\begin{aligned} &\mathbf{E} \int_0^T \Big\| \int_0^s \Big[k_g(s, \tau, x(\tau)) - k_g(s, \tau, y(\tau))\Big] d\tau \Big\|^2 ds \\ &\le K^2 T^2 L_2 \Big(\mathbf{E} \int_0^s \|x(\tau) - y(\tau)\|^2 d\tau\Big) \end{aligned} \tag{2.14}$$

$$\mathbf{E} \int_0^T \Big\| \int_0^s \Big[k_g(s, \tau, x(\tau)) d\tau\Big] \Big\|^2 ds \le K^2 T^3 L_5 \tag{2.15}$$

where $K = \max\{\|S(t)\| : 0 \le t \le T\}$ and $l_0 = \|AS(t)q\|$.

Lemma 2.2. *There exists a unique control $u_\alpha(\cdot) \in L_2^F(I,U)$ such that*

$$
\begin{aligned}
u_\alpha(t) = B^* S^*(T-t)\mathbf{E}\Big\{ R(\alpha, \Pi_0^T)\Big(b - S(T)[x_0 - h(x) - q(0)] - q(T) \\
- \int_0^T AS(T-s)q(s)ds - \int_0^T S(T-s)f(s)ds \\
- \int_0^T S(T-s)\Big[\int_0^s g(s,\tau)d\tau\Big]ds - \int_0^T S(T-s)\tilde{\sigma}(s)dw(s)\Big)|\mathcal{F}_t\Big\}
\end{aligned}
\tag{2.16}
$$

and

$$
\begin{aligned}
&x_\alpha(T) \\
&= b - \alpha R(\alpha, \Gamma_0^T)\Big(\mathbf{E}b - S(T)[x_0 - h(x) - q(0)] - q(T) \\
&- \mathbf{E}\int_0^T AS(T-s)q(s)ds - \mathbf{E}\int_0^T S(T-s)\Big[f(s)ds + \int_0^s g(s,\tau)d\tau\Big]ds\Big) \\
&- \alpha\int_0^T R(\alpha, \Gamma_s^T)\Big(k_b(s) - S(T-s)\tilde{\sigma}(s) - k_f(s) - k_q(s) - \int_0^s k_g(s,\tau)d\tau\Big)dw(s)
\end{aligned}
\tag{2.17}
$$

where

$$
\begin{gathered}
\mathbf{E}\{\int_0^T S(T-s)f(s)ds|\mathcal{F}_t\} = \mathbf{E}\int_0^T S(T-s)f(s)ds + \int_0^T k_f(s)dw(s), \\
\mathbf{E}\{\int_0^T AS(T-s)q(s)ds|\mathcal{F}_t\} = \mathbf{E}\int_0^T AS(T-s)q(s)ds + \int_0^T Ak_q(s)dw(s), \\
\mathbf{E}\{\int_0^T S(T-s)\Big[\int_0^s g(s,\tau)d\tau\Big]ds|\mathcal{F}_t\} \\
= \mathbf{E}\int_0^T S(T-s)\Big[\int_0^s g(s,\tau)d\tau\Big]ds + \int_0^T \Big[\int_0^s k_g(s,\tau)d\tau\Big]dw(s).
\end{gathered}
$$

Proof. The problem of minimizing the functional (2.1) has a unique solution $u_\alpha(\cdot) \in L_2^{\mathcal{F}}(I,U)$ which is completely characterized by the stochastic maximum principle (see [1]) and has the following form:

$$
u_\alpha(t) = -\alpha^{-1}B^*S^*(T-t)\mathbf{E}\{x_\alpha(T) - b|\mathcal{F}_t\}.
$$

Formula (2.17) shows that the linear system (2.2) is approximately controllable on $[0,T]$ if and only if $\alpha R(\alpha, \Pi_0^T)$ converges to zero operator as $\alpha \to 0^+$ in the strong topology [12]. □

3. Approximate Controllability

In this section sufficient conditions are established for the approximate controllability of the stochastic control system (1.2) under the assumption that the associated linear system is approximately controllable.

Definition. The stochastic system (1.2) is approximately controllable on the interval I if

$$
\overline{\mathcal{R}}_T(x_0) = L_2(\mathcal{F}_T, H),
$$

where $\mathcal{R}_T(x_0) = \{x(T; x_0, u) : u(\cdot) \in L_2^{\mathcal{F}}(I,U)\}$.

Define the control

$$
\begin{aligned}
u_\alpha(t) = B^*S^*(T-t)\mathbf{E}\Big\{ & R(\alpha,\Pi_0^T)\Big(b - S(T)[x_0 - h(x) - q(0,x(0))] - q(T,x(T)) \\
& - \int_0^T AS(T-s)q(s,x(s))ds - \int_0^T S(T-s)f(s,x(s))ds \\
& - \int_0^T S(T-s)\Big[\int_0^s g(s,\tau,x(\tau))d\tau\Big]ds \\
& - \int_0^T S(T-s)\sigma(s,x(s))dw(s)\Big)\Big|\mathcal{F}_t\Big\}.
\end{aligned}
\tag{3.1}
$$

To formulate the controllability problem in the form suitable for application of the Nussbaum fixed-point theorem, we put the control $u_\alpha(\cdot)$ into the stochastic control system (1.2) and obtain a nonlinear operator $\mathbb{P}^\alpha : C(I,L_2) \to C(I,L_2)$

$$
\begin{aligned}
(\mathbb{P}^\alpha x)(t) = S(t)&[x_0 - h(x) - q(0,x(0))] + q(t,x(t)) + \int_0^T AS(T-s)q(s,x(s))ds \\
& + \int_0^t S(t-s)f(s,x(s))ds + \int_0^t S(t-s)\sigma(s,x(s))dw(s) \\
& + \int_0^t S(t-s)\Big[\int_0^s g(s,\tau,x(\tau))d\tau\Big]ds + \Pi_0^t S^*(T-t)R(\alpha,\Pi_0^T) \\
& \times \Big(b - S(T)[x_0 - h(x) - q(0,x(0))] - q(t,x(t)) \\
& - \int_0^T AS(T-s)q(s,x(s))ds - \int_0^T S(T-s)f(s,x(s))ds \\
& - \int_0^T S(T-s)\Big[\int_0^s g(s,\tau,x(\tau))d\tau\Big]ds - \int_0^T S(T-s)\sigma(s,x(s))dw(s)\Big).
\end{aligned}
$$

It will be shown that the stochastic control system (1.2) is approximately controllable if for all $\alpha > 0$ there exists a fixed point of the operator $\mathbb{P}^\alpha$. To show that $\mathbb{P}^\alpha$ has a fixed point we employ the Nussbaum fixed-point theorem in $C(I,L_2)$.We now define the operators $\mathbb{P}_1^\alpha : C(I,L_2) \to C(I,L_2)$ and $\mathbb{P}_2^\alpha : C(I,L_2) \to C(I,H)$ as follows:

$$
\begin{aligned}
&(\mathbb{P}_1^\alpha x)(t) \\
&= S(t)[x_0 - h(x) - q(0,x(0))] + q(t,x(t)) + \int_0^T AS(T-s)q(s,x(s))ds \\
&\quad + \int_0^t S(t-s)f(s,x(s))ds + \int_0^t S(t-s)\sigma(s,x(s))dw(s) \\
&\quad + \int_0^t S(t-s)\Big[\int_0^s g(s,\tau,x(\tau))d\tau\Big]ds + \int_0^t \Gamma_s^t S^*(T-t)R(\alpha,\Gamma_s^T)\Big[k_b(s) \\
&\quad - S(T-s)\sigma(s,x(s)) - k_f(s,x(s)) - Ak_q(s,x(s)) - \int_0^s k_g(s,\tau,x(\tau))d\tau\Big]dw(s),
\end{aligned}
\tag{3.2}
$$

and

$$\begin{aligned}(\mathbb{P}_2^\alpha x)(t) = \Gamma_0^t S^*(T-t)R(\alpha,\Gamma_0^T)\Big(&\mathbf{E}b - S(T)[x_0 - h(x) - q(0,x(0))] - q(T,x(T)) \\ &- \mathbf{E}\int_0^T AS(T-s)q(s,x(s))ds - \mathbf{E}\int_0^T S(T-s)f(s,x(s))ds \\ &- \mathbf{E}\int_0^T S(T-s)\Big[\int_0^s g(s,\tau,x(\tau))d\tau\Big]\Big),\end{aligned} \tag{3.3}$$

where $k_b(s)$, $k_f(s,x)$, $k_q(s,x)$ and $k_g(s,\tau,x)$ are defined by (2.3), (2.7), (2.8) and (2.9) respectively. By using (2.6) along with

$$\begin{aligned} z \equiv b - S(t)[x_0 - h(x) - q(0,x(0))] - q(T,x(T)) + \int_0^T AS(T-s)q(s,x(s))ds \\ - \int_0^T S(T-s)f(s,x(s)ds - \int_0^T S(T-s)\int_0^s g(s,\tau,x(\tau))d\tau\Big]ds \\ - \int_0^T S(T-s)\sigma(s,x(s))dw(s),\end{aligned}$$

it is easy to observe that $\mathbb{P}^\alpha x = (\mathbb{P}_1^\alpha + \mathbb{P}_2^\alpha)x$. Define the set

$$Y_r = \{x(\cdot) \in C(I, L_2) : \mathbf{E}\|x(t)\|^2 \le r\},$$

where r is a positive constant. Let us take

$$M = \|B\|, \quad N = T\max\{\|S(t)BB^*S^*(t)\| : 0 \le t < T\}.$$

Theorem 3.1. *Assume that* (H1)-(H2), (AC) *hold. Then the system* (1.2) *is approximately controllable on* $[0,T]$.

Proof. The proof is done by the several steps.

Step 1. For arbitrary $\alpha > 0$ there is a positive constant $r_0 = r_0(\alpha)$ such that $\mathbb{P} : Y_{r_0} \to Y_{r_0}$. From the definition of $\mathbb{P}_1^\alpha$ and $\mathbb{P}_2^\alpha$, for any $x(\cdot) \in Y_{r_0}$, we have

$$\begin{aligned}&\|\mathbb{P}_1^\alpha x\|_* \\ &\le K[\|x_0\| + l_1 + \sqrt{L_6}] + \sqrt{L_6} + l_0KT\sqrt{L_6} + KT\sqrt{L_4} + K\sqrt{TL_4} \\ &\quad + KT\sqrt{TL_5} + \frac{1}{\alpha}NK\Big(\int_0^T \mathbf{E}\|k_b(s)\|^2ds + \int_0^T \mathbf{E}\|A\|\|k_q(s,x(s))\|^2ds \\ &\quad + \int_0^T \mathbf{E}\|k_f(s,x(s))\|^2ds + \int_0^T\Big[\int_0^s \mathbf{E}\|k_g(s,\tau,x(\tau))\|^2d\tau\Big]ds + K^2TL_3\Big)^{1/2} \\ &\le K\|x_0\| + Kl_1 + (K+1+KTl_0)\sqrt{L_6} + K(\sqrt{T}+1)\sqrt{TL_4} + KT\sqrt{TL_5} \\ &\quad + \frac{1}{\alpha}NK\Big(\int_0^T \mathbf{E}\|k_b(s)\|^2ds + K^2T^2l_0^2L_6 + K^2T^3L_5 + K^2T\Big(T+1\Big)L_4\Big)^{1/2},\end{aligned}$$

$$\|\mathbb{P}_2^\alpha x\| \le \frac{1}{\alpha}NK\Big(\|\mathbf{E}b\| + K(\|x_0\|+l_1) + (K+1+KTl_0)\sqrt{L_6} + KT(\sqrt{L_4} + \sqrt{TL_5})\Big),$$

which implies for sufficiently large $r_0 = r_0(\alpha)$

$$\|\mathbb{P}^\alpha x\|_* \le \|\mathbb{P}_1^\alpha x\|_* + \|\mathbb{P}_2^\alpha x\| \le r_0(\alpha).$$

Hence, $\mathbb{P}^\alpha$ maps Y_{r_0} into itself for some r_0.

Step 2. For arbitrary $\alpha > 0$ the operator $\mathbb{P}_2^\alpha$ maps Y_{r_0} into a relatively compact subset of Y_{r_0}. According to the infinite-dimensional version of the Ascoli-Arzela theorem we have to show that

(1) for arbitrary $t \in [0,T]$ the set
$$V(t) = \{(\mathbb{P}_2^\alpha x)(t) : x \in Y_{r_0}\} \subset X$$
is relatively compact.

(2) for arbitrary $\epsilon > 0$ there exists $\delta > 0$ such that
$$\|(\mathbb{P}_2^\alpha x)(t+\tau) - (\mathbb{P}_2^\alpha x)(t)\| < \epsilon,$$
if $\|x\| \le r$, $|\tau| \le \delta$, $t, t+\tau \in [0,T]$.

Notice that the uniform boundedness is proved in step 1.

Let us prove (1). In fact, the case where $t = 0$ is trivial, since $V(0) = \{x_0\}$, so let t, $0 < t \le T$, be fixed and let η be a given real number satisfying $0 < \eta < t$. Define
$$\begin{aligned}
&(\mathbb{P}_2^{\alpha,\eta} x)(t)\\
&= \int_0^{t-\eta} S(t-r)BB^*S^*(T-r)drR(\alpha,\Gamma_0^T)\Big(\mathbf{E}b - S(T)[x_0 - h(x) - q(0,x(0))]\\
&\quad - q(T,x(T)) - \mathbf{E}\int_0^T AS(T-s)q(s,x(s))ds - \mathbf{E}\int_0^T S(T-s)f(s,x(s))ds\\
&\quad - \mathbf{E}\int_0^T S(T-s)\Big[\int_0^s g(s,\tau,x(\tau))d\tau\Big]\Big)ds\\
&= S(\eta)(\mathbb{P}_2^\alpha x)(t-\eta).
\end{aligned}$$

Since $S(\eta)$ is compact and $(\mathbb{P}_2^\alpha x)(t-\eta)$ is bounded on Y_{r_0} the set
$$V_\eta(t) = \{(\mathbb{P}_2^{\alpha,\eta} x)(t) : x(\cdot) \in Y_r\}$$
is relatively compact set in H, that is, we can find a finite set $\{y_i, 1 \le i \le m\}$ in H such that
$$V_\eta(t) \subset \bigcup_{i=1}^m N(y_i, \frac{\epsilon}{2}).$$

On the other hand, there exists $\eta > 0$ such that
$$\begin{aligned}
&\|(\mathbb{P}_2^\alpha x)(t) - (\mathbb{P}_2^{\alpha,\eta} x)(t)\|\\
&= \|\int_{t-\eta}^t S(t-r)BB^*S^*(T-r)drR(\alpha,\Gamma_0^T)\Big(\mathbf{E}b - S(T)[x_0 - h(x) - q(0,x(0))]\\
&\quad - q(T,x(T)) - \mathbf{E}\int_0^T AS(T-s)q(s,x(s))ds - \mathbf{E}\int_0^T S(T-s)f(s,x(s)ds\\
&\quad - \mathbf{E}\int_0^T S(T-s)\Big(\int_0^s g(s,\tau,x(\tau))d\tau\Big)ds\Big)\|\\
&\le \frac{1}{\alpha}K^2M^2\Big(\|\mathbf{E}b\| + K(\|x_0\| + l_1) + (K+1+KTl_0)\sqrt{L_6} + KT(\sqrt{L_4} + \sqrt{TL_5})\Big)\eta\\
&\le \frac{\epsilon}{2}.
\end{aligned}$$

Consequently,

$$V(t) \subset \bigcup_{i=1}^{m} N(y_i, \epsilon).$$

Hence, for each $t \in [0,T]$, $V(t)$ is relatively compact in X.

Next, we prove (2). We have to show that $V = \{\mathbb{P}_2^\alpha x)(\cdot) : x \in Y_{r_0}\}$ is equicontinuous on $[0,T]$. In fact, for $0 < t < t+\tau \le T$ and $0 < \eta \le t$.

$$\begin{aligned}
&\|(\mathbb{P}_2^\alpha x)(t+\tau) - (\mathbb{P}_2^\alpha x)(t)\| \\
&\le \big\|\Gamma_0^{t+\tau} S^*(T-t-\tau) - \Gamma_0^t S^*(T-t)\big\|\big\|R(\alpha, \Gamma_0^T) \\
&\times \Big(\mathbf{E}b - S(T)[x_0 - h(x) - q(0,x(0))] - q(T,x(T)) - \mathbf{E}\int_0^T AS(T-s)q(s,x(s))ds \\
&\quad - \mathbf{E}\int_0^T S(T-s)f(s,x(s)) - \mathbf{E}\int_0^T S(T-s)\Big[\int_0^s g(s,\tau,x(\tau))d\tau\Big]ds\Big)\| \\
&\le \| \int_t^{t+\tau} S(t+\tau-s)BB^*S^*(T-s)ds - [S(\tau)-I]\int_0^t S(t-s)BB^*S^*(T-s)ds\big\| \\
&\quad \times \frac{1}{\alpha}\Big(\|\mathbf{E}b\| + K[\|x_0\| + l_1] + (K+1+KTl_0)\sqrt{L_6} + KT\sqrt{L_4} + KT\sqrt{TL_5}\Big) \\
&\le \frac{1}{\alpha}\Big(\tau + \|S(\tau)-I\|\Big)K^2M^2\Big(\|\mathbf{E}b\| + K[\|x_0 + l_1]\| + (K+1+KTl_0)\sqrt{L_6} \\
&\quad + KT(\sqrt{L_4} + \sqrt{TL_5})\Big).
\end{aligned}$$

The right-hand side of the above inequality does not depend on particular choice of $x(\cdot)$ and approaches zero as $\tau \to 0^+$. The case $0 < t+\tau < t \le T$ can be considered in a similar manner. So, we obtain the equicontinuity of V. Thus, $\mathbb{P}_2^\alpha$ maps Y_{r_0} into an equicontinuous family of deterministic functions which are also bounded. By the Ascoli-Arzela theorem $\mathbb{P}_2^\alpha[y_{r_0}]$ is relatively compact in $C(I, L_2)$.

Step 3. Here we prove $\mathbb{P}_1^\alpha$ is a contraction mapping. In fact

$$\begin{aligned}
&\|\mathbb{P}_1^\alpha x - \mathbb{P}_1^\alpha y\|_* \\
&\le \| \int_0^t S(t-s)\Big[h(x(s)) - h(y(s))\Big]ds\big\|_* + \big\|q(t,x(t)) - q(t,y(t)\big\|_* \\
&\quad + \| \int_0^t AS(t-s)\Big[q(s,x(s)) - q(s,y(s))\Big]ds\big\|_* \\
&\quad + \| \int_0^t S(t-s)\Big[f(s,x(s)) - f(s,y(s))\Big]ds\big\|_* \\
&\quad + \| \int_0^t S(t-s)\Big[\sigma(s,x(s)) - \sigma(s,y(s))\Big]dw(s)\big\|_* \\
&\quad + \| \int_0^t S(t-s)\Big[\int_0^s \Big[g(s,\tau,x(\tau)) - g(s,\tau,y(\tau))\Big]d(\tau)\Big]ds\big\|_* \\
&\quad + NK\big\|\int_0^t R(\alpha,\Gamma_s^T)S(T-s)\Big[\sigma(s,x(s)) - \sigma(s,x(s))\Big]dw(s)\big\|_* \\
&\quad + NK\big\|\int_0^t R(\alpha,\Gamma_s^T)S(T-s)\Big[k_f(s,x(s)) - k_f(s,y(s))\Big]dw(s)\big\|_*
\end{aligned}$$

$$
\begin{aligned}
&+ NK\Big\| \int_0^t R(\alpha,\Gamma_s^T) AS(T-s)\Big[k_q(s,x(s)) - k_q(s,y(s))\Big] dw(s)\Big\|_* \\
&+ NK\Big\| \int_0^t R(\alpha,\Gamma_s^T) S(T-s)\Big[\int_0^s \Big[k_g(s,\tau,x(\tau)) - k_g(s,\tau,y(\tau))\Big] d\tau\Big] dw(s)\Big\|_* \\
&\le Kl + (1+KTl_0)\sqrt{L_3} + K\sqrt{TL_1}(\sqrt{T}+1) + KT\sqrt{TL_2} \\
&+ \frac{1}{\alpha} NK^2\Big[2\sqrt{TL_1} + l_0\sqrt{TL_3} + T\sqrt{L_2}\Big] \|x(s)-y(s)\|_* \,.
\end{aligned}
$$

Here we used the inequality (2.11) and (2.12). So, if

$$
\begin{aligned}
&Kl + (1+KTl_0)\sqrt{L_3} + K\sqrt{TL_1}(\sqrt{T}+1) \\
&+ KT\sqrt{TL_2} + \frac{1}{\alpha} NK^2\Big[2\sqrt{TL_1} + l_0\sqrt{TL_3} + T\sqrt{L_2}\Big] < 1
\end{aligned}
\tag{3.4}
$$

Thus $\mathbb{P}_1^\alpha$ is a contraction mapping.

Step 4. Now we prove $\mathbb{P}_2^\alpha$ is continuous on $C(I,H)$. To apply the Nussbaum fixed-point theorem it remains to show that $\mathbb{P}_2^\alpha$ is continuous on $C(I,L_2)$. Let $\{x^n(\cdot)\} \subset C(I,L_2)$ with $x^n(\cdot) \to x(\cdot) \in C(I,L_2)$. Then the Lebesgue-dominated convergence theorem implies

$$
\begin{aligned}
&\|\mathbb{P}_2^\alpha x^n(t) - \mathbb{P}_2^\alpha x(t)\| \\
&\le \frac{1}{\alpha} NK\Big[\big\|S(T)(h(x^n)-h(x))\big\| + \big\|q(t,x^n(t)) - q(t,x(t))\big\| \\
&+ \mathbf{E}\int_0^T \big\|AS(T-s)\Big(q(s,x^n(s)) - q(s,x(s))\Big)\big\| ds \\
&+ \mathbf{E}\int_0^T \big\|S(T-s)\Big(f(s,x^n(s)) - f(s,x(s))\Big)\big\| ds \\
&+ \mathbf{E}\int_0^T \big\|S(T-s)\int_0^s \Big(g(s,\tau,x^n(\tau)) - g(s,\tau,x(\tau))\Big) d\tau\big\| ds\Big] \\
&\le \frac{1}{\alpha} NK(Kl+\sqrt{L_3}) + \frac{1}{\alpha} NK^2\sqrt{T}\Big[l_0^2\int_0^T \mathbf{E}\|q(s,x^n(s)) - q(s,x(s))\|^2 ds \\
&+ \int_0^T \mathbf{E}\|f(s,x^n(s)) - f(s,x(s))\|^2 ds \\
&+ \int_0^T \Big(\int_0^s \mathbf{E}\big\|g(s,\tau,x^n(\tau)) - g(s,\tau,x(\tau))\big\|^2 d\tau\Big) ds\Big]^{1/2} \\
&\le \frac{1}{\alpha} NK(Kl+\sqrt{L_3}) + \frac{1}{\alpha} NK^2\sqrt{T}\Big(l_0^2 L_3 + L_1 + TL_2\Big) \\
&\times \Big[\int_0^T \mathbf{E}\Big(\|x^n(s)-x(s)\|^2 ds + \|x^n(s)-x(s)\|^2 ds + \|x^n(s)-x(s)\|^2 ds\Big)\Big]^{1/2} \\
&\le \frac{1}{\alpha} NK(Kl+\sqrt{L_3}) + \frac{3}{\alpha} NK^2 T\Big(l_0^2 L_3 + L_1 + TL_2\Big)\|x^n - x\| \to 0
\end{aligned}
$$

as $n \to \infty$. Thus $\mathbb{P}_2^\alpha$ is continuous on $C(I,L_2)$.

Step 5. From the Nussbaum fixed point theorem $\mathbb{P}_\alpha$ has a fixed point provided that the inequality (3.4) is satisfied. It is easily seen that this fixed point is a solution of the system (1.2). The extra condition (3.4) can easily be removed by considering (1.2) on intervals $[0,\tilde{T}], [\tilde{T}, 2\tilde{T}], \dots,$ with $\tilde{T}$ satisfying (3.4). Let $x_\alpha^*(\cdot)$ be a fixed point of the operator $\mathbb{P}^\alpha$ in Y_{r_0}. Any fixed point of $\mathbb{P}^\alpha$ is a mild solution of (1.1) on $[0,T]$ under the control $u_\alpha(t)$ defined by (3.1), where x is replaced by x^*_α and, by Lemma 2.2 satisfies

$$
\begin{aligned}
(\mathbb{P}^\alpha x^*_\alpha)(T) &= x^*_\alpha(T)\\
&= b + \alpha R(\alpha, \Pi_0^T)\Big(S(T)[x_0 - h(x) - q(0,x(0))] + q(T,x(T))\\
&\quad + \int_0^T AS(T-s)q(s,x^*_\alpha(s))ds + \int_0^T S(T-s)f(s,x^*_\alpha(s))ds\\
&\quad + \int_0^T S(T-s)\Big[\int_0^s g(s,\tau,x^*_\alpha(\tau))d\tau\Big]ds\\
&\quad + \int_0^T S(T-s)\sigma(s,x^*_\alpha(s))dw(s) - b\Big).
\end{aligned}
\tag{3.5}
$$

Set

$$
\begin{aligned}
z_\alpha &= S(T)[x_0 - h(x) - q(0,x(0))] + q(T,x(T))\\
&\quad + \int_0^T AS(T-s)q(s,x^*_\alpha(s))ds + \int_0^T S(T-s)f(s,x^*_\alpha(s))ds\\
&\quad + \int_0^T S(T-s)\Big[\int_0^s g(s,\tau,x^*_\alpha(\tau))d\tau\Big]ds + \int_0^T S(T-s)\sigma(s,x^*_\alpha(s))dw(s) - b\Big).
\end{aligned}
$$

By (H2), and then there is a subsequence, still denoted by

$$
\{f(s,x^*_\alpha(s)), q(s,x^*_\alpha(s)), \int_0^s g(s,\tau,x^*_\alpha(\tau))d\tau, \sigma(s,x^*_\alpha)\},
$$

weakly converging to, say, $(f(s,\omega), (q(s,\omega), \sigma(s,\omega))$ in $H \times L_0^2$ and $g(s,\tau,\omega)$ in $H \times H \times L_0^2$. The compactness of $S(t)$, $t > 0$ implies

$$
\begin{gathered}
S(T-s)f(s,x^*_\alpha(s)) \to S(T-s)f(s,\omega),\\
S(T-s)q(s,x^*_\alpha(s)) \to q(T-s)q(s,\omega),\\
S(T-s)g(s,\tau,x^*_\alpha(\tau)) \to S(T-s)g(s,\tau,\omega),\\
S(T-s)\sigma(s,x^*_\alpha(s)) \to S(T-s)\sigma(s,\omega) \quad \text{a.e. in } I \times \Omega.
\end{gathered}
$$

On the other hand

$$
\begin{gathered}
\|S(T-s)f(s,x^*_\alpha(s))\|^2 + \|S(T-s)\sigma(s,x^*_\alpha(s))\|^2 \le K^2 L_4,\\
\|S(T-s)g(s,\tau,x^*_\alpha(\tau))\|^2 \le K^2 L_5,\\
\|S(T-s)q(s,x^*_\alpha(s))\|^2 \le K^2 L_6 \quad \text{a.e. in } I \times \Omega.
\end{gathered}
$$

Thus by the Lebesgue-dominated convergence theorem

$$
\mathbf{E}\|z_\alpha - z\|^2 \to 0 \quad \text{as } \alpha \to 0^+,
$$

where

$$
\begin{aligned}
z &= S(T)[x_0 - h(x) - q(0)] + q(T) + \int_0^T AS(T-s)q(s)ds + \int_0^T S(T-s)f(s)ds\\
&\quad + \int_0^T S(T-s)\Big[\int_0^s g(s,\tau)d\tau\Big]ds + \int_0^T S(T-s)\sigma(s)dw(s) - b.
\end{aligned}
$$

Then having in mind that $\mathbf{E}\|\alpha R(\alpha,\Pi_0^T)\|^2 \le 1$ and $\alpha R(\alpha,\Pi_0^T) \to 0$ strongly by the assumption (AC), from (3.5) we obtain

$$\begin{aligned}\sqrt{\mathbf{E}\|x_\alpha^*(T)-h\|^2} &\le \sqrt{\mathbf{E}\|\alpha R(\alpha,\Pi_0^T)(z_\alpha - z)\|^2} + \sqrt{\mathbf{E}\|\alpha R(\alpha,\Pi_0^T)(z)\|^2} \\ &\le \sqrt{\mathbf{E}\|z_\alpha - z\|^2} + \sqrt{\mathbf{E}\|\alpha R(\alpha,\Pi_0^T)(z)\|^2} \to 0\end{aligned}$$

as $\alpha \to 0^+$. This gives the approximate controllability of (1.2). Hence the proof is complete. □

Corollary 3.2. *Assume that* (H2) *holds. If the semigroup $S(t)$ is analytic and the deterministic linear system corresponding to* (1.1) *is approximately controllable on* $[0,T]$ *then the stochastic system* (1.1) *is approximately controllable on* $[0,T]$.

Proof. It is known that (see [12, Theorem 4.3]) when the semigroup $S(t)$ is analytic the linear stochastic system (1.3) is approximately controllable on $[0,T]$ if and only if the corresponding deterministic linear system is approximately controllable on $[0,T]$. Then by Theorem 3.1, the system (1.1) is approximately controllable on $[0,T]$. □

4. Applications

Consider the following stochastic classical heat equation for material with memory

$$\begin{gathered} d[z(t,\theta) - m(t,z(t,\theta))] \\ = \Big[z_{\theta\theta}(t,\theta) + Bu(t,\theta) + p(t,z(t,\theta)) + \int_0^t q(t,s,z(s,\theta)ds\Big]dt + k(t,z(t,\theta)dw(t), \\ \text{for } (t,\theta)\in I\times[0,\pi]=\Omega, \\ z(t,\theta)=0 \quad \text{for } I\times\partial\Omega, \\ z(0,\theta) + \sum_{i=1}^p c_i z(t_i,\theta) = z_0(\theta) \quad \text{for } \theta\in\Omega,\ 0<t_i\le T, \end{gathered} \tag{4.1}$$

where Ω is an open bounded subset of R^n with smooth boundary $\partial\Omega$, and B is a bounded linear operator from a Hilbert space U into H. We assume that $p: I\times H \to H$, $m: I\times H\to H$, $k: I\times H \to L_2^0$, $q: I\times I\times H \to H$, and $c_i \in C(I,H)$ are all continuous and uniformly bounded, $u(t)$ is a feedback control and w is an Q-Wiener process. Let $H = L_2[0,\pi]$, and let $A: H\to H$ be an operator defined by

$$Ax = x_{\theta\theta}$$

with domain

$$D(A) = \{x\in H\ | x, x_\theta \text{ are absolutely continuous, } x_{\theta\theta}\in H, x(0)=x(\pi)=0\}.$$

Let $f: I\times H\to H$ be defined by

$$f(t,x)(\theta) = p(t,x(\theta)),\quad (t,x)\in I\times H,\quad \theta\in[0,\pi]$$

Let $q: I\times H\to H$ be defined by

$$q(t,x)(\theta) = m(t,x(\theta)),$$

Let $g : I \times I \times H \to H$ be defined by

$$g(t,s,x)(\theta) = q(t,s,x(\theta)),$$

Let $h : C(I,H) \to H$ be defined by

$$h(x)(\theta) = \sum_{i=1}^{p} c_i x(t_i)(\theta),$$

Let $\sigma : I \times H \to L_2^0$ be defined by

$$\sigma(t,x)(\theta) = k(t,x(\theta)).$$

With this choice of A, B, f, q, g, h and σ, (1.2) is the abstract formulation of (4.1), be such that the condition in (H2) is satisfied. Then

$$Ax = \sum_{n=1}^{\infty} \Big(-n^2 \Big) (x, e_n) e_n(\theta), \quad x \in D(A),$$

where $e_n(\theta) = \sqrt{\frac{2}{\pi}} \sin n\theta$, $0 \le \theta \le \pi$, $n = 1, 2, \dots$ It is known that A generates an analytic semigroup $S(t)$, $t > 0$ in H and is given by

$$S(t)x = \sum_{n=1}^{\infty} e^{-n^2 t} (x, e_n) e_n(\theta), \quad x \in H.$$

Now define an infinite-dimensional space

$$U = \Big\{ u = \sum_{n=2}^{\infty} u_n e_n(\theta) : \sum_{n=2}^{\infty} u_n^2 < \infty \Big\}$$

with a norm defined by $\|u\| = (\sum_{n=2}^{\infty} u_n^2)^{\frac{1}{2}}$ and a linear continuous mapping B from U to H as follows:

$$Bu = 2u_2 e_1(\theta) + \sum_{n=2}^{\infty} u_n e_n(\theta).$$

It is obvious that for $u(t,\theta,\omega) = \sum_{n=2}^{\infty} u_n(t,\omega) e_n(\theta) \in L_2^{\mathcal{F}}(I,U)$

$$Bu(t) = 2u_2(t) e_1(\theta) + \sum_{n=2}^{\infty} u_n(t) e_n(\theta) \in L_2^{\mathcal{F}}(I,H).$$

Moreover,

$$B^* v = (2v_1 + v_2) e_2(\theta) + \sum_{n=3}^{\infty} v_n e_n(\theta),$$

$$B^* S^*(t) z = \Big(2z_1 e^{-t} + z_2 e^{-4t} \Big) e_2(\theta) + \sum_{n=3}^{\infty} z_n e^{-n^2 t} e_n(\theta)$$

for $v = \sum_{n=1}^{\infty} v_n e_n(\theta)$ and $z = \sum_{n=1}^{\infty} z_n e_n(\theta)$. Let

$$\|B^* S^*(t) z\| = 0, \quad t \in [0,T],$$

it follows that

$$\Big\| 2z_1 e^{-t} + z_2 e^{-4t} \Big\|^2 + \sum_{n=3}^{\infty} \Big\| z_n e^{-n^2 t} \Big\|^2 = 0, \quad t \in [0,T]$$

$$\Longrightarrow z_n = 0, \quad n = 1, 2, \dots \Longrightarrow z = 0.$$

Thus, by [7, Theorem 4.1.7], the deterministic linear system corresponding to (4.1) is approximately controllable on $[0, T]$ and by Corollary 3.1, the system (4.1) is approximately controllable on $[0, T]$ provided that f, q, g, h and σ satisfies the assumptions (H2).

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