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SHARP ASYMPTOTIC ESTIMATES FOR VORTICITY SOLUTIONS OF THE 2D NAVIER-STOKES EQUATION

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ABSTRACT. The asymptotic dynamics of high-order temporal-spatial derivatives of the two-dimensional vorticity and velocity of an incompressible, viscous fluid flow in $\mathbb{R}^2$ are studied, which is equivalent to the 2D Navier-Stokes equation. It is known that for any integrable initial vorticity, the 2D vorticity solution converges to the Oseen vortex. In this paper, sharp exterior decay estimates of the temporal-spatial derivatives of the vorticity solution are established. These estimates are then used and combined with similarity and L^p compactness to show the asymptotical attraction rates of temporal-spatial derivatives of generic 2D vorticity and velocity solutions by the Oseen vortices and velocity solutions respectively. The asymptotic estimates and the asymptotic attraction rates of all the derivatives obtained in this paper are independent of low or high Reynolds numbers.

1. INTRODUCTION

Vorticity is a vector field defined by $\omega = \nabla \times v$ for a flow with velocity v. For any two-dimensional flow, if we orient the z-axis of a Cartesian coordinate system normal to the velocity field, then the one-dimensional vorticity is expressed by

$$\omega = \omega_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}.$$

For an incompressible, viscous fluid motion in the domain of $\mathbb{R}^2$, the velocity field of the fluid is governed by the 2D Navier-Stokes equation:

$$\begin{gathered} \frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = 0, \\ \nabla \cdot u = 0, \end{gathered} \tag{1.1}$$

for $t > 0$, $x \in \mathbb{R}^2$, where $u = u(t, x)$ is a two-dimensional vector field of the fluid velocity, $p(t, x)$ is the pressure, and we assume that the external force on the fluid is zero. The kinematic viscosity coefficient ν is a positive constant. The initial value condition for $u(t, x)$ is

$$u(0, x) = u_0(x), \quad \text{for } x \in \mathbb{R}^2.$$

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Taking the curl of the equation (1.1), we can consider the equation for the vorticity $\omega(t,x)$ as well,

$$\frac{\partial \omega}{\partial t} = \nu \Delta \omega - (u \cdot \nabla)\omega, \quad t > 0, \quad x \in \mathbb{R}^2. \tag{1.2}$$

By the Biot-Savart law, cf. [15, 28, 29], the velocity $u(t,x)$ can be reconstructed in terms of the vorticity $\omega(t,x)$,

$$u(t,x) = \int_{\mathbb{R}^2} G(x-y)\omega(t,y)\,dy, \quad t > 0,\ x \in \mathbb{R}^2, \tag{1.3}$$

where

$$G(x) = G(x_1,x_2) = \frac{x^\perp}{2\pi|x|^2} = \frac{(-x_2,x_1)}{2\pi|(x_1,x_2)|^2}. \tag{1.4}$$

The system of equations (1.2) and (1.3) is equivalent to the Navier-Stokes equation (1.1). We shall call (1.2)–(1.3) the two-dimensional *vorticity equation*, which is the vorticity formulation of the Navier-Stokes equation.

In [31], it was found that there is a special type of solutions to (1.2)–(1.3),

$$\omega^*(t,x) = \frac{\kappa}{4\pi\nu t} \exp\big(-\frac{|x|^2}{4\nu t}\big), \tag{1.5}$$

where κ is a constant called the strength of the vortex. These special solutions are called the *Oseen vortices*, whose initial data are $\omega^*(0,x) = \kappa\delta(x)$ in the distributional sense, where $\delta(x)$ is the Dirac function.

It has been proved in [18] that for any initial vorticity $\omega_0 \in L^1(\mathbb{R}^2)$ with small Renolds numbers $\int_{\mathbb{R}^2} |\omega_0(x)|dx/\nu$, the vorticity equation (1.2)–(1.3) has a unique solution

$$\omega \in C([0,\infty), L^1(\mathbb{R}^2)) \cap C((0,\infty), L^\infty(\mathbb{R}^2)),$$
$$u \in C([0,\infty), L^1(\mathbb{R}^2) \times L^1(\mathbb{R}^2)) \cap C((0,\infty), L^\infty(\mathbb{R}^2) \times L^\infty(\mathbb{R}^2)),$$

with the asymptotic convergence properties at specific decaying rates:

$$\lim_{t\to\infty} t^{1-\frac{1}{p}} \|\omega(t,\cdot) - \omega^*(t,\cdot)\|_p = 0, \quad \text{for } 1 \le p \le \infty, \tag{1.6}$$

$$\lim_{t\to\infty} t^{\frac{1}{2}-\frac{1}{q}} \|u(t,\cdot) - u^*(t,\cdot)\|_q = 0, \quad \text{for } 2 < q \le \infty, \tag{1.7}$$

where $\omega^*(t,x)$ is the Oseen vortex (1.5) with the coefficient of strength

$$\kappa = \kappa(\omega_0) = \int_{\mathbb{R}^2} \omega_0(x)\,dx, \quad \omega_0 \in L^1(\mathbb{R}^2), \tag{1.8}$$

and the corresponding Oseen velocity $u^*(t,x)$ is given by

$$u^*(t,x) = \int_{\mathbb{R}^2} G(x-y)\omega^*(t,y)\,dy. \tag{1.9}$$

This asymptotical convergence result (1.6) and (1.7) was then extended in [11, 16].

As indicated in [4, 5, 7, 11, 14, 18, 22, 23, 30], it is relatively easier to understand the asymptotics of the velocity solutions of the Navier-Stokes equation (1.1) and the associated Euler equation by studying the long-time dynamics of the vorticity solutions of the vorticity equation, especially for two-dimensional flows due to the scalar vorticity. Once the vorticity field on $\mathbb{R}^2$ is known or approximately known, one can always reconstruct the velocity field precisely or approximately by the singular integral with the Riesz potential specified in the Biot-Savart law. Therefore, the study of the regularity and asymptotical dynamics of the 2D and 3D vorticity equations is important in the mathematical theory of fluid mechanics. Some of the established results, related topics, and numerical simulations in recent years can be found in [2, 3, 4, 5, 7, 11, 13, 14, 16, 18, 22, 23, 24, 25, 27, 32, 33, 36].

In this work we shall prove certain sharp estimates on the temporal-spatial decaying dynamics of all the derivatives of the vorticity solution $\omega(t,x)$ and velocity solution $u(t,x)$ to the vorticity equation (1.2)–(1.3) and then indicate the applications to the global attraction regularity of the Oseen vortex and velocity solutions. We emphasize that the asymptotic estimates and the asymptotic attraction rates of all the temporal-spatial derivatives obtained in this paper are independent of low or high Reynolds numbers.

This application of the estimates shows that the set of all the Oseen vortices,

$$\Psi = \{\omega^*(t,x) : \kappa \in \mathbb{R}\},$$

is a global attractor for the 2D vorticity semiflow in the full regularity sense with respect to L^p topology. Moreover, it also shows that the attraction rates of Ψ for the temporal-spatial derivatives are getting faster and faster in the specific and uniform order of $O(t^{-(\alpha+|\beta|/2)})$, which depends on the time derivative order α and the spatial derivative order β. Therefore, it reveals that regularity asymptotics of the 2D vorticity solutions on $\mathbb{R}^2$ is determined by the unique Oseen vortex pattern completely.

Below we shall use the notation $\partial_t^\alpha = (\partial/\partial t)^\alpha$ for $\alpha \in Z^+$, $\partial_x^\beta = \partial^{|\beta|}/(\partial x_1^{\beta_1}\partial x_2^{\beta_2})$ for $\beta = (\beta_1,\beta_2) \in Z^+ \times Z^+$, where Z^+ stands for the set of all nonnegative integers. As notational remarks, we have

$$|\beta| = \beta_1 + \beta_2, \quad \beta! = \beta_1!\beta_2!, \quad \binom{\beta}{\gamma} = \frac{\beta!}{\gamma!(\beta-\gamma)!} \le 2^{|\beta|} \quad \text{for } \gamma \le \beta.$$

By convention, if $p = \infty$, then $1/p = 0$. We use $|\cdot|$ to denote either an absolute value or a vector norm. The norm of $L^p(\mathbb{R}^2)$ will be denoted by $\|\cdot\|_p, 1 \le p \le \infty$. For simplicity, we shall always use $\|\cdot\|$ for the norm of $L^1(\mathbb{R}^2)$ instead of using $\|\cdot\|_1$.

For any function g defined on $\mathbb{R}^2$, we shall use g^r and g_r to denote the functions

$$g^r(x) = \vartheta_r(x)g(x) \quad \text{and} \quad g_r(x) = (1-\vartheta_r(x))g(x),$$

respectively, where ϑ_r is the characteristic function of the set $\mathbb{R}^2 \cap \{|x| > r\}$.

The following convolution property [8, Théorème IV.15] will be used: for any $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n), 1 \le p \le \infty$, one has $f * g \in L^p(\mathbb{R}^n)$ and

$$\|f * g\|_{L^p(\mathbb{R}^n)} \le \|f\|_{L^1(\mathbb{R}^n)}\|g\|_{L^p(\mathbb{R}^n)}. \tag{1.10}$$

One of the basic L^p estimates for convolutions is the Hardy-Littlewood-Sobolev inequality [21, Theorem 4.5.3]: Let integer $n \ge 1$ and

$$\Phi(x) = |x|^{-n/a}, \quad x \in \mathbb{R}^n.$$

If $1 < a < \infty, 1 < p < q < \infty$, and $1/p + 1/a = 1 + 1/q$, then there exists a uniform constant $S_{n,p,q} > 0$, such that

$$\|\Phi * g\|_q \le S_{n,p,q}\|g\|_p, \quad \text{for } g \in L^p(\mathbb{R}^n). \tag{1.11}$$

In Section 2, we prove the exterior decay estimates of the L^p norms for vorticity and velocity. In Section 3, based on the mild solution formula and semigroup expressions of the derivatives of vorticity solutions, the exterior decay rates are established for the linear terms and the nonlinear integrands by decomposition methods. Then in Section 4, these nested estimates are utilized to prove the global decay rates of temporal-spatial derivatives of vorticity solutions in Theorem 4.1 and Theorem 4.2 via the bootstrap approach. Finally in Section 5, an application result on the regularity of asymptotical attraction of the Oseen vortex and velocity solutions is shown.

2. Exterior Decay Estimates of Vorticity and Velocity

By the optimal smoothing inequalities [9, Theorems 1 and 2] and the ultracontractivity (or called L^1 to L^∞ regularity) of the parabolic semigroups shown in [20, 12, 34] for unbounded domains, a solution $(\omega(t,x), u(t,x))$ of the 2D vorticity equations (1.2)–(1.3) with integrable initial data $\omega_0(x)$ satisfies the following estimates:

$$\|\omega(t,\cdot)\|_\infty \le \frac{1}{4\pi\nu t}\|\omega_0\|, \quad \|u(t,\cdot)\|_\infty \le \frac{1}{\sqrt{2\pi^2\nu t}}\|\omega_0\|, \quad t>0. \tag{2.1}$$

Based on these estimates, it has been shown in [9, Theorem 3] that for the 2D Navier-Stokes vorticity, the following pointwise decay holds,

$$|\omega(t,x)| \le \int_{\mathbb{R}^2} H(t,x,y)|\omega_0(y)|\,dy, \quad t>0,\ x\in\mathbb{R}^2, \tag{2.2}$$

where, for any given $0<b<1$,

$$H(t,x,y) \le \frac{C_b}{t}\exp\big(-\frac{b|x-y|^2}{4\nu t}\big), \tag{2.3}$$

with the constant C_b given by

$$C_b = \frac{1}{4\pi\nu}\exp\Big(\frac{b\|\omega_0\|^2}{2\pi^2\nu(1-b)}\Big).$$

First we prove an estimate of the L^p norm of vorticity in the exterior of a bounded ball.

Lemma 2.1. *Let $\omega(t,x)$ be a vorticity solution of (1.2)–(1.3) with initial data $\omega_0\in L^1(\mathbb{R}^2)$. Then for any $p\in[1,\infty]$ and any given $R>1$, the following estimate is satisfied by ω,*

$$\|\omega^R(t,\cdot)\|_p \le L_0(p,\|\omega_0\|)\Big(\frac{1}{R}t^{-(\frac12-\frac1p)}\|\omega_0\| + t^{-(1-\frac1p)}\|\omega_0^{R/2}\|\Big), \tag{2.4}$$

where $L_0(p,\|\omega_0\|)$ is a constant depending only on p and $\|\omega_0\|$.

Proof. By (2.2)–(2.3) where $b\in(0,1)$ is arbitrarily fixed, for any given $R>1$ and for $|x|>R$, we have

$$\begin{aligned}|\omega(t,x)| &\le \int_{\mathbb{R}^2\cap\{|y|\le R/2\}} \frac{C_b}{t}\exp\Big(-\frac{b|x-y|^2}{4\nu t}\Big)|\omega_0(y)|\,dy \\ &\quad + \int_{\mathbb{R}^2\cap\{|y|>R/2\}} \frac{C_b}{t}\exp\Big(-\frac{b|x-y|^2}{4\nu t}\Big)|\omega_0(y)|\,dy\end{aligned} \tag{2.5}$$

and we shall denote the two integrals in (2.5) by $J_1(t,x)$ and $J_2(t,x), |x|>R$, respectively. Note that for the function

$$\varphi(t) = \frac{1}{t^{1/2}}\exp\big(-\frac{\zeta^2}{t}\big),$$

where ζ is a constant, one can check that

$$\max_{t>0} \varphi(t) = \varphi(2\zeta^2) = \frac{1}{\sqrt{2}\zeta} e^{-1/2}. \tag{2.6}$$

By (2.6), the first integral in (2.5) satisfies the estimate

$$\begin{aligned} J_1(t,x) &\le C_b \int_{\{|y|\le R/2\}} \frac{1}{t} \exp\Big(-\frac{b(|x|-|y|)^2}{4\nu t}\Big)|\omega_0(y)|\,dy \\ &\le C_b \int_{\{|y|\le R/2\}} \frac{1}{t^{1/2}} \Big[\frac{1}{t^{1/2}} \exp\Big(-\frac{b(|x|/2)^2}{4\nu t}\Big)\Big]|\omega_0(y)|\,dy \\ &= C_b \int_{\{|y|\le R/2\}} \frac{1}{t^{1/2}} \exp\Big(-\frac{b|x|^2}{32\nu t}\Big)\Big[\frac{1}{t^{1/2}} \exp\Big(-\frac{b|x|^2}{32\nu t}\Big)\Big]|\omega_0(y)|\,dy \\ &\le C_1(b)\frac{1}{t^{1/2}|x|} \exp\Big(-\frac{b|x|^2}{32\nu t}\Big)\|\omega_0\|, \end{aligned}$$

due to that $|x| > R > R/2 \ge |y|$, where

$$C_1(b) = \frac{4\sqrt{\nu}C_b e^{-1/2}}{\sqrt{b}}.$$

Hence it follows that, for $p \ge 1$,

$$\begin{aligned} \|J_1(t,\cdot)\|_p &\le \frac{C_1(b)}{t^{1/2}}\Big(\int_{\{|x|>R\}} \frac{1}{|x|^p} \exp\Big(-\frac{pb|x|^2}{32\nu t}\Big)\,dx\Big)^{1/p}\|\omega_0\| \\ &\le \frac{C_1(b)}{t^{1/2}R}\Big(\int_{\{|x|>R\}} \exp\Big(-\frac{pb|x|^2}{32\nu t}\Big)\,dx\Big)^{1/p}\|\omega_0\| \\ &\le \frac{C_2(b,p)}{R}\, t^{-(\frac12-\frac1p)}\|\omega_0\|, \end{aligned} \tag{2.7}$$

where

$$C_2(b,p) = C_1(b)\Big(\int_{\mathbb{R}^2} \exp\big(-\frac{pb|\xi|^2}{32\nu}\big)\,d\xi\Big)^{1/p}.$$

According to the convolution property (1.10), the second integral $J_2(t,x), |x| > R$, in (2.5) is estimated as follows,

$$\begin{aligned} \|J_2(t,\cdot)\|_p &\le \frac{C_b}{t}\Big(\int_{\{|y|>R/2\}} \exp\big(-\frac{pb|x-y|^2}{4\nu t}\big)\,dy\Big)^{1/p}\|\omega_0^{R/2}\| \\ &\le \frac{C_b}{t}\Big(\int_{\mathbb{R}^2} \exp\big(-\frac{pb|y|^2}{4\nu t}\big)\,dy\Big)^{1/p}\|\omega_0^{R/2}\| \\ &\le C_3(b,p)t^{-(1-\frac1p)}\|\omega_0^{R/2}\|, \end{aligned} \tag{2.8}$$

where

$$C_3(b,p) = C_b\Big(\int_{\mathbb{R}^2} \exp\big(-\frac{pb|\xi|^2}{4\nu}\big)\,d\xi\Big)^{1/p}.$$

Finally, let $L_0(p, \|\omega_0\|) = \max\{C_2(b,p), C_3(b,p)\}$. Then the estimate (2.4) follows from (2.7) and (2.8). The lemma is proved. $\square$

Next we shall estimate the exterior decay of the velocity solution $u(t,x)$ in (1.3). The key to this estimation is the Riesz potential corresponding to the kernel $G(x-y)$

in (1.4). For any given $\theta \in (0,n)$, define a Riesz potential Γ_θ to be

$$\Gamma_\theta(f)(x) = \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\theta}} f(y)\, dy, \quad f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n). \tag{2.9}$$

Lemma 2.2. *Let $1 < p < q < \infty$ and $\theta = n(\frac{1}{p} - \frac{1}{q})$. Then there exist positive constants $M_1(n,\theta,q)$ and $M_2(n,\theta,q)$ such that for any given $R > 1$, it holds that*

$$\|\Gamma_\theta(f)^R(\cdot)\|_q \le \frac{M_1}{R^{n-\theta-\frac{n}{q}}}\|f\| + M_2\|f^{R/2}\|_p, \quad f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n). \tag{2.10}$$

Proof. By the similar decomposition as in the proof of Lemma 2.1, we have

$$|\Gamma_\theta(f)(x)| \le \int_{\{|y| \le R/2\}} \frac{1}{|x-y|^{n-\theta}}|f(y)|\, dy + \int_{\{|y|>R/2\}} \frac{1}{|x-y|^{n-\theta}}|f(y)|\, dy,$$

where the two integrals are denoted by $I_1(x)$ and $I_2(x)$ respectively. First we can get

$$I_1(x) \le \frac{2^{n-\theta}}{|x|^{n-\theta}}\|f\|, \quad \text{for } |x| > R. \tag{2.11}$$

Since $\theta = n(\frac{1}{p} - \frac{1}{q})$ implies $n - \theta - \frac{n}{q} = n(1 - \frac{1}{p}) > 0$, it follows that

$$\|I_1^R(\cdot)\|_q \le 2^{n-\theta}\|f\| \Big(\int_{\{|x|>R\}} \frac{1}{|x|^{q(n-\theta)}}\, dx\Big)^{1/q} = \frac{M_1(n,\theta,q)}{R^{n-\theta-\frac{n}{q}}}\|f\|, \tag{2.12}$$

where

$$M_1(n,\theta,q) = 2^{n-\theta}\Big(\frac{\Omega_n}{q(n-\theta)-n}\Big)^{1/q},$$

in which Ω_n is the surface area of the unit sphere in $\mathbb{R}^n$.

By the Hardy-Littlewood-Sobolev inequality (1.11), we readily have

$$\|I_2^R(\cdot)\|_q \le M_2(n,\theta,q)\|f^{R/2}\|_p, \tag{2.13}$$

with $M_2(n,\theta,q) = S_{n,p,q}$ in (1.11). Combining (2.12) and (2.13), we obtain (2.10). □

Lemma 2.3. *For any given $\theta \in (0,n)$, there exist positive constants $M_{1,\infty}(n,\theta)$ and $M_{2,\infty}(n,\theta)$ such that for any given $R > 1$, it holds that*

$$\|\Gamma_\theta(f)^R(\cdot)\|_\infty \le \frac{M_{1,\infty}}{R^{n-\theta}}\|f\| + M_{2,\infty}\|f^{R/2}\|^{\frac{\theta}{n}}\|f^{R/2}\|_\infty^{1-\frac{\theta}{n}}, \quad f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n). \tag{2.14}$$

Proof. Using the same notation as in the proof of Lemma 2.2, by (2.11) we can get

$$\|I_1^R(\cdot)\|_\infty \le \frac{M_{1,\infty}}{R^{n-\theta}}\|f\|,$$

with $M_{1,\infty} = 2^{n-\theta}$. For $I_2(x), |x| > R$, we get

$$\begin{aligned} I_2(x) &= \int_{\{|y|>R/2\}} \frac{1}{|x-y|^{n-\theta}}|f(y)|\, dy \\ &= \int_{\{|y|>R/2\}\cap\{|x-y|>W\}} \frac{1}{|x-y|^{n-\theta}}|f(y)|\, dy \\ &\quad + \int_{\{|y|>R/2\}\cap\{|x-y|\le W\}} \frac{1}{|x-y|^{n-\theta}}|f(y)|\, dy \end{aligned}$$

$$\le \frac{1}{W^{n-\theta}}\|f^{R/2}\| + \int_{\{|z|\le W\}} \frac{1}{|z|^{n-\theta}}\,dz\|f^{R/2}\|_\infty$$
$$= \frac{1}{W^{n-\theta}}\|f^{R/2}\| + \frac{\Omega_n}{\theta}W^\theta\|f^{R/2}\|_\infty.$$

In the above inequality we choose constant

$$W = \Big(\frac{\|f^{R/2}\|}{\|f^{R/2}\|_\infty}\Big)^{1/n}$$

to obtain

$$\|I_2^R(\cdot)\|_\infty \le M_{2,\infty}\|f^{R/2}\|^{\frac{\theta}{n}}\|f^{R/2}\|_\infty^{1-\frac{\theta}{n}},$$

where $M_{2,\infty} = 1 + \Omega_n/\theta$. Thus the estimate (2.14) is proved. □

By the Biot-Savart law (1.3), Lemma 2.2 and Lemma 2.3 directly lead to the exterior decay estimates of the velocity field $u(t,x)$, as stated in the following lemma.

Lemma 2.4. *Let $1 < p < q < \infty$ satisfy $\frac{1}{2} = \frac{1}{p} - \frac{1}{q}$. Let $\omega(t,x)$ and $u(t,x)$ be the vorticity and velocity solution of the vorticity equation (1.2)–(1.3) with $\omega_0 \in L^1(\mathbb{R}^2)$. Then there exist positive constants $L_1(p), L_2(p), L_{1,\infty}$, and $L_{2,\infty}$ such that for any given $R > 1$, the following estimates hold,*

$$\|u^R(t,\cdot)\|_q \le \frac{L_1(p)}{R^{2(\frac{1}{2}-\frac{1}{q})}}\|\omega(t,\cdot)\| + L_2(p)\|\omega^{R/2}(t,\cdot)\|_p, \tag{2.15}$$

and

$$\|u^R(t,\cdot)\|_\infty \le \frac{L_{1,\infty}}{R}\|\omega(t,\cdot)\| + L_{2,\infty}\|\omega^{R/2}(t,\cdot)\|^{1/2}\|\omega^{R/2}(t,\cdot)\|_\infty^{1/2}. \tag{2.16}$$

Proof. Simply let $n = 2, \theta = 1$ in Lemma 2.2 and Lemma 2.3 and add the coefficient $1/(2\pi)$ to the Riesz potential (2.9). Then the conclusion follows. □

3. Exterior Decay Estimates of Spatial Derivatives

The vorticity equation (1.2) is a semilinear evolutionary equation. By the semigroup theory, the vorticity solution $\omega(t,x)$ satisfies the variation-of-constant formula for all mild solutions [34, Chapter 3], which is the following nonlinear integral equation

$$\omega(t,\cdot) = e^{At}\omega_0 - \int_0^t e^{A(t-s)}(u(s,\cdot)\cdot\nabla)\omega(s,\cdot)ds, \quad t \ge 0, \tag{3.1}$$

where $e^{At}, t \ge 0$, is the C_0-semigroup on the Banach space $Y = C_B(\mathbb{R}^2)$, generated by the closed linear operator $A = \nu\Delta : \mathcal{D}(A) = C_B^2(\mathbb{R}^2) \to C_B(\mathbb{R}^2)$ and given by

$$(e^{At}g)(x) = \frac{1}{4\pi\nu t}\int_{\mathbb{R}^2} \exp\Big(-\frac{|x-y|^2}{4\nu t}\Big)g(y)dy, \quad t \ge 0,\ g \in Y. \tag{3.2}$$

Here $C_B^k(\mathbb{R}^2)$ stands for the Banach space of functions whose derivatives up to the k-th order are all continuous and bounded, with $C_B^0(\mathbb{R}^2)$ denoted by $C_B(\mathbb{R}^2)$.

In this section we shall study the spatial derivatives of the linear term and the nonlinear integrand in (3.1). We have

$$\partial_x^\beta(e^{At}g)(x) = \frac{1}{4\pi\nu t}\int_{\mathbb{R}^2} \partial_x^\beta \exp\Big(-\frac{|x-y|^2}{4\nu t}\Big)g(y)dy. \tag{3.3}$$

The following two regularity estimates for vorticity and velocity solutions $\omega(t,x)$ and $u(t,x)$ of (1.2)–(1.3) have been shown in [19, 24],

$$\|\partial_t^\alpha \partial_x^\beta \omega(t,\cdot)\|_p \le N_\omega(\alpha,|\beta|,p) t^{-(\alpha+\frac{|\beta|}{2}+1-\frac{1}{p})}\|\omega_0\|, \quad \text{for } p\in[1,\infty], \tag{3.4}$$

$$\|\partial_t^\alpha \partial_x^\beta u(t,\cdot)\|_q \le N_u(\alpha.|\beta|,q) t^{-(\alpha+\frac{|\beta|}{2}+\frac{1}{2}-\frac{1}{q})}\|\omega_0\|, \quad \text{for } q\in(2,\infty], \tag{3.5}$$

where $N_\omega(\alpha,|\beta|,p)$ and $N_u(\alpha,|\beta|,p)$ are positive constants depending on $\alpha,|\beta|,p$, and $\|\omega_0\|$. Now we prove a key lemma, which shows the exterior decay estimates of these spatial derivatives in (3.3).

Lemma 3.1. *For $p\in[1,\infty]$ and any index $\beta=(\beta_1,\beta_2)$ with $|\beta|=k\ge 0$, there exist positive constants $L_3(k,p)$ and $L_4(k)$ such that for any given $R>1$,*

$$\|\partial_x^\beta (e^{At}g)^R\|_p \le L_3(k,p) R^{-(3-\frac{2}{p})} t^{-\frac{1}{2}(k-1)}\|g\| + L_4(k) t^{-\frac{k}{2}}\|g^{R/2}\|_p, \tag{3.6}$$

for any $g\in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$.

Proof. Similar to (2.6), for any given $\ell>0$ and a constant $\zeta>0$, the function

$$\psi(t) = \frac{1}{t^\ell}\exp\big(-\frac{\zeta^2}{t}\big)$$

has the maximum

$$\max_{t>0}\psi(t) = \psi(\zeta^2) = \frac{e^{-1}}{\zeta^{2\ell}}. \tag{3.7}$$

By (3.3) and direct differentiation, we have

$$\begin{aligned} |\partial_x^\beta (e^{At}g)^R(x)| &\le \frac{1}{4\pi\nu t}\sum_{\beta_1=0}^{k}\int_{\mathbb{R}^2}\Big|\partial_{x_1}^{\beta_1}\partial_{x_2}^{k-\beta_1}\exp\Big(-\frac{|x-y|^2}{4\nu t}\Big)\Big||g(y)|\,dy \\ &= \frac{1}{4\pi\nu t}\sum_{j=[\frac{k+1}{2}]}^{k} C_j(k)\int_{\mathbb{R}^2}\frac{|x-y|^i}{t^j}\exp\Big(-\frac{|x-y|^2}{4\nu t}\Big)|g(y)|\,dy \\ &= \frac{1}{4\pi\nu t}\sum_{j=[\frac{k+1}{2}]}^{k} C_j(k)\int_{\mathbb{R}^2}\frac{|x-y|^{2j-k}}{t^j}\exp\Big(-\frac{|x-y|^2}{4\nu t}\Big)|g(y)|\,dy, \end{aligned} \tag{3.8}$$

in which we have $0\le i\le j$ and $j+(j-i)=k$ so that $0\le i=2j-k$, and $C_j(k)$ is a positive constant depending on j and $k=|\beta|$. For each term $I_j(t,x)$, $j=[\frac{k+1}{2}],\cdots,k$, in (3.8), a decomposition yields

$$I_j(t,x) = \frac{C_j(k)}{4\pi\nu t}\int_{\mathbb{R}^2}\frac{|x-y|^{2j-k}}{t^j}\exp\big(-\frac{|x-y|^2}{4\nu t}\big)|g(y)|dy = I_{j,1}(t,x)+I_{j,2}(t,x). \tag{3.9}$$

The first integral in (3.9) satisfies the estimate

$$\begin{aligned} I_{j,1}(t,x) &= \frac{C_j(k)}{4\pi\nu t}\int_{\{|y|\le R/2\}} \frac{|x-y|^{2j-k}}{t^j}\exp\big(-\frac{|x-y|^2}{4\nu t}\big)|g(y)|dy \\ &= \frac{C_j(k)}{4\pi\nu t^{(k-1)/2}}\int_{\{|y|\le R/2\}} \frac{|x-y|^{2j-k}}{t^{\frac32+j-\frac k2}}\exp\big(-\frac{|x-y|^2}{4\nu t}\big)|g(y)|dy \\ &\le \frac{C_j(k)e^{-1}}{4\pi\nu t^{(k-1)/2}}\int_{\{|y|\le R/2\}} \frac{|x-y|^{2j-k}(4\nu)^{\frac32+j-\frac k2}}{|x-y|^{2(\frac32+j-\frac k2)}}|g(y)|dy \\ &\le \frac{\widetilde{C}(j,k)}{|x|^3 t^{(k-1)/2}}\int_{\{|y|\le R/2\}}|g(y)|dy \\ &\le \frac{\widetilde{C}(j,k)}{|x|^3 t^{(k-1)/2}}\|g\|, \quad \text{for } |x|>R, \end{aligned}$$

where we used (3.7) and the fact that $|x|>R>R/2\ge|y|$, and the constant $\widetilde{C}(j,k)$ is given by

$$\widetilde{C}(j,k) = 8C_j(k)(\pi e)^{-1}(4\nu)^{\frac12(1-k)+j}.$$

Then it follows that

$$\begin{aligned} \|I^R_{j,1}(t,\cdot)\|_p &\le \frac{\widetilde{C}(j,k)}{t^{(k-1)/2}}\Big(\int_{\{|x|>R\}}\frac{1}{|x|^{3p}}\,dx\Big)^{1/p}\|g\| \\ &= \widetilde{C}(j,k)\big(\frac{2\pi}{3p-2}\big)^{1/p}R^{-(3-\frac2p)}t^{-\frac12(k-1)}\|g\| \\ &= M_3(j,k,p)R^{-(3-\frac2p)}t^{-\frac12(k-1)}\|g\|, \end{aligned} \tag{3.10}$$

where $M_3(j,k,p)=\widetilde{C}(j,k)(2\pi/(3p-2))^{1/p}$. The second integral in (3.9) is given by

$$I^R_{j,2}(t,x) = \frac{C_j(k)}{4\pi\nu t}\int_{\{|y|>R/2\}}\frac{|x-y|^{2j-k}}{t^j}\exp\Big(-\frac{|x-y|^2}{4\nu t}\Big)|g(y)|\,dy,\ |x|>R.$$

By the convolution property (1.10) we can assert that

$$\|I^R_{j,2}(t,\cdot)\|_p \le \frac{C_j(k)(4\nu)^{\frac12(2j-k)}}{\pi t^{j-\frac12(2j-k)}}\Big(\int_{\mathbb{R}^2}|z|^{2j-k}e^{-|z|^2}dz\Big)\|g^{R/2}\|_p = \frac{M_4(j,k)}{t^{k/2}}\|g^{R/2}\|_p, \tag{3.11}$$

where, as we remarked after (3.8), the exponents $2j-k\ge 0$ for $j=[\frac{k+1}{2}],\cdots,k$, and the constant $M_4(j,k)$ is given by

$$M_4(j,k) = C_j(k)\pi^{-1}(4\nu)^{j-\frac k2}\Big(\int_{\mathbb{R}^2}|z|^{2j-k}e^{-|z|^2}dz\Big).$$

Substituting (3.10) and (3.11) into (3.9) and (3.8), we obtain

$$\begin{aligned} \|\partial_x^\beta(e^{At}g)^R\|_p &\le \sum_{j=[\frac{k+1}{2}]}^{k}\big(\|I^R_{j,1}(t,\cdot)\|_p+\|I^R_{j,2}(t,\cdot)\|_p\big) \\ &\le \sum_{j=[\frac{k+1}{2}]}^{k}\Big(M_3(j,k,p)R^{-(3-\frac2p)}t^{-\frac12(k-1)}\|g\|+M_4(j,k)t^{-\frac k2}\|g^{R/2}\|_p\Big) \\ &= L_3(k,p)R^{-(3-\frac2p)}t^{-\frac12(k-1)}\|g\|+L_4(k)t^{-\frac k2}\|g^{R/2}\|_p, \end{aligned}$$

with $L_3(k,p)=\sum_{j=[\frac{k+1}{2}]}^{k} M_3(j,k,p)$ and $L_4(k)=\sum_{j=[\frac{k+1}{2}]}^{k} M_4(j,k)$. Thus (3.6) is proved for $p\in[1,\infty)$. The proof of (3.6) for $p=\infty$ is similar and even easier. □

Next is another key lemma showing the regularity estimates of the spatial derivatives of the nonlinear integrand function in (3.1), which involves the nonlinear term $v(t,x)$ in the vorticity equation (1.2). Here

$$v(t,x)=(u(t,x)\cdot\nabla)\omega(t,x). \tag{3.12}$$

Lemma 3.2. *Let $\omega(t,x)$ and $u(t,x)$ be a solution of the vorticity equations (1.2)–(1.3) with initial data $\omega_0\in L^1(\mathbb{R}^2)$. Let $p\in[1,\infty]$. Then for any index $\beta\in Z^+\times Z^+$ and any given $R>1$, the following estimate is satisfied,*

$$\begin{aligned}
&\|\partial_x^\beta(e^{A(t-s)}v(s,\cdot))^R\|_p\\
&\le L_5\Big(\frac{1}{R^{3-\frac{2}{p}}(t-s)^{\frac12(|\beta|-1)}s^{2-\frac1p}}+\frac{1}{R(t-s)^{\frac{|\beta|}{2}}s^{\frac32-\frac1p}}+\frac{\|\omega_0^{R/8}\|}{(t-s)^{\frac{|\beta|}{2}}s^{2-\frac1p}}\Big),
\end{aligned}\tag{3.13}$$

where $0<s<t$, and $L_5=L_5(|\beta|,p,\|\omega_0\|)$ is a positive constant.

Proof. Let $|\beta|=k\ge 0$. By Lemma 3.1, for $t>0$, we have

$$\begin{aligned}
&\|\partial_x^\beta(e^{A(t-s)}v(s,\cdot))^R\|_p\\
&=\|\partial_x^\beta(e^{A(t-s)}(u(s,\cdot)\cdot\nabla)\omega(s,\cdot))^R\|_p\\
&\le L_3(k,p)R^{-(3-\frac2p)}(t-s)^{-\frac12(k-1)}\|v(s,\cdot)\|+L_4(k)(t-s)^{-\frac{k}{2}}\|v^{R/2}(s,\cdot)\|_p,
\end{aligned}\tag{3.14}$$

where by (3.4) and (3.5) we have

$$\begin{aligned}
\|v(s,\cdot)\|&=\int_{\mathbb{R}^2}|(u(s,x)\cdot\nabla)\omega(s,x)|\,dx\\
&\le\|u_1(s,\cdot)\|_\infty\|\partial_{x_1}\omega(s,\cdot)\|+\|u_2(s,\cdot)\|_\infty\|\partial_{x_2}\omega(s,\cdot)\|\\
&\le\Big(N_u(0,0,\infty)s^{-\frac12}\|\omega_0\|\Big)\Big(2N_\omega(0,1,1)s^{-(\frac32-\frac1p)}\|\omega_0\|\Big)\\
&=N_{v,1}s^{-(2-\frac1p)},
\end{aligned}\tag{3.15}$$

in which $N_{v,1}=2N_u(0,0,\infty)N_\omega(0,1,1)\|\omega_0\|^2$, and

$$\begin{aligned}
&\|v^{R/2}(s,\cdot)\|_p\\
&=\Big(\int_{\{|x|>R/2\}}|(u(s,x)\cdot\nabla)\omega(s,x)|^p\,dx\Big)^{1/p}\\
&\le\|u^{R/2}(s,\cdot)\|_\infty\Big(\|\partial_{x_1}\omega^{R/2}(s,\cdot)\|_p+\|\partial_{x_2}\omega^{R/2}(s,\cdot)\|_p\Big)\\
&\le 2N_\omega(0,1,p)s^{-(\frac32-\frac1p)}\|\omega_0\|\|u^{R/2}(s,\cdot)\|_\infty=N^*s^{-(\frac32-\frac1p)}\|u^{R/2}(s,\cdot)\|_\infty,
\end{aligned}\tag{3.16}$$

in which $N^*=2N_\omega(0,1,p)\|\omega_0\|$. Note that by Lemma 2.4 and (2.16), one has

$$\|u^R(s,\cdot)\|_\infty\le\frac{L_{1,\infty}}{R}\|\omega(s,\cdot)\|+L_{2,\infty}\|\omega^{R/2}(s,\cdot)\|^{1/2}\|\omega^{R/2}(s,\cdot)\|_\infty^{1/2}. \tag{3.17}$$

Substituting (3.17) with R replaced by $R/2$ into (3.16), by Lemma 2.1 and (3.4), we obtain

$$
\begin{aligned}
&\|v^{R/2}(s,\cdot)\|_p \\
&\le N^* s^{-(\frac{3}{2}-\frac{1}{p})}\Big\{\frac{2L_{1,\infty}}{R}\|\omega(s,\cdot)\| + L_{2,\infty}\|\omega^{R/4}(s,\cdot)\|^{1/2}\|\omega^{R/4}(s,\cdot)\|_\infty^{1/2}\Big\} \\
&\le N^* s^{-(\frac{3}{2}-\frac{1}{p})}\Big\{\frac{2L_{1,\infty}}{R}N_\omega(0,0,1)\|\omega_0\| + L_{2,\infty}\Big[\frac{4L_0(1)}{R}s^{1/2}\|\omega_0\| + \|\omega_0^{R/8}\|\Big]^{1/2} \\
&\quad \times \Big[\frac{4L_0(\infty)}{R}s^{-1/2}\|\omega_0\| + s^{-1}\|\omega_0^{R/8}\|\Big]^{1/2}\Big\} \\
&\le \frac{N^*}{s^{\frac{3}{2}-\frac{1}{p}}}\Big\{2L_{1,\infty}N_\omega(0,0,1)\frac{\|\omega_0\|}{R} + L_{2,\infty}(4L_0(1)\|\omega_0\|+1)^{1/2} \\
&\quad \times (4L_0(\infty)\|\omega_0\|+1)^{1/2}\Big(R^{-1}s^{1/2} + \|\omega_0^{R/8}\|\Big)^{1/2}\Big(R^{-1}s^{-1/2} + s^{-1}\|\omega_0^{R/8}\|\Big)^{1/2}\Big\} \\
&\le \frac{N_{v,2}}{s^{\frac{3}{2}-\frac{1}{p}}}\Big(\frac{1}{R} + \frac{\|\omega_0^{R/8}\|}{s^{\frac{1}{2}}}\Big)
\end{aligned}
\tag{3.18}
$$

where $N_{v,2}$ is a constant given by

$$
\begin{aligned}
N_{v,2} = N^*\big[&2L_{1,\infty}N_\omega(0,0,1)\|\omega_0\| \\
&+ L_{2,\infty}(4L_0(1)\|\omega_0\|+1)^{1/2}(4L_0(\infty)\|\omega_0\|+1)^{1/2}\big].
\end{aligned}
$$

Finally we substitute (3.15) and (3.18) into (3.14) to obtain

$$
\begin{aligned}
&\|[\partial_x^\beta(e^{A(t-s)}(u(s,\cdot)\cdot\nabla)\omega(s,\cdot))]^R\|_p \\
&\le \frac{L_3(k,p)N_{v,1}}{R^{3-\frac{2}{p}}(t-s)^{\frac{1}{2}(k-1)}s^{2-\frac{1}{p}}} + \frac{L_4(k)N_{v,2}}{(t-s)^{\frac{k}{2}}s^{\frac{3}{2}-\frac{1}{p}}}\Big(\frac{1}{R} + \frac{\|\omega_0^{R/8}\|}{s^{\frac{1}{2}}}\Big) \\
&\le L_5\Big(\frac{1}{R^{3-\frac{2}{p}}(t-s)^{\frac{1}{2}(k-1)}s^{2-\frac{1}{p}}} + \frac{1}{R(t-s)^{\frac{k}{2}}s^{\frac{3}{2}-\frac{1}{p}}} + \frac{\|\omega_0^{R/8}\|}{(t-s)^{\frac{k}{2}}s^{2-\frac{1}{p}}}\Big),
\end{aligned}
$$

where the constant $L_5 = L_5(k,p,\|\omega_0\|) = \max\{L_3(k,p)N_{v,1}, L_4(k)N_{v,2}\}$ and $k = |\beta|$. Therefore, (3.13) is proved. □

4. Exterior Decay Estimates of Temporal-Spatial Derivatives

In this section we establish the exterior decay estimates of the temporal-spatial derivatives of the vorticity solution $\omega(t,x)$ of (1.2)–(1.3). These estimates of L^p-norms of the derivatives are dominated by the combination of the spatial decay rates $1/R$ and $\|\omega_0^{R/16}\|$ and by the temporal decay rate $t^{-(\alpha+\frac{1}{2}(|\beta|-1))}$, as described in the following two theorems. First we deal with the spatial derivatives.

Theorem 4.1. *Let $p \in [1,\infty]$. For any given $R > 1$ and any $\beta \in Z^+ \times Z^+$, the spatial derivative of the vorticity solution $\omega(t,x)$ of (1.2)–(1.3) with any initial data $\omega_0 \in L^1(\mathbb{R}^2)$ has the following decay property,*

$$
\|\partial_x^\beta \omega^R(t,\cdot)\|_p \le \frac{B(|\beta|,p,\|\omega_0\|)}{t^{\frac{1}{2}(|\beta|-1)}}\Big(\frac{1}{R} + \frac{1}{Rt^{1-\frac{1}{p}}} + \frac{\|\omega_0^{R/16}\|}{t^{\frac{3}{2}-\frac{1}{p}}}\Big), \tag{4.1}
$$

where $B(|\beta|,p,\|\omega_0\|)$ is a positive constant depending on $|\beta|$, p, and $\|\omega_0\|$.

Proof. We have shown in Lemma 2.1 that (4.1) holds for $|\beta| = 0$. Now we prove that (4.1) is valid for spatial derivatives of $\omega(t,x)$ of any order β by induction.

The variation-of-constant formula (3.1) implies that

$$\partial_x^\beta \omega(t,\cdot) = \partial_x^\beta \Big(e^{At/2}\omega(t/2,\cdot)\Big) - \int_{t/2}^t \partial_x^\beta \Big(e^{A(t-s)}(u(s,\cdot)\cdot\nabla)\omega(s,\cdot)\Big)\,ds, \quad t>0. \tag{4.2}$$

Step 1. Let $|\beta| = k \ge 1$. By Lemma 3.1 and (3.6), the linear part in (4.2) satisfies

$$\begin{aligned}&\|\partial_x^\beta (e^{At/2}\omega(t/2,\cdot))^R\|_p \\ &\le L_3(k,p)R^{-(3-\frac{2}{p})}t^{-\frac12(k-1)}\|\omega(t/2,\cdot)\| + L_4(k)t^{-\frac{k}{2}}\|\omega(t/2,\cdot)^{R/2}\|_p, \quad \text{for } p\in[1,\infty].\end{aligned} \tag{4.3}$$

Then using Lemma 2.1 and the estimates (3.4) and (3.5), we get from (4.3) that

$$\begin{aligned}&\|\partial_x^\beta (e^{At/2}\omega(t/2,\cdot))^R\|_p \\ &\le 2L_3(k,p)N_\omega(0,0,1)R^{-(3-\frac{2}{p})}t^{-[\frac12(k-1)+(1-\frac1p)]}\|\omega_0\| \\ &\quad + 2^{1-\frac1p}L_4(k)t^{-\frac{k}{2}}L_0(p,\|\omega_0\|)\Big(\frac{1}{R}t^{-(\frac12-\frac1p)}\|\omega_0\| + t^{-(1-\frac1p)}\|\omega_0^{R/2}\|\Big) \\ &\le \frac{L_6(k,p,\|\omega_0\|)}{t^{\frac12(k-1)}}\Big(\frac{\|\omega_0\|}{Rt^{1-\frac1p}} + \frac{\|\omega_0^{R/2}\|}{t^{\frac32-\frac1p}}\Big),\end{aligned} \tag{4.4}$$

where we used $R^{-2(1-1/p)} \le 1$ due to $R>1$, and the constant L_6 is given by

$$L_6(k,p,\|\omega_0\|) = 2(L_3(k,p)N_\omega(0,0,1) + 2^{1-\frac1p}L_4(k)L_0(p,\|\omega_0\|)).$$

Note that the estimate (4.4) for the linear part is valid for all $\beta \in Z^+\times Z^+$.

The next two steps are devoted to the nonlinear integral term in (4.2).

Step 2. Since the nonlinear term in (1.2), which is denoted by $v(t,x)$ in (3.12), involves $\nabla\omega(t,x)$, first we have to show that (4.1) is valid for the gradient $\nabla\omega(t,x)$ with $|\beta| = 1$ in this step. Indeed we show that

$$\|\nabla\omega^R(t,\cdot)\|_p \le B_0(p,\|\omega_0\|)\Big(\frac{1}{Rt^{1-\frac1p}} + \frac{\|\omega_0^{R/8}\|}{t^{\frac32-\frac1p}}\Big), \tag{4.5}$$

where $B_0(p,\|\omega_0\|)$ is a constant. According to (4.2), $\nabla\omega(t,x)$ consists of the linear term and the nonlinear integral term.

For the linear term, we apply the estimate (4.4) established in Step 1 with $k=1$ to get

$$\|\nabla(e^{At/2}\omega(t/2,\cdot))^R\|_p \le L_6(1,p,\|\omega_0\|)\Big(\frac{\|\omega_0\|}{Rt^{1-\frac1p}} + \frac{\|\omega_0^{R/2}\|}{t^{\frac32-\frac1p}}\Big). \tag{4.6}$$

For the nonlinear term in (4.2) for the gradient $\nabla\omega(t,x)$, by the Jensen inequality and the convexity of the L^p norm, we can apply Lemma 3.2 to assert that for $k=1$,

$$
\begin{aligned}
&\| \int_{t/2}^{t} \nabla (e^{A(t-s)}(u(s,\cdot)\cdot\nabla)\omega(s,\cdot))^R \, ds\|_p \\
&\le L_5(1,p,\|\omega_0\|) \int_{t/2}^{t} \Big(\frac{1}{R^{3-\frac{2}{p}} s^{2-\frac{1}{p}}} + \frac{1}{R(t-s)^{\frac12} s^{\frac32-\frac1p}} + \frac{\|\omega_0^{R/8}\|}{(t-s)^{\frac12} s^{2-\frac1p}} \Big) ds \\
&\le L_7(p,\|\omega_0\|) \Big(\frac{1}{Rt^{1-\frac1p}} + \frac{\|\omega_0^{R/8}\|}{t^{\frac32-\frac1p}} \Big),
\end{aligned} \tag{4.7}
$$

where

$$
L_7(p,\|\omega_0\|) = 2^{3-\frac1p} L_5(1,p,\|\omega_0\|).
$$

Collecting (4.6) and (4.7), due to that $\|\omega_0^{R/2}\| \le \|\omega_0^{R/8}\|$, we obtain (4.5) with the constant B_0 given by

$$
B_0(p,\|\omega_0\|) = L_6(1,p,\|\omega_0\|)\|\omega_0\| + L_7(p,\|\omega_0\|).
$$

Therefore, (4.1) is valid for $|\beta| = 1$, i.e. for the gradient $\nabla\omega(t,x)$.

Step 3. Now we prove that (4.1) is valid for spatial derivatives of $\omega(t,x)$ of any order by induction. Assuming that (4.1) holds for all the spatial derivatives of $\omega(t,x)$ up to the order k, and let $\beta \in Z^+ \times Z^+$ has $|\beta| = k$. We want to show that (4.1) remains valid for $\nabla\partial_x^\beta\omega(t,x)$.

By (4.2) and variable changes in the integral part, we can write

$$
\nabla(\partial_x^\beta \omega^R(t,\cdot)) = \nabla[\partial_x^\beta (e^{At/2}\omega(t/2,\cdot))^R] - \int_{t/2}^{t} \nabla[e^{A(t-s)}\partial_x^\beta (u(s,\cdot)\cdot\nabla)\omega(s,\cdot))]^R \, ds, \tag{4.8}
$$

where $t > 0$. By (4.4) shown in Step 1, for the linear term in (4.8) we have readily

$$
\|\nabla[\partial_x^\beta (e^{At/2}\omega(t/2,\cdot))^R]\|_p \le \frac{L_6(k,p,\|\omega_0\|)(\|\omega_0\|+1)}{t^{k/2}} \Big(\frac{1}{Rt^{1-\frac1p}} + \frac{\|\omega_0^{R/2}\|}{t^{\frac32-\frac1p}} \Big), \quad t > 0. \tag{4.9}
$$

Below we focus on estimating the nonlinear integral term in (4.8).

Using the notation (3.12) and applying Lemma 3.1, the L^p norm of the integrand function in (4.8) admits the estimate,

$$
\begin{aligned}
&\|\nabla[e^{A(t-s)}\partial_x^\beta (u(s,\cdot)\cdot\nabla)\omega(s,\cdot))]^R\|_p \\
&= \|\nabla(e^{A(t-s)}\partial_x^\beta v(s,\cdot))^R\|_p \\
&\le \frac{L_3(k,p)}{R^{3-\frac2p}} \|\partial_x^\beta v(s,\cdot)\| + \frac{L_4(k)}{(t-s)^{1/2}} \|\partial_x^\beta v^{R/2}(s,\cdot)\|_p.
\end{aligned} \tag{4.10}
$$

By Leibniz formula in product differentiation and using (3.4) with $p = 1$ and (3.5) with $q = \infty$, we have

$$
\begin{aligned}
\|\partial_x^\beta v(s,\cdot)\| &= \|\sum_{\gamma\le\beta}\binom{\beta}{\gamma}\partial_x^\gamma u(s,\cdot)\cdot\partial_x^{\beta-\gamma}(\nabla\omega(s,\cdot))\| \\
&\le 2^{|\beta|}\sum_{\gamma\le\beta}\|\partial_x^\gamma u(s,\cdot)\|_\infty\|\partial_x^{\beta-\gamma}\nabla\omega(s,\cdot)\| \\
&\le 2^{|\beta|}\sum_{\gamma\le\beta}\frac{N_u(0,|\gamma|,\infty)\|\omega_0\|}{s^{\frac{|\gamma|}{2}+\frac12}}\cdot\frac{N_\omega(0,|\beta-\gamma|,1)\|\omega_0\|}{s^{\frac{|\beta-\gamma|}{2}+\frac12}} \\
&= \frac{L_8(|\beta|,\|\omega_0\|)}{s^{\frac{|\beta|}{2}+1}},
\end{aligned}
\tag{4.11}
$$

where

$$
L_8(|\beta|,\|\omega_0\|) = 2^{|\beta|}\|\omega_0\|^2\sum_{\gamma\le\beta}N_u(0,|\gamma|,\infty)N_\omega(0,|\beta-\gamma|,1).
$$

Next we have

$$
\begin{aligned}
&\|\partial_x^\beta v^{R/2}(s,\cdot)\|_p \\
&= \|\sum_{\gamma\le\beta}\binom{\beta}{\gamma}\partial_x^\gamma u^{R/2}(s,\cdot)\cdot\partial_x^{\beta-\gamma}(\nabla\omega^{R/2}(s,\cdot))\|_p \\
&\le \sum_{\gamma\le\beta}\binom{\beta}{\gamma}\|\partial_x^\gamma u^{R/2}(s,\cdot)\|_\infty\|\partial_x^{\beta-\gamma}\nabla\omega^{R/2}(s,\cdot)\|_p \\
&= \|u^{R/2}(s,\cdot)\|_\infty\|\partial_x^\beta\nabla\omega^{R/2}(s,\cdot)\|_p + \|\partial_x^\beta u^{R/2}(s,\cdot)\|_\infty\|\nabla\omega^{R/2}(s,\cdot)\|_p \\
&\quad + \sum_{0<|\gamma|<|\beta|}\binom{\beta}{\gamma}\|\partial_x^\gamma u(s,\cdot)\|_\infty\|\partial_x^{\beta-\gamma}\nabla\omega(s,\cdot)\|_p = P_1(s)+P_2(s)+P_3(s),
\end{aligned}
\tag{4.12}
$$

where we use $P_i(s), i = 1,2,3$, to denote the three parts in the entire sum of (4.12).

By using (2.16), (3.4) and (3.5), we have

$$
\begin{aligned}
P_1(s) &= \|u^{R/2}(s,\cdot)\|_\infty\|\partial_x^\beta\nabla\omega^{R/2}(s,\cdot)\|_p \\
&\le \frac{N_\omega(0,|\beta|+1,p)\|\omega_0\|}{s^{\frac{|\beta|}{2}+\frac32-\frac1p}} \\
&\quad\times\Big(\frac{L_{1,\infty}}{R}\|\omega(s,\cdot)\| + L_{2,\infty}\|\omega^{R/4}(s,\cdot)\|^{1/2}\|\omega^{R/4}(s,\cdot)\|_\infty^{1/2}\Big)
\end{aligned}
\tag{4.13}
$$

where by Lemma 2.1,

$$
\begin{aligned}
&\|\omega^{R/4}(s,\cdot)\|^{1/2}\|\omega^{R/4}(s,\cdot)\|_\infty^{1/2} \\
&\le \Big(L_0(1,\|\omega_0\|)\Big(\frac{2s^{1/2}}{R}\|\omega_0\| + \|\omega_0^{R/8}\|\Big)L_0(\infty,\|\omega_0\|) \\
&\quad\times\Big(\frac{2s^{-1/2}}{R}\|\omega_0\| + s^{-1}\|\omega_0^{R/8}\|\Big)\Big)^{1/2} \\
&\le L_9(\|\omega_0\|)\Big(\frac1R + s^{-1/2}\|\omega_0^{R/8}\|\Big),
\end{aligned}
$$

in which

$$
L_9(\|\omega_0\|) = (2L_0(1,\|\omega_0\|)\|\omega_0\|+1)^{1/2}(2L_0(\infty,\|\omega_0\|)\|\omega_0\|+1)^{1/2}.
$$

From (4.13) it follows that

$$P_1(s) \le \frac{B_1(|\beta|, p, \|\omega_0\|)}{s^{\frac{|\beta|}{2}+\frac{3}{2}-\frac{1}{p}}}\Big(\frac{1}{R} + s^{-1/2}\|\omega_0^{R/8}\|\Big), \tag{4.14}$$

where

$$B_1(|\beta|, p, \|\omega_0\|) = N_\omega(0, |\beta|+1, p)\|\omega_0\|\left\{L_{1,\infty}N_\omega(0,0,1)\|\omega_0\| + L_{2,\infty}L_9(\|\omega_0\|)\right\}.$$

Next by (3.5) and (4.5) for $\nabla\omega(t,x)$ proved in Step 2, we have

$$\begin{aligned} P_2(s) &= \|\partial_x^\beta u^{R/2}(s,\cdot)\|_\infty \|\nabla\omega^{R/2}(s,\cdot)\|_p \\ &\le \frac{N_u(0,|\beta|,\infty)\|\omega_0\|}{s^{\frac{1}{2}(|\beta|+1)}} B_0(p,\|\omega_0\|)\Big(\frac{1}{Rs^{1-\frac{1}{p}}} + \frac{\|\omega_0^{R/16}\|}{s^{\frac{3}{2}-\frac{1}{p}}}\Big) \\ &\le \frac{B_2(|\beta|, p, \|\omega_0\|)}{s^{\frac{|\beta|}{2}+\frac{3}{2}-\frac{1}{p}}}\Big(\frac{1}{R} + s^{-1/2}\|\omega_0^{R/16}\|\Big), \end{aligned} \tag{4.15}$$

where the constant $B_2(|\beta|, p, \|\omega_0\|)$ is given by

$$B_2(|\beta|, p, \|\omega_0\|) = N_u(0,|\beta|,\infty)\|\omega_0\| B_0(p,\|\omega_0\|).$$

Moreover, by the assumption of induction in this step that (4.1) holds for all the spatial derivatives of $\omega(t,x)$ up to the order k, and here $|\beta| = k$, we have

$$\begin{aligned} P_3(s) &= \sum_{0<|\gamma|<|\beta|} \binom{\beta}{\gamma} \|\partial_x^\gamma u(s,\cdot)\|_\infty \|\partial_x^{\beta-\gamma}\nabla\beta(s,\cdot)\|_p \\ &\le \sum_{0<|\gamma|<|\beta|} \binom{\beta}{\gamma} \frac{N_u(0,|\gamma|,\infty)\|\omega_0\|}{s^{\frac{1}{2}(|\gamma|+1)}} \frac{B(|\beta-\gamma|, p, \|\omega_0\|)}{s^{\frac{1}{2}(|\beta-\gamma|+1-1)}} \\ &\quad \times \Big(\frac{1}{R} + \frac{1}{Rs^{1-\frac{1}{p}}} + \frac{\|\omega_0^{R/16}\|}{s^{\frac{3}{2}-\frac{1}{p}}}\Big) \\ &\le \frac{B_3(|\beta|, p, \|\omega_0\|)}{s^{\frac{1}{2}(|\beta|+1)}}\Big(\frac{1}{R} + \frac{1}{Rs^{1-\frac{1}{p}}} + \frac{\|\omega_0^{R/16}\|}{s^{\frac{3}{2}-\frac{1}{p}}}\Big), \end{aligned} \tag{4.16}$$

where the constant $B_3(|\beta|, p, \|\omega_0\|)$ is given by

$$B_3(|\beta|, p, \|\omega_0\|) = \sum_{0<|\gamma|<|\beta|} \binom{\beta}{\gamma} N_u(0,|\gamma|,\infty)\|\omega_0\| B(|\beta-\gamma|, p, \|\omega_0\|).$$

Now substituting (4.14). (4.15) and (4.16) into (4.12), we get

$$\begin{aligned} \|\partial_x^\beta v^{R/2}(s,\cdot)\|_p &\le P_1(s) + P_2(s) + P_3(s) \\ &\le \frac{B_1+B_2+B_3}{s^{\frac{|\beta|}{2}+\frac{3}{2}-\frac{1}{p}}}\left(\frac{1}{R} + \frac{\|\omega_0^{R/16}\|}{s^{\frac{1}{2}}}\right) + \frac{B_3}{Rs^{\frac{1}{2}(|\beta|+1)}}, \end{aligned} \tag{4.17}$$

Finally we substitute (4.11) and (4.17) into (4.10) and then we can estimate the nonlinear integral term in (4.8) as follows, noting that $|\beta| = k \geq 0$ and $R > 1$,

$$
\begin{aligned}
&\| \int_{t/2}^{t} \nabla [e^{A(t-s)} \partial_x^\beta (u(s,\cdot)\cdot\nabla)\omega(s,\cdot))]^R \, ds \|_p \\
&\leq \int_{t/2}^{t} \|\nabla [e^{A(t-s)} \partial_x^\beta (u(s,\cdot)\cdot\nabla)\omega(s,\cdot))]^R \|_p \, ds \\
&\leq \int_{t/2}^{t} \Big(\frac{L_3(|\beta|,p)}{R^{3-\frac{2}{p}}} \|\partial_x^\beta v(s,\cdot)\| + \frac{L_4(|\beta|)}{(t-s)^{1/2}} \|\partial_x^\beta v^{R/2}(s,\cdot)\|_p \Big) ds \\
&\leq \int_{t/2}^{t} \frac{L_3(|\beta|,p)}{R^{3-\frac{2}{p}}} \cdot \frac{L_8(|\beta|, \|\omega\|_0)}{s^{\frac{|\beta|}{2}+1}} \, ds \\
&\quad + \int_{t/2}^{t} \frac{L_4(|\beta|)}{(t-s)^{1/2}} \Big[\frac{B_1+B_2+B_3}{s^{\frac{|\beta|}{2}+\frac{3}{2}-\frac{1}{p}}} \Big(\frac{1}{R} + \frac{\|\omega_0^{R/16}\|}{s^{\frac{1}{2}}} \Big) + \frac{B_3}{R s^{\frac{1}{2}(|\beta|+1)}} \Big] ds \\
&\leq L_3(|\beta|,p) \frac{L_8(|\beta|,\|\omega\|_0)}{R(t/2)^{\frac{|\beta|}{2}}} + L_4(|\beta|) \frac{2B_3 (t/2)^{1/2}}{R(t/2)^{\frac{1}{2}(|\beta|+1)}} \\
&\quad + L_4(|\beta|) \frac{2(B_1+B_2+B_3)(t/2)^{1/2}}{(t/2)^{\frac{|\beta|}{2}+\frac{3}{2}-\frac{1}{p}}} \Big(\frac{1}{R} + \frac{\|\omega_0^{R/16}\|}{(t/2)^{\frac{1}{2}}} \Big) \\
&\leq \frac{B_4(|\beta|+1,p,\|\omega_0\|)}{t^{\frac{1}{2}[(|\beta|+1)-1]}} \Big(\frac{1}{R} + \frac{1}{R t^{1-\frac{1}{p}}} + \frac{\|\omega_0^{R/16}\|}{t^{\frac{3}{2}-\frac{1}{p}}} \Big),
\end{aligned}
\tag{4.18}
$$

where

$$
\begin{aligned}
B_4(|\beta|+1,p,\|\omega_0\|) = \max \big\{ & 2^{\frac{|\beta|}{2}} L_3(|\beta|,p) L_8(|\beta|,\|\omega\|_0) + 2^{\frac{|\beta|}{2}+1} L_4(|\beta|) B_3, \\
& 2^{\frac{|\beta|}{2}+2-\frac{1}{p}} L_4(|\beta|)(B_1+B_2+B_3) \big\}.
\end{aligned}
$$

We assemble (4.8), (4.9) (noting that $|\beta| = k$ and $\|\omega_0^{R/2}\| \leq \|\omega_0^{R/16}\|$) and (4.18) to obtain

$$
\|\nabla \partial_x^\beta \omega^R(t,x)\|_p \leq \frac{B(|\beta|+1,p,\|\omega_0\|)}{t^{\frac{|\beta|}{2}}} \Big(\frac{1}{R} + \frac{1}{R t^{1-\frac{1}{p}}} + \frac{\|\omega_0^{R/16}\|}{t^{\frac{3}{2}-\frac{1}{p}}} \Big), \tag{4.19}
$$

where

$$
B(|\beta|+1,p,\|\omega_0\|) = \max\{ L_6(|\beta|,p,\|\omega_0\|)(\|\omega_0\|+1),\ B_4(|\beta|+1,p,\|\omega_0\|) \}.
$$

The inequality (4.19) shows that (4.1) is valid for $|\beta| = k+1$. Thus by induction we have proved that (4.1) is valid for all order $\beta \in Z^+ \times Z^+$. □

Based on Theorem 4.1, we now prove the following result on the exterior decay estimates of all the temporal-spatial derivatives of the vorticity.

Theorem 4.2. *For any given $R > 1$, $\alpha \in Z^+$ and $\beta \in Z^+ \times Z^+$, the temporal-spatial derivative of the vorticity solution $\omega(t,x)$ of (1.2)–(1.3) with initial data $\omega_0 \in L^1(\mathbb{R}^2)$ has the following decay property,*

$$
\|\partial_t^\alpha \partial_x^\beta \omega^R(t,\cdot)\|_p \leq \frac{K(\alpha,|\beta|,p,\|\omega_0\|)}{t^{\alpha+\frac{1}{2}(|\beta|-1)}} \Big(\frac{1}{R} + \frac{1}{R t^{1-\frac{1}{p}}} + \frac{\|\omega_0^{R/16}\|}{t^{\frac{3}{2}-\frac{1}{p}}} \Big), \tag{4.20}
$$

where $K(\alpha,|\beta|,p,\|\omega_0\|)$ is a positive constant depending on $\alpha, |\beta|, p,$ and $\|\omega_0\|$.

Proof. We shall prove (4.20) by induction with respect to the order α of temporal derivatives of $\partial_x^\beta \omega(t,x)$. In Theorem 4.1, we have shown that (4.20) is valid for $\alpha = 0$ and any $\beta \in Z^+ \times Z^+$. Now assuming that (4.20) is valid for an $\alpha \geq 0$ and any $\beta \in Z^+ \times Z^+$, we prove that (4.20) holds for $\alpha + 1$ and any β.

By the C^∞ regularity of the vorticity and velocity in (1.2)–(1.3), one has

$$\partial_t^{\alpha+1}\partial_x^\beta \omega = \partial_t^\alpha \partial_x^\beta \Delta\omega - \partial_t^\alpha \partial_x^\beta ((u\cdot\nabla)\omega) \tag{4.21}$$

and, for any given $R > 1$,

$$\|\partial_t^{\alpha+1}\partial_x^\beta \omega^R(t,\cdot)\|_p \leq \|\partial_t^\alpha \partial_x^\beta \Delta\omega^R(t,\cdot)\|_p + \|\partial_t^\alpha \partial_x^\beta ((u^R\cdot\nabla)\omega^R(t,\cdot))\|_p. \tag{4.22}$$

By the assumption of induction, for the first term on the right-hand side of (4.22) we have

$$\|\partial_t^\alpha \partial_x^\beta \Delta\omega^R(t,\cdot)\|_p \leq \frac{K(\alpha,|\beta|+2,p,\|\omega_0\|)}{t^{\alpha+\frac{1}{2}(|\beta|+1)}}\Big(\frac{1}{R} + \frac{1}{Rt^{1-\frac{1}{p}}} + \frac{\|\omega_0^{R/16}\|}{t^{\frac{3}{2}-\frac{1}{p}}}\Big). \tag{4.23}$$

For the second term on the righr-hand side of (4.22), from (3.5) and the assumption of induction we get

$$\begin{aligned}
&\|\partial_t^\alpha \partial_x^\beta ((u^R(t,\cdot)\cdot\nabla)\omega^R(t,\cdot))\|_p \\
&\leq \sum_{\eta\leq\alpha}\sum_{\gamma\leq\beta}\binom{\alpha}{\eta}\binom{\beta}{\gamma}\|\partial_t^\eta\partial_x^\gamma u^R(t,\cdot)\|_\infty \|\partial_t^{\alpha-\eta}\partial_x^{\beta-\gamma}\nabla\omega^R(t,\cdot)\|_p \\
&\leq \sum_{\eta\leq\alpha}\sum_{\gamma\leq\beta}\binom{\alpha}{\eta}\binom{\beta}{\gamma}\frac{N_u(\eta,|\gamma|,\infty)\|\omega_0\|}{t^{\eta+\frac{|\gamma|}{2}+\frac{1}{2}}}\cdot\frac{K(\alpha-\eta,|\beta-\gamma|+1,p,\|\omega_0\|)}{t^{\alpha-\eta+\frac{1}{2}|\beta-\gamma|}} \\
&\quad\times\Big(\frac{1}{R} + \frac{1}{Rt^{1-\frac{1}{p}}} + \frac{\|\omega_0^{R/16}\|}{t^{\frac{3}{2}-\frac{1}{p}}}\Big) \\
&\leq \frac{L_{10}(\alpha,|\beta|,p,\|\omega_0\|)}{t^{\alpha+\frac{1}{2}(|\beta|+1)}}\Big(\frac{1}{R} + \frac{1}{Rt^{1-\frac{1}{p}}} + \frac{\|\omega_0^{R/16}\|}{t^{\frac{3}{2}-\frac{1}{p}}}\Big),
\end{aligned} \tag{4.24}$$

where

$$\begin{aligned}
&L_{10}(\alpha,|\beta|,p,\|\omega_0\|) \\
&= \sum_{\eta\leq\alpha}\sum_{\gamma\leq\beta}\binom{\alpha}{\eta}\binom{\beta}{\gamma}N_u(\eta,|\gamma|,\infty)\|\omega_0\|K(\alpha-\eta,|\beta-\gamma|+1,p,\|\omega_0\|).
\end{aligned}$$

Substituting (4.23) and (4.24) into (4.22), we get the following estimate,

$$\|\partial_t^{\alpha+1}\partial_x^\beta \omega^R(t,\cdot)\|_p \leq \frac{K(\alpha+1,|\beta|,p,\|\omega_0\|)}{t^{\alpha+1+\frac{1}{2}(|\beta|-1)}}\Big(\frac{1}{R} + \frac{1}{Rt^{1-\frac{1}{p}}} + \frac{\|\omega_0^{R/16}\|}{t^{\frac{3}{2}-\frac{1}{p}}}\Big), \tag{4.25}$$

with

$$K(\alpha+1,|\beta|,p,\|\omega_0\|) = K(\alpha,|\beta|+2,p,\|\omega_0\|) + L_{10}(\alpha,|\beta|,p,\|\omega_0\|). \tag{4.26}$$

It follows from (4.25) and (4.26) that the estimate (4.20) is valid for $\alpha+1$ and any $\beta \in Z^+ \times Z^+$. Thus by induction this theorem is proved. $\square$

5. Attraction of Oseen Vortex and Velocity Solutions

As an application of the obtained derivative estimates of the 2D vorticity and velocity solutions, we shall sketch the proof of the following theorem, which shows that the temporal-spatial derivatives of the 2D vorticity and velocity solutions on $\mathbb{R}^2$ behave as the derivatives of the same order of the corresponding Oseen solutions asymptotically in the sense of $L^p (p \geq 1)$ and $L^q (q \geq 2)$ convergence, respectively.

Theorem 5.1. *Let $\omega(t,x)$ be the vorticity solution of (1.2)–(1.3) with initial vorticity $\omega_0 \in L^1(\mathbb{R}^2)$. Let $\omega^*(t,x)$ be the Oseen vortex (1.5) with $\kappa = \kappa(\omega_0)$ given in (1.8). Then for any $\alpha \in Z^+$ and any $\beta \in Z^+ \times Z^+$, it hold that*

$$\lim_{t\to\infty} t^{\alpha+\frac{|\beta|}{2}+1-\frac{1}{p}} \|\partial_t^\alpha \partial_x^\beta \omega(t,\cdot) - \partial_t^\alpha \partial_x^\beta \omega^*(t,\cdot)\|_p = 0, \quad p \in [1,\infty], \tag{5.1}$$

and

$$\lim_{t\to\infty} t^{\alpha+\frac{|\beta|}{2}+\frac{1}{2}-\frac{1}{q}} \|\partial_t^\alpha \partial_x^\beta u(t,\cdot) - \partial_t^\alpha \partial_x^\beta u^*(t,\cdot)\|_q = 0, \quad q \in (2,\infty], \tag{5.2}$$

where $u(t,x)$ and $u^*(t,x)$ are the velocity solutions given by (1.3) corresponding to $\omega(t,x)$ and the Oseen vortex $\omega^*(t,x)$, respectively.

The proof of theorem 5.1 uses the shown estimates and takes the approach of the parabolic similarity [10, 16, 9, 38] and the compactness processing in L^p spaces.

Let (ω, u) be a vorticity-velocity solution of (1.2)–(1.3) with any initial vorticity $\omega_0 \in L^1(\mathbb{R}^2)$. For any scalar constant $\lambda > 0$, we define

$$\begin{aligned} \omega_\lambda(t,x) &= \lambda^2 \omega(\lambda^2 t, \lambda x), \\ u_\lambda(t,x) &= \lambda u(\lambda^2 t, \lambda x), \\ \omega_{0,\lambda}(x) &= \lambda^2 \omega_0(\lambda x) \ (= \omega_\lambda(0,x)\ ). \end{aligned} \tag{5.3}$$

For any $\lambda > 0, t > 0, x \in \mathbb{R}^2$, it is easy to check that $(\omega_\lambda, u_\lambda)$ satisfies the vorticity equation (1.2)–(1.3) and that

$$\begin{gathered} \|\omega_{0,\lambda}\| = \|\omega_0\|, \quad \kappa(\omega_{0,\lambda}) = \kappa(\omega_0), \\ \omega_\lambda^*(t,x) = \omega^*(t,x), \quad u_\lambda^*(t,x) = u^*(t,x). \end{gathered} \tag{5.4}$$

Since

$$\partial_t^\alpha \partial_x^\beta \omega_\lambda(t,x) - \partial_t^\alpha \partial_x^\beta \omega^*(t,x) = \lambda^{2+2\alpha+|\beta|}[\partial_t^\alpha \partial_x^\beta \omega(\lambda^2 t, \lambda x) - \partial_t^\alpha \partial_x^\beta \omega^*(\lambda^2 t, \lambda x)],$$

we find that

$$\|\partial_t^\alpha \partial_x^\beta \omega_\lambda(1,\cdot) - \partial_t^\alpha \partial_x^\beta \omega^*(1,\cdot)\|_p = \lambda^{2+2\alpha+|\beta|-\frac{2}{p}} \|\partial_t^\alpha \partial_x^\beta \omega_\lambda(\lambda^2,\cdot) - \partial_t^\alpha \partial_x^\beta \omega^*(\lambda^2,\cdot)\|_p. \tag{5.5}$$

Taking $t = \lambda^2$, we have the *equivalence* that (5.1) holds for $p \in [1,\infty]$ if and only if

$$\lim_{\lambda\to\infty} \|\partial_t^\alpha \partial_x^\beta \omega_\lambda(1,\cdot) - \partial_t^\alpha \partial_x^\beta \omega^*(1,\cdot)\|_p = 0, \quad \text{for } p \in [1,\infty]. \tag{5.6}$$

Without confusion $\partial_t^\alpha \partial_x^\beta w(1,x)$ means $(\partial_t^\alpha \partial_x^\beta w(t,x))|_{t=1}$, for $w = \omega_\lambda, \omega^*, u_\lambda$, or u^*.

Below we prove (5.6), which is equivalent to (5.1). The proof involves the following lemma, which crucially relies on the exterior decay estimates of temporal-spatial derivatives of the vorticity solutions shown in Theorem 4.2 and the Riesz criterion [1, Theorem 2.21] of compactness of subsets in $L^p(\mathbb{R}^n)$.

Lemma 5.2. *The set of functions $\{\omega_\lambda(1,x)\}_{\lambda\geq 1}$ defined in (5.3) is a precompact set in the Sobolev space $W^{|\beta|,p}(\mathbb{R}^2)$, for any $p \in [1,\infty]$ and any integer $|\beta| \geq 0$.*

Proof. First let $p \in [1, \infty)$. Let $\omega_0 \in L^1(\mathbb{R}^2)$ be arbitrarily given. In order to prove this lemma, it suffices to show that for any given integer $|\beta| = k$,

$$\{\partial_x^\gamma \omega_\lambda(1,x)\}_{\lambda \geq 1} \text{is precompact in } L^p(\mathbb{R}^2), \text{ for all } 0 \ \leq \gamma \leq \beta. \tag{5.7}$$

By the Riesz criterion of compactness, we just check the following three conditions. Let $\gamma \leq \beta$.

(a) $\{\partial_x^\gamma \omega_\lambda(1,x)\}_{\lambda \geq 1}$ is a bounded subset in $L^p(\mathbb{R}^2)$. In fact, (3.4) and (5.4) yield

$$\|\partial_x^\gamma \omega_\lambda(1,\cdot)\|_p \leq N_\omega(0,|\gamma|,p)\|\omega_{0,\lambda}\| = N_\omega(0,|\gamma|,p)\|\omega_0\| \leq \max_{\gamma \leq \beta}\{N_\omega(0,|\gamma|,p)\}\|\omega_0\|,$$

for all $\lambda \geq 1$, which confirms this assertion.

(b) The uniform convergence to zero at infinity for $\{\partial_x^\gamma \omega_\lambda(1,x)\}_{\lambda \geq 1}$ is attested by applying Theorem 4.1 and (4.1) to $\omega_\lambda(t,\cdot)$ with $t = 1$ and choosing $R > 1$ to be sufficiently large such that

$$B(|\gamma|,p,\|\omega_0\|)\Big(\frac{2}{R} + \|\omega_0^{R/16}\|\Big) < \varepsilon. \tag{5.8}$$

(c) The equicontinuity of $\{\partial_x^\gamma \omega_\lambda(1,x)\}_{\lambda \geq 1}$ in $L^p(\mathbb{R}^2)$ is shown below. By the formula (4.2) on the time interval $[\frac{1}{2}, 1]$ for the vorticity solutions $\omega_\lambda(t,x)$ and $\omega_\lambda(t,x-y)$, where y is a parameter, we need to estimate

$$\begin{aligned} \|\partial_x^\gamma \omega_\lambda(1,\cdot - y) - \partial_x^\gamma \omega_\lambda(1,\cdot)\|_p &\leq \|\partial_x^\gamma e^{A(1/2)}(\omega_\lambda(1/2,\cdot - y) - \omega_\lambda(1/2,\cdot))\|_p \\ &\quad + \int_{1/2}^1 \|\partial_x^\gamma e^{A(1-s)}(v_\lambda(s,\cdot - y) - v_\lambda(s,\cdot))\|_p \, ds, \end{aligned} \tag{5.9}$$

where, as in (3.12), $v_\lambda(t,x) = (u_\lambda(t,x) \cdot \nabla)\omega_\lambda(t,x)$ and $v_\lambda(t,x-y) = (u_\lambda(t,x-y) \cdot \nabla)\omega_\lambda(t,x-y)$. We shall estimate the two terms in (5.9) respectively.

(i) First we estimate the linear term on the right-hand side of (5.9). Let

$$E(x) = E(x_1.x_2) = \frac{1}{2\pi\nu}\exp(-\frac{|x|^2}{2\nu}), \quad x \in \mathbb{R}^2.$$

According to (3.3) and by variable changes, we have

$$\begin{aligned} &\|\partial_x^\gamma e^{A(1/2)}(\omega_\lambda(1/2,\cdot - y) - \omega_\lambda(1/2,\cdot))\|_p \\ &= \Big\|\frac{1}{2\pi\nu}\int_{\mathbb{R}^2} \partial_x^\gamma \exp\big(-\frac{|x-\xi|^2}{2\nu}\big)(\omega_\lambda(1/2,\xi - y) - \omega_\lambda(1/2,\xi))\,d\xi\Big\|_p \\ &\text{(by the variable change } \zeta = \xi - x) \\ &= \Big\|\frac{1}{2\pi\nu}\int_{\mathbb{R}^2} \exp\big(-\frac{|\zeta|^2}{2\nu}\big)\partial_\zeta^\gamma(\omega_\lambda(1/2,\zeta + x - y) - \omega_\lambda(1/2,\zeta + x))\,d\zeta\Big\|_p \\ &\text{(by the variable changes } \eta = \zeta + x - y \text{ and } \eta = \zeta + x, \text{ respectively)} \\ &= \Big\|\frac{1}{2\pi\nu}\int_{\mathbb{R}^2} \big[\exp\big(-\frac{|x-y-\eta|^2}{2\nu}\big) - \exp\big(-\frac{|x-\eta|^2}{2\nu}\big)\big]\partial_\eta^\gamma \omega_\lambda(1/2,\eta)\,d\eta\Big\|_p \\ &= \|(E(x-y) - E(x)) * (\partial_x^\gamma \omega_\lambda(1/2,x))\|_p. \end{aligned} \tag{5.10}$$

By the convolution property (1.10), (3.4), and (5.4), it follows that

$$\begin{aligned} &\|(E(x-y) - E(x)) * (\partial_x^\gamma \omega_\lambda(1/2,x))\|_p \\ &\leq \|E(x-y) - E(x)\|N_\omega(0,|\gamma|,p)2^{|\gamma|/2}\|\omega_0\|. \end{aligned}$$

Let ε be arbitrarily given. By the continuity of integrable functions with respect to the L^1-norm [37, Chapter VIII, Proposition 1.4], here for the function $E(x)$ there exists a uniform $\delta_1(\varepsilon) > 0$ independent of $\lambda \geq 1$, such that if $|y| < \delta_1$, then

$$\begin{aligned} &\|\partial_x^\gamma e^{A(1/2)}(\omega_\lambda(1/2, \cdot - y) - \omega_\lambda(1/2, \cdot))\|_p \\ &\leq \|E(x-y) - E(x)\| N_\omega(0, |\gamma|, p) 2^{|\gamma|/2} \|\omega_0\| < \frac{\varepsilon}{2}. \end{aligned} \tag{5.11}$$

(ii) Now we estimate the nonlinear integral term in (5.9). Let

$$F(x,s) = \frac{1}{4\pi\nu(1-s)} \exp\Big(-\frac{|x|^2}{4\nu(1-s)}\Big), \quad 0 < s < 1.$$

By variable changes similar to what we did in deriving (5.10), we have

$$\begin{aligned} &\int_{1/2}^1 \|\partial_x^\gamma e^{A(1-s)}(v_\lambda(s, \cdot - y) - v_\lambda(s, \cdot))\|_p \, ds \\ &\leq \int_{1-d}^1 (\|F(\cdot - y, s)\| + \|F(\cdot, s)\|) \|\partial_x^\gamma v_\lambda(s, \cdot)\|_p \, ds \\ &\quad + \int_{1/2}^{1-d} \|F(\cdot - y, s) - F(\cdot, s)\| \, \|\partial_x^\gamma v_\lambda(s, \cdot)\|_p \, ds \end{aligned} \tag{5.12}$$

where $0 < d < 1/2$. By (3.4), (3.5), (5.4), and since $s \geq 1/2$ in (5.12), we get

$$\begin{aligned} &\|\partial_x^\gamma v_\lambda(s,x)\|_p \\ &\leq \sum_{\mu \leq \gamma} \binom{\gamma}{\mu} \|(\partial_x^\mu u_\lambda(s,x)) \partial_x^{\gamma-\mu}(\nabla \omega_\lambda(s,x))\|_p \\ &\leq \sum_{\mu \leq \gamma} \binom{\gamma}{\mu} N_u(0, |\mu|, \infty) s^{-(\frac{|\mu|}{2} + \frac{1}{2})} \|\omega_{0,\lambda}\| N_\omega(0, |\gamma - \mu| + 1, p) s^{-\frac{|\gamma-\mu|+1}{2}} \|\omega_{0,\lambda}\| \\ &\leq \sum_{\mu \leq \gamma} \binom{\gamma}{\mu} 2^{\frac{|\gamma|}{2}+1} N_u(0, |\mu|, \infty) N_\omega(0, |\gamma - \mu| + 1, p) \|\omega_0\|^2 \\ &= D(|\gamma|, p, \|\omega_0\|), \end{aligned}$$

where $D(|\gamma|, p, \|\omega_0\|)$ is a uniform constant independent of $\lambda \geq 1$. For the arbitrarily given $\varepsilon > 0$ mentioned in (i), there exists a $d(\varepsilon) \in (0, 1/2)$ such that $d(\varepsilon) D(|\gamma|, p, \|\omega_0\|) < \varepsilon/8$, which yields

$$\int_{1-d}^1 (\|F(\cdot - y, s)\| + \|F(\cdot, s)\|) \|\partial_x^\gamma v_\lambda(s, \cdot)\|_p \, ds \leq 2d(\varepsilon) D(|\gamma|, p, \|\omega_0\|) < \frac{\varepsilon}{4}, \tag{5.13}$$

due to $\|F(\cdot - y, s)\| = \|F(\cdot, s)\| = 1$. Moreover, we have

$$\begin{aligned} &\|F(x-y,s) - F(x,s)\| \\ &= \int_{\mathbb{R}^2} \frac{1}{4\pi\nu(1-s)} \Big| \exp\Big(-\frac{|x-y|^2}{4\nu(1-s)}\Big) - \exp\Big(-\frac{|x-y|^2}{4\nu(1-s)}\Big) \Big| dx \\ &= \int_{\mathbb{R}^2} \frac{1}{\pi} \big| \exp(-|\widetilde{x} - \widetilde{y}|^2) - \exp(-|\widetilde{x}|^2) \big| \, d\widetilde{x}, \end{aligned}$$

where

$$\widetilde{x} = \frac{x}{\sqrt{4\nu(1-s)}} \quad \text{and} \quad \widetilde{y} = \frac{y}{\sqrt{4\nu(1-s)}}.$$

By the continuity of the integrable function $\exp(-|\widetilde{x}|^2)$ with respect to the L^1-norm, for the arbitrarily given $\varepsilon > 0$ there exists a uniform $\delta_2(\varepsilon) > 0$ independent of $\lambda \geq 1$, such that if

$$\frac{|y|}{\sqrt{4\nu(1-s)}} = |\widetilde{y}| < \delta_2(\varepsilon), \quad \text{for any } s \in [1/2, 1-d(\varepsilon)], \tag{5.14}$$

then

$$\|F(x-y,s) - F(x,s)\| < \frac{\varepsilon}{2D(|\gamma|, p, \|\omega_0\|)},$$

so that

$$\int_{1/2}^{1-d} \|F(\cdot - y, s) - F(\cdot, s)\| \, \|\partial_x^\gamma v_\lambda(s,\cdot)\|_p \, ds < \frac{\varepsilon}{4}. \tag{5.15}$$

Note that (5.14) holds if $|y| < \sqrt{4\nu d(\varepsilon)}\delta_2(\varepsilon)$. ¿From (5.12), (5.13), and (5.15), we have shown that if $|y| < \sqrt{4\nu d(\varepsilon)}\delta_2(\varepsilon)$, then

$$\int_{1/2}^{1} \|\partial_x^\gamma e^{A(1-s)}(v_\lambda(s, \cdot - y) - v_\lambda(s,\cdot))\|_p < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}\,. \tag{5.16}$$

In turn by (5.9), (5.11) and (5.16), there exists a uniform constant

$$\delta = \delta(\varepsilon) = \min\{\delta_1(\varepsilon), \sqrt{4\nu d(\varepsilon)}\delta_2(\varepsilon)\},$$

which is independent of $\lambda \geq 1$, such that if $|y| < \delta$, then

$$\|\partial_x^\gamma \omega_\lambda(1, \cdot - y) - \partial_x^\gamma \omega_\lambda(1,\cdot)\|_p < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \tag{5.17}$$

Therefore, the equicontinuity of $\{\partial_x^\gamma \omega_\lambda(1,x)\}_{\lambda \geq 1}$ in $L^p(\mathbb{R}^2)$ is proved.

By the Riesz criterion, we have proved the statement (5.7) and consequently $\{\omega_\lambda(1,x)\}_{\lambda\geq 1}$ is a precompact set in the space $W^{|\beta|,p}(\mathbb{R}^2)$ for all $\beta \in Z^+ \times Z^+$ and for $p \in [1,\infty)$. For $p = \infty$, by using Ascoli-Arzelà Theorem, we can also prove that $\{\omega_\lambda(1,x)\}_{\lambda\geq 1}$ is a precompact set in the space $W^{|\beta|,\infty}(\mathbb{R}^2)$ for all $\beta \in Z^+ \times Z^+$. □

Proof of Theorem 5.1. By the result shown in Lemma 5.2, for $p \in [1,\infty]$, and for any $\beta \in Z^+ \times Z^+, \{\omega_\lambda(1,x)\}_{\lambda\geq 1}$ is precompact in $W^{|\beta|,p}(\mathbb{R}^2)$. Thus there exist a sequence $\{\omega_{\lambda_\ell}(1,x)\}_{\ell=1}^\infty$ of $\{\omega_\lambda(1,x)\}_{\lambda\geq 1}$, where $\lambda_\ell < \lambda_{\ell+1} \to \infty$ as $\ell \to \infty$, and a function $h(x) \in W^{|\beta|,p}(\mathbb{R}^2)$ such that

$$\lim_{\ell\to\infty} \omega_{\lambda_\ell}(1,\cdot) = h(\cdot) \quad \text{in } W^{|\beta|,p}(\mathbb{R}^2). \tag{5.18}$$

It has been shown (as mentioned in Section 1) that

$$\lim_{t\to\infty} t^{1-\frac{1}{p}} \|\omega(t,\cdot) - \omega^*(t,\cdot)\|_p = 0,$$

which implies

$$\lim_{\ell\to\infty} \|\omega_{\lambda_\ell}(1,\cdot) - \omega^*(1,\cdot)\|_p = 0 \tag{5.19}$$

due to the equivalence of (5.1) and (5.6) here for $\alpha = 0, |\beta| = 0$. By the uniqueness of limit in $L^p(\mathbb{R}^2)$, (5.18) and (5.19) imply that

$$h(x) = \omega^*(1,x) \quad \text{in } L^p(\mathbb{R}^2) \text{ and also in } W^{|\beta|,p}(\mathbb{R}^2). \tag{5.20}$$

From (5.18) and (5.20) it follows that

$$\lim_{\ell\to\infty} \|\partial_x^\beta \omega_{\lambda_\ell}(1,\cdot) - \partial_x^\beta \omega^*(1,\cdot)\|_p = 0. \tag{5.21}$$

Since any sequence itself of $\{\omega_\lambda(1,\cdot)\}_{\lambda\geq 1}$ contains a subsequence convergent to the same limit $\omega^*(1,\cdot)$ in $W^{|\beta|,p}(\mathbb{R}^2)$, we have proved that (5.6) holds for $\alpha = 0$, namely,

$$\lim_{\lambda\to\infty} \|\partial_x^\beta \omega_\lambda(1,\cdot) - \partial_x^\beta \omega^*(1,\cdot)\|_p = 0 \text{ for any } \beta \in Z^+ \times Z^+,\ p \in [1,\infty]. \tag{5.22}$$

Note that (5.22) is stronger than the sequence limit (5.21).

Now we take the bootstrap approach to prove (5.6) and equivalently (5.1) for any $\alpha \geq 1$, based on the result (5.22). Suppose that (5.6) is valid for an $\alpha \geq 0$ with any $\beta \in Z^+ \times Z^+$ and $p \in [1,\infty]$. From (1.2) we have

$$\begin{aligned}
\partial_t^{\alpha+1}\partial_x^\beta \omega_\lambda &= \partial_t^\alpha \partial_x^\beta \Delta\omega_\lambda - \partial_t^\alpha\partial_x^\beta (u_\lambda \cdot \nabla)\omega_\lambda, \\
\partial_t^{\alpha+1}\partial_x^\beta \omega^* &= \partial_t^\alpha \partial_x^\beta \Delta\omega^* - \partial_t^\alpha\partial_x^\beta (u^* \cdot \nabla)\omega^*.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\|\partial_t^{\alpha+1}\partial_x^\beta \omega_\lambda(1,\cdot) - \partial_t^{\alpha+1}\partial_x^\beta \omega^*(1,\cdot)\|_p \\
&\leq \|\partial_t^\alpha\partial_x^\beta(\Delta\omega_\lambda(1,\cdot) - \Delta\omega^*(1,\cdot))\|_p + \|\partial_t^\alpha\partial_x^\beta(v_\lambda(1,\cdot) - v^*(1,\cdot))\|_p
\end{aligned} \tag{5.23}$$

where $v_\lambda(t,x) = (u_\lambda(t,x)\cdot\nabla)\omega_\lambda(t,x)$ and $v^*(t,x) = (u^*(t,x)\cdot\nabla)\omega^*(t,x)$. By the assumption of induction, we readily have

$$\lim_{\lambda\to\infty} \|\partial_t^\alpha\partial_x^\beta(\Delta\omega_\lambda(1,\cdot) - \Delta\omega^*(1,\cdot))\|_p = 0. \tag{5.24}$$

Moreover,

$$\begin{aligned}
&\|\partial_t^\alpha\partial_x^\beta(v_\lambda(1,\cdot) - v^*(1,\cdot))\|_p \\
&\leq \sum_{\theta\leq\alpha}\sum_{\gamma\leq\beta}\binom{\alpha}{\theta}\binom{\beta}{\gamma}(\|\partial_t^\theta\partial_x^\gamma(u_\lambda(1,\cdot) - u^*(1,\cdot))\|_\infty \|\partial_t^{\alpha-\theta}\partial_x^{\beta-\gamma}\nabla\omega_\lambda(1,\cdot)\|_p \\
&\quad + \|\partial_t^\theta\partial_x^\gamma u^*(1,\cdot)\|_\infty \|\partial_t^{\alpha-\theta}\partial_x^{\beta-\gamma}\nabla(\omega_\lambda(1,\cdot) - \omega^*(1,\cdot))\|_p).
\end{aligned} \tag{5.25}$$

Since $\|\partial_t^\theta\partial_x^\gamma u^*(1,\cdot)\|_\infty$ is a constant and by the assumption of induction we obtain

$$\lim_{\lambda\to\infty} \|\partial_t^{\alpha-\theta}\partial_x^{\beta-\gamma}\nabla(\Delta\omega_\lambda(1,\cdot) - \Delta\omega^*(1,\cdot))\|_p = 0\,.$$

It follows that

$$\lim_{\lambda\to\infty}\sum_{\theta\leq\alpha}\sum_{\gamma\leq\beta}\binom{\alpha}{\theta}\binom{\beta}{\gamma}\|\partial_t^\theta\partial_x^\gamma u^*(1,\cdot)\|_\infty \|\partial_t^{\alpha-\theta}\partial_x^{\beta-\gamma}\nabla(\omega_\lambda(1,\cdot) - \omega^*(1,\cdot))\|_p = 0\,. \tag{5.26}$$

For another part in (5.25), note that (3.4) and (5.4) yield

$$\|\partial_t^{\alpha-\theta}\partial_x^{\beta-\gamma}\nabla\omega_\lambda(1,\cdot)\|_p \leq N_\omega(\alpha-\theta, |\beta-\gamma|+1, p)\|\omega_0\|. \tag{5.27}$$

The rest issue is the convergence of $\|\partial_t^\theta\partial_x^\gamma(u_\lambda(1,\cdot) - u^*(1,\cdot))\|_\infty$. By the Sobolev imbedding theorem, there is a uniform constant $C_0 > 0$ such that $\|\varphi\|_\infty \leq C_0\|\varphi\|_{H^2}$, where H^2 stands for the Sobolev space $H^2(\mathbb{R}^2)$. By the Biot-Savart law, we get

$$\begin{aligned}
&\|\partial_t^\theta \partial_x^\gamma (u_\lambda(1,\cdot) - u^*(1,\cdot))\|_\infty \\
&\le C_0 \|\partial_t^\theta \partial_x^\gamma (u_\lambda(1,\cdot) - u^*(1,\cdot))\|_{H^2} \\
&\le C_0 \sum_{|\rho|\le 2} \|\partial_t^\theta \partial_x^{\gamma+\rho} (u_\lambda(1,\cdot) - u^*(1,\cdot))\|_2 \\
&= \frac{C_0}{2\pi} \sum_{|\rho|\le 2} \Big\| \int_{\mathbb{R}^2} \partial_t^\theta \partial_x^{\gamma+\rho} \frac{(x-y)^\perp}{|x-y|^2} (\omega_\lambda(1,y) - \omega^*(1,y))\, dy \Big\|_2 \\
&\text{(by a variable changes as in (5.10))} \\
&= \frac{C_0}{2\pi} \sum_{|\rho|\le 2} \Big\| \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \partial_t^\theta \partial_y^{\gamma+\rho} (\omega_\lambda(1,y) - \omega^*(1,y))\, dy \Big\|_2 \\
&\le \frac{C_0}{2\pi} \sum_{|\rho|\le 2} \Big\| \int_{\mathbb{R}^2} \frac{1}{|x-y|} \left|\partial_t^\theta \partial_y^{\gamma+\rho} (\omega_\lambda(1,y) - \omega^*(1,y))\right| dy \Big\|_2 \\
&\le \frac{C_0}{2\pi} \sum_{|\rho|\le 2} S_{2,1,2} \|\partial_t^\theta \partial_x^{\gamma+\rho} (\omega_\lambda(1,x) - \omega^*(1,x))\| \longrightarrow 0, \quad \text{as } \lambda \to \infty,
\end{aligned} \tag{5.28}$$

by the assumption of induction and the Hardy-Littlewood-Sobolev inequality (1.11) here with $n = 2, a = 2, p = 1, q = 2$. Thus we can combine (5.27) and (5.28) to get

$$\lim_{\lambda\to\infty} \sum_{\theta\le\alpha} \sum_{\gamma\le\beta} \binom{\alpha}{\theta} \binom{\beta}{\gamma} \|\partial_t^\theta \partial_x^\gamma (u_\lambda(1,\cdot) - u^*(1,\cdot))\|_\infty \|\partial_t^{\alpha-\theta} \partial_x^{\beta-\gamma} \nabla\omega_\lambda(1,\cdot)\|_p = 0. \tag{5.29}$$

Finally, substituting (5.26) and (5.29) into (5.25), we obtain

$$\lim_{\lambda\to\infty} \|\partial_t^\alpha \partial_x^\beta (v_\lambda(1,\cdot) - v^*(1,\cdot))\|_p = 0\,. \tag{5.30}$$

Then (5.23), (5.24) and (5.30) altogether imply that (5.6) holds for $\alpha + 1$; i.e.,

$$\lim_{\lambda\to\infty} \|\partial_t^{\alpha+1} \partial_x^\beta (\omega_\lambda(1,\cdot) - \omega^*(1,\cdot))\|_p = 0, \quad \beta \in Z^+ \times Z^+,\ p \in [1,\infty],$$

and equivalently (5.1) holds for $\alpha + 1$ and any $\beta \in Z^+ \times Z^+,\ p \in [1,\infty]$. This bootstrap induction shows that the convergence statement (5.1) is valid.

By the Biot-Savart law and utilizing the corresponding equivalence of (5.2) and the following statement,

$$\lim_{\lambda\to\infty} \|\partial_t^\alpha \partial_x^\beta (u_\lambda(1,\cdot) - u^*(1,\cdot))\|_q = 0, \quad \alpha \ge 0,\ \beta \in Z^+ \times Z^+,\ q \in (2,\infty],$$

we can also prove that (5.2) is valid. The detail is omitted. □

Remark. For the three-dimensional Navier-Stokes equation [34, Chapter 6], the global existence and regularity of strong solutions is an extremely challenging open problem, that is defined by the Clay Mathematics Institute as one of the century problems in mathematics for the new millennium. Nevertheless, the study of 3D vorticity plays an important role. In this regard, one of the major differences between 2D and 3D vorticity behavior arises from the fact that the 3D vorticity is governed by a different evolutionary equation:

$$\frac{\partial\omega}{\partial t} = \nu\Delta\omega - (u\cdot\nabla)\omega + (\omega\cdot\nabla)u, \quad t > 0,\ x \in \mathbb{R}^3, \tag{5.31}$$

where the additional nonlinear term $(\omega \cdot \nabla)u$ known as the vorticity stretching amplifies the vorticity strength by the interaction with the gradient of velocity field. Therefore, the dynamics, regularity, and asymptotics of the three-dimensional vorticity evolution are much more complicated than the two-dimensional case.

The extension of Oseen vortices in the three-dimensional case is Burgers vortices. A Burgers vortex is an exact solution of the 3D Navier-Stokes equation that is a superposition of a background irrotational strain field/flow and a 2D vortex, just like the Oseen vortex, in the plane perpendicular to the strain axis. In recent years some results from the ongoing researches are emerging in regard to the local stability and asymptotic structures of the 3D Burgers vortices. In [17] it has been shown that the whole family of axisymmetric and non-axisymmetric Burgers vortex solutions parametrized by the total circulation numbers of the 3D Navier-Stokes equation is locally stable for small Reynolds numbers. Based on several numerical simulation results of turbulent fluid flows shown in [35] and others, very recently in [26] the existence of Burgers vortices as the swirling, tube-like structures in 3D turbulence is rigorously proved for high Reynolds numbers with asymmetric parameters less than one half. However, the stability of the 3D Burgers vortices for high Reynolds numbers is still not known even though some numerical studies indicate some kind of tendency of persistence possibly toward stability.

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