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MULTIPLE SOLUTIONS FOR QUASILINEAR ELLIPTIC PROBLEMS WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. Using a recent result by Bonanno [2], we obtain a multiplicity result for the quasilinear elliptic problem

$$\begin{gathered}-\Delta_p u + |u|^{p-2}u = \lambda f(u) \quad \text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \mu g(u) \quad \text{on } \partial\Omega,\end{gathered}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 3$ with smooth boundary $\partial\Omega$, $\frac{\partial}{\partial \nu}$ is the outer unit normal derivative, the functions f, g are $(p-1)$-sublinear at infinity $(1 < p < N)$, λ and μ are positive parameters.

1. Introduction and Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 3$ with smooth boundary $\partial\Omega$ and a constant p with $1 < p < N$. In this paper, we consider the quasilinear elliptic problems

$$\begin{gathered}-\Delta_p u + |u|^{p-2}u = \lambda f(u) \quad \text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \mu g(u) \quad \text{on } \partial\Omega.\end{gathered} \tag{1.1}$$

Such problems were studied in many works, for example [1, 3, 4, 5, 6]. In [3], Bonder studied the problem in the case: $f \equiv 0$ and g is a sign-changing Carathéodory function. Then, using the variational techniques in [8] the author obtained at least two solutions in the space $W^{1,p}(\Omega)$ provided that μ is large enough. In [4], the author considered a more general situation, where the functions f, g are involved, but not the parameters λ and μ. Using the Lusternik - Schnirelman method for non-compact manifolds, the author showed the existence of at least three solutions, and the sign of the solutions are also well-defined. We also find that the lower and upper solutions and variational methods were combined with together in [1] to obtain multiplicity results for the problems of (1.1) type. Finally, in the papers [5], [6] and [12], existence results of infinitely many solutions were investigated and the corresponding Neumann problems involving the $p(x)$-Laplacian operator were also studied in [7] and [11]. In the present paper, we are interested in the case: the functions f, g are $(p-1)$-sublinear at infinity. Hence, our main ingredient is a

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recent critical point result due to G. Bonanno [2]. Using this interesting result we show that problem (1.1) has at least two nontrivial solutions provided that λ and μ are suitable. In order to state our main result we introduce some hypotheses.

We assume that the functions f and $g : \mathbb{R} \to \mathbb{R}$ satisfy the following conditions:

(H1) There exist constants $M_1, M_2 > 0$ such that for all $t \in \mathbb{R}^N$,
$$|f(t)| \le M_1(1 + |t|^{p-1}), \quad |g(t)| \le M_2|t|^{p-1};$$
(H2) f is superlinear at zero; i.e.,
$$\lim_{t\to 0} \frac{f(t)}{|t|^{p-1}} = 0;$$
(H3) if we set $F(t) = \int_0^t f(t)dt$ and $G(t) = \int_0^t g(t)dt$, then there exists $t_0 \in \mathbb{R}$ such that
$$F(t_0) = \int_0^{t_0} f(t)dt > 0 \quad \text{or} \quad G(t_0) = \int_0^{t_0} g(t)dt > 0.$$

Let $W^{1,p}(\Omega)$ be the usual Sobolev space with respect to the norm
$$\|u\|_{1,p}^p = \int_\Omega (|\nabla u|^p + |u|^p)dx$$
and $W_0^{1,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$. For any $1 < p < N$ and $1 \le q \le p^\star = \frac{Np}{N-p}$, we denote by $S_{q,\Omega}$ the best constant in the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ and for all $1 \le q \le p_\star = \frac{(N-1)p}{N-p}$, we also denote by $S_{q,\partial\Omega}$ the best constant in the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$, i.e.
$$S_{q,\partial\Omega} = \inf_{u\in W^{1,p}(\Omega)\setminus W_0^{1,p}(\Omega)} \frac{\int_\Omega(|\nabla u|^p + |u|^p)dx}{\big(\int_{\partial\Omega} |u|^q d\sigma\big)^{p/q}}.$$
Moreover, if $1 \le q < p^\star$, then the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact and if $1 \le q < p_\star$, then the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$ is compact. As a consequence, we have the existence of extremals, i.e. functions where the infimum is attained (see [3, 6]).

Definition 1.1. We say that $u \in W^{1,p}(\Omega)$ is a weak solution of problem (1.1) if and only if
$$\int_\Omega (|\nabla u|^{p-2}\nabla u \nabla\varphi + |u|^{p-2}u\varphi)dx - \lambda \int_\Omega f(u)\varphi dx - \mu \int_{\partial\Omega} g(u)\varphi d\sigma = 0$$
for all $\varphi \in W^{1,p}(\Omega)$.

Theorem 1.2. *Assuming hypotheses* (H1)–(H3) *are fulfilled then there exist an open interval Λ_μ and a constant $\delta_\mu > 0$ such that for all $\lambda \in \Lambda_\mu$, problem (1.1) has at least two weak solutions in $W^{1,p}(\Omega)$ whose $\|\cdot\|_{1,p}$-norms are less than δ_μ.*

We emphasize that the condition (H3) cannot be omitted. Indeed, if for instance $f \equiv 0$ and $g \equiv 0$, then (H1) and (H2) clearly hold, but problem (1.1) has only the trivial solution. Theorem 1.2 will be proved by using a recent result on the existence of at least three critical points by Bonanno [2] which is actually a refinement of a general principle of Ricceri (see [9, 10]). For the reader's convenience, we describe it as follows.

Lemma 1.3 (see [2, Theorem 2.1])**.** *Let $(X, \|\cdot\|)$ be a separable and reflexive real Banach space, $\mathcal{A}, \mathcal{F} : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals. Assume that there exists $x_0 \in X$ such that $\mathcal{A}(x_0) = \mathcal{F}(x_0) = 0$, $\mathcal{A}(x) \geq 0$ for all $x \in X$ and there exist $x_1 \in X$, $\rho > 0$ such that*

(i) $\rho < \mathcal{A}(x_1)$,
(ii) $\sup_{\{\mathcal{A}(x)<\rho\}} \mathcal{F}(x) < \rho \frac{\mathcal{F}(x_1)}{\mathcal{A}(x_1)}$.

Further, put

$$\overline{a} = \frac{\xi\rho}{\rho\frac{\mathcal{F}(x_1)}{\mathcal{A}(x_1)} - \sup_{\{\mathcal{A}(x)<\rho\}} \mathcal{F}(x)}, \quad \text{with } \xi > 1,$$

and assume that the functional $\mathcal{A} - \lambda\mathcal{F}$ is sequentially weakly lower semicontinuous, satisfies the Palais-Smale condition and

(iii) $\lim_{\|x\|\to\infty}[\mathcal{A}(x) - \lambda\mathcal{F}(x)] = +\infty$ *for every* $\lambda \in [0, \overline{a}]$.

Then, there exist an open interval $\Lambda \subset [0, \overline{a}]$ and a positive real number δ such that each $\lambda \in \Lambda$, the equation

$$D\mathcal{A}(u) - \lambda D\mathcal{F}(u) = 0$$

has at least three solutions in X whose $\|\cdot\|$-norms are less than δ.

2. Multiple solutions

Throughout this section, we suppose that all assumptions of Theorem 1.2 are satisfied. For λ and $\mu \in \mathbb{R}$, we define the functional $\Phi_{\mu,\lambda} : W^{1,p}(\Omega) \to \mathbb{R}$ by

$$\Phi_{\mu,\lambda}(u) = \mathcal{I}_\mu(u) - \lambda\mathcal{J}(u) \text{ for all } u \in W_0^{1,p}(\Omega),$$

where

$$\mathcal{I}_\mu(u) = \int_\Omega (|\nabla u|^p + |u|^p)dx - \mu\int_{\partial\Omega} G(u)d\sigma, \quad \mathcal{J}(u) = \int_\Omega F(u)dx \tag{2.1}$$

with $F(t) = \int_0^t f(t)dt$ and $G(t) = \int_0^t g(t)dt$.

A simple computation implies that the functional $\Phi_{\mu,\lambda}$ is of C^1-class and hence weak solutions of (1.1) correspond to the critical points of $\Phi_{\mu,\lambda}$. To prove Theorem 1.2, we shall apply Lemma 1.3 by choosing $X = W^{1,p}(\Omega)$ as well as $\mathcal{A} = \mathcal{I}_\mu$ and $\mathcal{F} = \mathcal{J}$ as in (2.1). Now, we shall check all assumptions of Lemma 1.3. For each $\mu \in [0, \frac{pS_{p,\partial\Omega}}{M_2})$ we have $\mathcal{I}_\mu(u) \geq 0$ for all $u \in W^{1,p}(\Omega)$ and $\mathcal{I}_\mu(0) = \mathcal{J}(0) = 0$ since the assumption (H1) holds. Moreover, by the compact embeddings $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ and $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$, a simple computation helps us to conclude the following lemma.

Lemma 2.1. *For every $\mu \in [0, \frac{pS_{p,\partial\Omega}}{M_2})$ and all $\lambda \in \mathbb{R}$, the functional $\Phi_{\mu,\lambda}$ is sequentially weakly lower semicontinuous on $W^{1,p}(\Omega)$.*

Lemma 2.2. *There exist two positive constants $\overline{\mu}$ and $\overline{\lambda}$ such that for all $\mu \in [0, \overline{\mu})$ and all $\lambda \in [0, \overline{\lambda})$, the functional $\Phi_{\lambda,\mu}$ is coercive and satisfies the Palais-Smale condition in $W^{1,p}(\Omega)$.*

Proof. By (H1), we have

$$\begin{aligned}\Phi_{\mu,\lambda}(u) &= \int_\Omega (|\nabla u|^p + |u|^p)dx - \lambda\int_\Omega F(u)dx - \mu\int_{\partial\Omega} G(u)d\sigma \\ &\geq \|u\|_{1,p}^p - \lambda M_1\int_\Omega (|u| + \frac{|u|^p}{p})dx - \mu\frac{M_2}{p}\int_{\partial\Omega}|u|^p d\sigma \\ &\geq \|u\|_{1,p}^p\Big(1 - \lambda\frac{M_1}{pS_{p,\Omega}} - \mu\frac{M_2}{pS_{p,\partial\Omega}}\Big) - \lambda\frac{M_1}{S_{1,\Omega}}\|u\|_{1,p}. \end{aligned} \tag{2.2}$$

Since relation (2.2) holds, by choosing

$$\overline{\mu} = \overline{\lambda} = \min\big\{\frac{pS_{p,\Omega}}{2M_1}, \frac{pS_{p,\partial\Omega}}{2M_2}\big\},$$

where M_1, M_2 are given in (H1), we conclude that for all $\lambda \in [0, \overline{\lambda})$ and all $\mu \in [0, \overline{\mu})$, the functional $\Phi_{\mu,\lambda}$ is coercive.

Now, let $\{u_m\}$ be a Palais-Smale sequence for the functional $\Phi_{\mu,\lambda}$ in $W^{1,p}(\Omega)$; i.e.,

$$|\Phi_{\mu,\lambda}(u_m)| \leq \overline{c}, \quad D\Phi_{\mu,\lambda}(u_m) \to 0 \text{ in } W^{-1,p}(\Omega), \tag{2.3}$$

where $W^{-1,p}(\Omega)$ is the dual space of $W^{1,p}(\Omega)$. Since $\Phi_{\mu,\lambda}$ is coercive, the sequence $\{u_m\}$ is bounded in $W^{1,p}(\Omega)$. Therefore, there exists a subsequence of $\{u_m\}$, denoted by $\{u_m\}$ such that $\{u_m\}$ converges weakly to some $u \in W^{1,p}(\Omega)$ and hence converges strongly to u in $L^p(\Omega)$ and in $L^p(\partial\Omega)$. We shall prove that $\{u_m\}$ converges strongly to u in $W^{1,p}(\Omega)$. Indeed, we have

$$\begin{aligned}\|u_m - u\|_{1,p}^p &\leq \int_\Omega (|\nabla u_m|^{p-2}\nabla u_m - |\nabla u|^{p-2}\nabla u)(\nabla u_m - \nabla u)dx \\ &\quad + \int_\Omega (|u_m|^{p-2}u_m - |u|^{p-2}u)(u_m - u)dx \\ &= [D\Phi_{\mu,\lambda}(u_m) - D\Phi_{\mu,\lambda}(u)](u_m - u) + \lambda\int_\Omega [f(u_m) - f(u)](u_m - u)dx \\ &\quad + \mu\int_{\partial\Omega}[g(u_m) - g(u)](u_m - u)dx.\end{aligned}$$

On the other hand, the compact embeddings and (H1) imply

$$\begin{aligned}&|\int_\Omega [f(u_m) - f(u)](u_m - u)dx| \\ &\leq \int_\Omega |f(u_m) - f(u)||u_m - u|dx \\ &\leq M_1\int_\Omega (2 + |u_m|^{p-1} + |u|^{p-1})|u_m - u|dx \\ &\leq M_1(2\,\mathrm{meas}(\Omega)^{\frac{p-1}{p}} + \|u_m\|_{L^p(\Omega)}^{p-1} + \|u\|_{L^p(\Omega)}^{p-1})\|u_m - u\|_{L^p(\Omega)}^p\end{aligned}$$

which approaches 0 as $m \to \infty$. Similarly, we obtain

$$\begin{aligned}|\int_{\partial\Omega}[g(u_m) - g(u)](u_m - u)dx| &\leq \int_{\partial\Omega}|g(u_m) - g(u)||u_m - u|dx \\ &\leq M_2\int_{\partial\Omega}(|u_m|^{p-1} + |u|^{p-1})|u_m - u|dx \\ &\leq M_2(\|u_m\|_{L^p(\partial\Omega)}^{p-1} + \|u\|_{L^p(\partial\Omega)}^{p-1})\|u_m - u\|_{L^p(\partial\Omega)}^p\end{aligned}$$

which approaches zero as $m \to \infty$. Hence, by (2.3) we have $\|u_m - u\|_{1,p} \to 0$ as $m \to \infty$; i.e., the functional $\Phi_{\mu,\lambda}$ satisfies the Palais-Smale condition. □

Lemma 2.3. *For every $\mu \in [0, \overline{\mu})$ with $\overline{\mu}$ as in Lemma 2.2, we have*

$$\lim_{\rho \to 0^+} \frac{\sup\{\mathcal{J}(u) : \mathcal{I}_\mu(u) < \rho\}}{\rho} = 0.$$

Proof. Let $\lambda \in [0, \overline{\lambda})$ and $\mu \in [0, \overline{\mu})$ be fixed. By (H2), for any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that

$$|f(s)| < \epsilon p S_{p,\Omega}\Big(1 - \mu \frac{M_2}{p S_{p,\partial\Omega}}\Big)|s|^{p-1} \text{ for all } |s| < \delta.$$

We first fix $q \in (p, p^\star)$. Combining the above inequalities with (H1) we deduce that

$$|F(s)| \le \epsilon S_{p,\Omega}\Big(1 - \mu \frac{M_2}{p S_{p,\partial\Omega}}\Big)|s|^p + C_\delta |s|^q, \tag{2.4}$$

for all $s \in \mathbb{R}$, where C_δ is a constant depending on δ. Now, for every $\rho > 0$, we define the sets

$$\mathcal{B}^1_\rho = \{u \in W^{1,p}(\Omega) : \mathcal{I}_\mu(u) < \rho\}$$

and

$$\mathcal{B}^2_\rho = \big\{u \in W^{1,p}(\Omega) : \Big(1 - \mu \frac{M_2}{p S_{p,\partial\Omega}}\Big)\|u\|^p_{1,p} < \rho\big\}.$$

Then $\mathcal{B}^1_\rho \subset \mathcal{B}^2_\rho$. From (2.4) we get

$$|\mathcal{J}(u)| \le \epsilon\Big(1 - \mu \frac{M_2}{p S_{p,\partial\Omega}}\Big)\|u\|^p_{1,p} + \frac{C_\delta}{S^{\frac{q}{p}}_{q,\Omega}}\|u\|^q_{1,p}. \tag{2.5}$$

It is clear that $0 \in \mathcal{B}^1_\rho$ and $\mathcal{J}(0) = 0$. Hence, $0 \le \sup_{u \in \mathcal{B}^1_\rho} \mathcal{J}(u)$, using (2.5) we get

$$0 \le \frac{\sup_{u \in \mathcal{B}^1_\rho} \mathcal{J}(u)}{\rho} \le \frac{\sup_{u \in \mathcal{B}^2_\rho} \mathcal{J}(u)}{\rho} \le \epsilon + \frac{C_\delta}{S^{\frac{q}{p}}_{q,\Omega}}\Big(1 - \mu \frac{M_2}{p S_{p,\partial\Omega}}\Big)^{-\frac{q}{p}} \rho^{\frac{q}{p}-1}. \tag{2.6}$$

We complete the proof of the lemma by letting $\rho \to 0^+$, since $\epsilon > 0$ is arbitrary. □

Proof of Theorem 1.2 completed. Let s_0 be as in (H3). We choose a constant $r_0 > 0$ such that $r_0 < \operatorname{dist}(0, \partial\Omega)$. For each $\delta \in (0, 1)$ we define the function

$$u_\delta(x) = \begin{cases} 0, & \text{if } x \in \mathbb{R}^N \backslash B_{r_0}(0) \\ s_0, & \text{if } x \in B_{\delta r_0}(0) \\ \frac{s_0}{r_0(1-\delta)}(r_0 - |x|), & \text{if } x \in B_{r_0}(0) \backslash B_{\delta r_0}(0), \end{cases}$$

where $B_{r_0}(0)$ denotes the open ball with center 0 and radius $r_0 > 0$. Then, it is clear that $u_\delta \in W^{1,p}_0(\Omega)$. Moreover, we have

$$\|u_\delta\|^p_{1,p} \ge \frac{|s_0|^p(1 - \delta^N)}{(1 - \delta)^p} r_0^{N-p} \omega_N > 0, \tag{2.7}$$

$$\mathcal{J}(u_\delta) \ge [F(s_0)\delta^N - \max_{|t| \le |s_0|} |F(t)|(1 - \delta^N)]\omega_N r_0^N, \tag{2.8}$$

where ω_N is the volume of the unit ball in $\mathbb{R}^N$. From (2.8), there is $\delta_0 > 0$ such that $\|u_{\delta_0}\|_{1,p} > 0$ and $\mathcal{J}(u_{\delta_0}) > 0$. Now, by Lemma 2.3, we can choose $\rho_0 \in (0,1)$ such that

$$\rho_0 < \Big(1 - \mu\frac{M_2}{pS_{p,\partial\Omega}}\Big)\|u_{\delta_0}\|_{1,p}^p \le \mathcal{I}_\mu(u_{\delta_0})$$

and satisfies

$$\frac{\sup\{\mathcal{J}(u) : \mathcal{I}_\mu(u) < \rho_0\}}{\rho_0} < \frac{\mathcal{J}(u_{\delta_0})}{2\mathcal{I}_\mu(u_{\delta_0})}.$$

To apply Lemma 1.3, we choose $x_1 = u_{\delta_0}$ and $x_0 = 0$. Then, the assumptions (i) and (ii) of Lemma 1.3 are satisfied. Next, we define

$$a_\mu = \frac{1+\rho_0}{\frac{\mathcal{J}(u_{\delta_0})}{\mathcal{I}_\mu(u_{\delta_0})} - \frac{\sup\{\mathcal{J}(u):\mathcal{I}_\mu(u)<\rho_0\}}{\rho_0}} > 0 \quad \text{and} \quad \overline{a}_\mu = \min\{a_\mu, \overline{\lambda}\}.$$

A simple computation implies that (**iii**) are verified. Hence, there exist an open interval $\Lambda_\mu \subset [0, \overline{a}_\mu]$ and a real positive number δ_μ such that for each $\lambda \in \Lambda_\mu$, the equation $D\Phi_{\mu,\lambda}(u) = D\mathcal{I}_\mu(u) - \lambda D\mathcal{J}(u) = 0$ has at least three solutions in $W^{1,p}(\Omega)$ whose $\|\cdot\|_{1,p}$-norms are less than δ_μ. By (H1) and (H2), one of them may be the trivial one. Thus, (1.1) has at least two weak solutions in $W^{1,p}(\Omega)$. The proof is complete. □

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