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SOLUTIONS TO BOUNDARY-VALUE PROBLEMS FOR SECOND-ORDER IMPULSIVE DIFFERENTIAL EQUATIONS AT RESONANCE

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Abstract. In this paper, we investigate the existence and uniqueness of solutions to boundary-value problems for second-order impulsive differential equations at resonance. To obtain these results, we apply fixed point methods and new differential inequalities.

1. Introduction

We consider the uniqueness for the nonlinear impulsive boundary value problem (IBVP)

$$\begin{gathered} x'' = f(t, x, x'), \quad t \in [0, T],\, t \neq t_1, \\ \Delta x(t_1) = I(x(t_1)), \quad \Delta x'(t_1) = J(x'(t_1)), \quad t_1 \in (0, T), \end{gathered} \tag{1.1}$$

where t_1 is a fixed value and

$$x \in \beta_0, \tag{1.2}$$

where $I \in C(\mathbb{R}^n, \mathbb{R}^n)$, $J \in C(\mathbb{R}^n, \mathbb{R}^n)$, $\Delta x(t_1) = x(t_1^+) - x(t_1^-)$, $\Delta x'(t_1) = x'(t_1^+) - x'(t_1^-)$, where $x^{(i)}(t_1^+)$ (respectively $x^{(i)}(t_1^-)$) denote the right limit (respectively left limit) of $x^{(i)}(t)$ at $t = t_1$, $i = 0, 1$. The function $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a L^2-Carathédory nonlinear function, that is f satisfies

- (i) the map $(y_0, y_1) \to f(t, y_0, y_1)$ is continuous for a.e. $t \in [0, T] \setminus \{t_1\}$,
- (ii) the map $t \to f(t, y_0, y_1)$ is measurable for all $(y_0, y_1) \in \mathbb{R}^n \times \mathbb{R}^n$,
- (iii) for each $r > 0$, there exists an $\alpha_r \in L^2[0, T]$ such that $|f(t, x, y)| \le \alpha_r(t)$ for a.e. $t \in [0, T]$ and every $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$.

Boundary condition (1.2) represents periodic boundary condition

$$x(0) = x(T), \quad x'(0) = x'(T) \tag{1.3}$$

or the Neumann boundary condition

$$x'(0) = x'(T) = 0. \tag{1.4}$$

In recent years, impulsive differential equations have been studied extensively because of its wide application in many fields such as: chemotherapy; population

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dynamics; optimal control; ecology; biotechnology and physics. We refer the reader to [1, 2, 3, 4, 6, 7, 8] and the references therein for nice examples and applications. For a general theory on impulsive differential equations, see the monographs [2] and [9].

There are many authors who have considered the solvability of boundary value problems with impulses. Nieto [6] studied the existence of solutions to the first order periodic problem with impulse. Chen, Tisdell and Yuan [1] studied the solvability of the periodic problem with impulse. Lin and Jiang [3] studied the existence of positive solutions for the second order Dirichlet boundary value problem with impulse.

For the uniqueness of solutions, there are some papers related boundary value problems for first order differential equations. In [8], the uniqueness of solutions was obtained by nonlinear alternative of Leray-Schauder type in Fréchet spaces. In [10], the existence and uniqueness of solutions was obtained by employing the method of upper and lower solution coupled with the monotone iterative technique. Nieto and Tisdell [7, Section 4.2], obtained existence and uniqueness of solutions to first order IBVPs. Their methods included Schaefer's fixed-point theorem and differential inequalities.

As far as we know, there are few authors who study the uniqueness of solutions for second-order IBVPs. The aim of this work is to study the uniqueness of solutions for second-order impulsive differential equations with periodic condition and Neumann condition.

This paper is organized as follows. In section 2, we present some novel differential inequalities, which are useful to estimate *a priori* bounds on solutions. In section 3, we devote our attention to the uniqueness of solution to (1.1), (1.2). The proof of main results are divided into two parts. First we apply the Schaefer's fixed point theorem to prove the existence of at least one solution. Second we prove the uniqueness of solutions by contradiction.

We note that the main results of this paper are easy to extend to an arbitrary impulse $I_i, J_i,\ i = 1, 2, \dots, p$. However, for clarity and brevity, we restrict our attention to BVPs with one impulse.

For the remainder of the section, we introduce notations and definitions which are used throughout the paper. Let $J' = [0, T] \setminus \{t_1\}$, $t_0 = 0$, $t_2 = T$. The space

$$\begin{aligned} PC^1([0,T];\mathbb{R}^n) = \{ & x : [0,T] \to \mathbb{R}^n : x|_{(t_k,t_{k+1})} \in C^1(t_k, t_{k+1}),\ k = 0, 1, \\ & x(t_1^-) = x(t_1), x'(t_1^-) = x'(t_1),\ x(t_1^+),\ x'(t_1^+) \text{ exist}\} \end{aligned}$$

is a Banach space with the norm $\|x\|_{PC^1} = \max\{\|x\|_{PC}, \|x'\|_{PC}\}$, where $\|x\|_{PC} = \sup_{t\in[0,T]} |x(t)|$. Let

$$Y = \{x \in PC^1([0,T];\mathbb{R}^n) : x|_{(t_k,t_{k+1})} \in W^{2,2}(t_k, t_{k+1}),\ k = 0, 1\}.$$

Clearly, Y is a Banach space.

A function x is said to be a solution of (1.1), (1.2), if $x \in Y$ satisfies (1.1), (1.2).

For $x, y \in \mathbb{R}^n$, we denote by $\langle x, y\rangle$ the usual inner product and by $|x|$ the norm $(\sum_{i=1}^n x_i^2)^{1/2}$. In addition we denote $\|x\|_{L^2} = \big(\int_0^T |x(t)|^2 dt\big)^{1/2}$.

2. Related Lemmas

Lemma 2.1. *If $x \in Y$, then $\|x^{(i)}\|_{L^2} \le \frac{2T}{\pi}\|x^{(i+1)}\|_{L^2}$, $i = 0, 1$.*

Proof. Let

$$y(t) = \begin{cases} x(t), & t \in [0, T], \\ x(2T - t), & t \in [T, 2T], \\ -x(-t), & t \in [-T, 0], \\ -x(t + 2T), & t \in [-2T, -T]. \end{cases}$$

Then

- (i) y, y'' are odd functions on $[-2T, 2T]$, y' is an even function on $[-2T, 2T]$;
- (ii) $\int_{-2T}^{2T} y^{(i)}(s)ds = 0, i = 0, 1, 2;$
- (iii) $\int_{-2T}^{2T} |y^{(i)}(t)|^2 dt = 4 \int_0^T |x^{(i)}(t)|^2 dt, i = 0, 1, 2.$

So y has the Fourier expansion

$$y(t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi t}{2T}.$$

The right hand series converges to $\frac{y(t^+)+y(t^-)}{2}$ at the points $t = 0$, t_1, $2T - t_1$, $-t_1$, $-2T + t_1$, respectively. The Parseval equality implies

$$\int_{-2T}^{2T} |y'(t)|^2 dt = \sum_{n=1}^{\infty} \frac{n^2\pi^2}{2T} b_n^2 \geq \frac{\pi^2}{4T^2} \sum_{n=1}^{\infty} 2T b_n^2 = \frac{\pi^2}{4T^2} \int_{-2T}^{2T} |y(t)|^2 dt.$$

So $\int_{-2T}^{2T} |y(t)|^2 dt \leq \frac{4T^2}{\pi^2} \int_{-2T}^{2T} |y'(t)|^2 dt$. By (iii), we have $\|x\|_{L^2} \leq \frac{2T}{\pi} \|x'\|_{L^2}$.

On the other hand, since (ii) holds, the function y' has the Fourier expansion

$$y'(t) = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi t}{2T},$$

and the right hand series converges to $\frac{y'(t^+)+y'(t^-)}{2}$ at the points $t = 0, t_1, T, 2T - t_1, -t_1, -T, -2T + t_1$. The Parseval equality implies

$$\int_{-2T}^{2T} |y''(t)|^2 dt = \sum_{n=1}^{\infty} \frac{n^2\pi^2}{2T} a_n^2 \geq \frac{\pi^2}{4T^2} \sum_{n=1}^{\infty} 2T a_n^2 = \frac{\pi^2}{4T^2} \int_{-2T}^{2T} |y'(t)|^2 dt.$$

So $\int_{-2T}^{2T} |y'(t)|^2 dt \leq \frac{4T^2}{\pi^2} \int_{-2T}^{2T} |y''(t)|^2 dt$. By (iii), we have $\|x'\|_{L^2} \leq \frac{2T}{\pi} \|x''\|_{L^2}$. □

Lemma 2.2. *If $x \in Y$, then*

1. $\|x\|_{PC} \leq T^{1/2}\Gamma \|x'\|_{L^2}$,
2. $\|x^{(i)}\|_{PC} \leq (\frac{2T}{\pi})^{1-i} T^{1/2} \Gamma \|x''\|_{L^2}, i = 0, 1,$

where $\Gamma = \frac{2T}{\pi \min\{t_1, T-t_1\}} + 1$.

Proof. For $t \in [0, t_1]$. It follows from the mean value theorem that

$$x(\tau_1) = \frac{1}{t_1} \int_0^{t_1} x(s)ds$$

for some $\tau_1 \in [0, t_1]$. Hence for $t \in [0, t_1]$, using Hölder's inequality, we have

$$\begin{aligned}|x(t)| &= \big|x(\tau_1) + \int_{\tau_1}^{t} x'(s)ds\big| \\ &\le \frac{1}{t_1}\int_0^{t_1} |x(s)|ds + \int_0^T |x'(s)|ds \\ &\le \frac{1}{t_1}\int_0^{t_1} |x(s)|ds + \|x'\|_{L^2} T^{1/2}.\end{aligned}$$

For $t \in (t_1, T]$, it follows from the mean value theorem that

$$x(\tau_2) = \frac{1}{T - t_1}\int_{t_1}^{T} x(s)ds$$

for some $\tau_2 \in (t_1, T]$. Hence for $t \in (t_1, T]$, using Hölder inequality,

$$\begin{aligned}|x(t)| &= \big|x(\tau_2) + \int_{\tau_2}^{t} x'(s)ds\big| \\ &\le \frac{1}{T - t_1}\int_{t_1}^{T} |x(s)|ds + \int_0^T |x'(s)|ds \\ &\le \frac{1}{T - t_1}\int_{t_1}^{T} |x(s)|ds + \|x'\|_{L^2} T^{1/2}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|x\|_{PC} &\le \frac{1}{\min\{t_1, T - t_1\}}\int_0^T |x(s)|ds + \|x'\|_{L^2} T^{1/2} \\ &\le \frac{T^{1/2}}{\min\{t_1, T - t_1\}}\|x\|_{L^2} + \|x'\|_{L^2} T^{1/2}.\end{aligned}$$

By Lemma 2.1, we have

$$\|x\|_{PC} \le T^{1/2}\Gamma\|x'\|_{L^2}.$$

So 1) holds. Applying Lemma 2.1 again, we have $\|x\|_{PC} \le \frac{2T^{\frac{3}{2}}}{\pi}\Gamma\|x''\|_{L^2}$. Similar to the above process, we have

$$\|x'\|_{PC} \le \frac{T^{1/2}}{\min\{t_1, T - t_1\}}\|x'\|_{L^2} + \|x''\|_{L^2} T^{1/2} \le T^{1/2}\Gamma\|x''\|_{L^2}.$$

Therefore, 2) holds. □

3. Main results

Theorem 3.1. *Suppose that there exist $a_i \in C([0, T], R^+), p, q > 0$ such that*

$$|f(t, x_0, x_1) - f(t, y_0, y_1)| \le \sum_{i=0}^{1} a_i(t)|x_i - y_i| \tag{3.1}$$

for $(t, x_i, y_i) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$, $i = 0, 1$;

$$|I(x) - I(y)| \le p|x - y|, \quad |J(x) - J(y)| \le q|x - y|, \quad x, y \in \mathbb{R}^n; \tag{3.2}$$

and

$$\sum_{i=0}^{1} \|a_i\|_0\big(\frac{2T}{\pi}\big)^{2-i} + \Big[\big(\frac{2T}{\pi}\big)^2 + (p + q)\frac{2T^2}{\pi}\Gamma^2\Big] < 1, \tag{3.3}$$

where Γ *is defined in Lemma 2.2. Then* (1.1), (1.2) *has a unique solution in* Y.

Proof. Consider the following impulsive BVP, which is equivalent to (1.1)–(1.2),

$$\begin{gathered} x'' - x = f(t,x,x') - x, \quad t \in [0,T] \setminus \{t_1\} \\ \Delta x(t_1) = I(x(t_1)), \quad \Delta x'(t_1) = J(x'(t_1)), \\ x \in \beta. \end{gathered} \tag{3.4}$$

Since the linear boundary value problem

$$x'' - x = 0, \quad x \in \beta \tag{3.5}$$

has only the zero solution, there exists a unique, continuous once-differentiable Green's function $G : [0,T] \times [0,T] \to R$ such that (3.4) may be equivalently reformulated as the integral equation

$$\begin{aligned} x(t) = &\int_0^T G(t,s)[f(s,x(s),x'(s)) - x(s)]ds \\ &+ G(t,t_1)J(x'(t_1)) - (\partial G/\partial s)(t,t_1)I(x(t_1)), \quad t \in [0,T]. \end{aligned}$$

Define the operator $H : PC^1([0,T],\mathbb{R}^n) \to PC^1([0,T],\mathbb{R}^n)$ by

$$\begin{aligned} (Hx)(t) = &\int_0^T G(t,s)[f(s,x(s),x'(s)) - x(s)]ds \\ &+ G(t,t_1)J(x'(t_1)) - (\partial G/\partial s)(t,t_1)I(x(t_1)), \quad t \in [0,T]\,. \end{aligned} \tag{3.6}$$

Then consider the family of equations

$$x = \lambda Hx, \tag{3.7}$$

$\lambda \in (0,1)$. Since Hx actually belongs to Y for each $x \in PC^1([0,T],\mathbb{R}^n)$, H is a compact map. We will apply Schaefer's fixed point theorem [5] to prove that H has at least one fixed point in $PC^1([0,T],\mathbb{R}^n)$. Since $H : PC^1([0,T],\mathbb{R}^n) \to PC^1([0,T],\mathbb{R}^n)$ is compact, it remains to verify that all solutions to (3.6) are bounded independently of λ.

Since x is a solution of $x = \lambda Hx$, then $x \in Y$ satisfies

$$\begin{gathered} x'' - x = \lambda[f(t,x,x') - x], \quad t \in [0,T] \setminus \{t_1\}, \\ \Delta x(t_1) = \lambda I(x(t_1)), \quad \Delta x'(t_1) = \lambda J(x'(t_1)), \\ x \in \beta. \end{gathered} \tag{3.8}$$

Multiplying by x'' and integrating from 0 to T, we have

$$\|x''\|_{L^2}^2 \le \lambda \Big| \int_0^T f(t,x(t),x'(t))x''(t)dt \Big| + (1-\lambda) \Big| \int_0^T x(t)x''(t)dt \Big|. \tag{3.9}$$

By (3.1), we have

$$\begin{aligned}
&\big|\int_0^T f(t,x(t),x'(t))x''(t)dt\big| \\
&\le \int_0^T |f(t,x(t),x'(t)) - f(t,0,0)| \times |x''(t)|\,dt + \int_0^T |f(t,0,0)x''(t)|\,dt \\
&\le \sum_{i=0}^{1}\int_0^T |a_i(t)x^{(i)}(t)x''(t)|dt + \|f(t,0,0)\|_{L^2}\|x''\|_{L^2} \\
&\le \sum_{i=0}^{1} \|a_i\|_0\|x^{(i)}\|_{L^2}\|x''\|_{L^2} + \|f(t,0,0)\|_{L^2}\|x''\|_{L^2}.
\end{aligned} \tag{3.10}$$

By the impulsive condition

$$\begin{aligned}
\big|\int_0^T x(t)x''(t)dt\big| &= \big|\sum_{i=0}^{1} x(t)x'(t)\big|_{t_i^+}^{t_{i+1}^-}\big| + \int_0^T |x'(t)|^2 dt \\
&\le |x(T)x'(T) - x(0)x'(0) - \Delta(x(t_1)x'(t_1))| + \int_0^T |x'(t)|^2 dt.
\end{aligned} \tag{3.11}$$

The boundary condition (1.2), the impulsive condition, and condition (3.2) imply

$$\begin{aligned}
&\big|x(T)x'(T) - x(0)x'(0) - \Delta(x(t_1)x'(t_1))\big| + \int_0^T |x'(t)|^2 dt \\
&= \big|x'(t_1^+)\Delta x(t_1) + x(t_1)\Delta x'(t_1)\big| + \|x'\|_{L^2}^2 \\
&= \big|x'(t_1^+)(I(x(t_1)) - I(0)) + x(t_1)(J(x'(t_1)) - J(0))\big| \\
&\quad + \big|x'(t_1^+)I(0)\big| + |x(t_1)J(0)| + \|x'\|_{L^2}^2 \\
&\le (p+q)\|x\|_{PC}\|x'\|_{PC} + \|x'\|_{L^2}^2 + \|x\|_{PC}|J(0)| + \|x'\|_{PC}|I(0)|.
\end{aligned} \tag{3.12}$$

Substituting (3.10) (3.11) (3.12) into (3.9), we have

$$\begin{aligned}
\|x''\|_{L^2}^2 \le{}& \sum_{i=0}^{1} \|a_i\|_0\|x^{(i)}\|_{L^2}\|x''\|_{L^2} + \|f(t,0,0)\|_{L^2}\|x''\|_{L^2} \\
&+ (p+q)\|x\|_{PC}\|x'\|_{PC} + \|x'\|_{L^2}^2 + \|x\|_{PC}|J(0)| + \|x'\|_{PC}|I(0)|.
\end{aligned}$$

By Lemma 2.1 and Lemma 2.2 we have

$$\begin{aligned}
\|x''\|_{L^2}^2 \le{}& \sum_{i=0}^{1} \|a_i\|_0\big(\frac{2T}{\pi}\big)^{2-i}\|x''\|_{L^2}^2 + \|f(t,0,0)\|_{L^2}\|x''\|_{L^2} \\
&+ \big[\big(\frac{2T}{\pi}\big)^2 + (p+q)\frac{2T^2}{\pi}\Gamma^2\big]\|x''\|_{L^2}^2 + M\|x''\|_{L^2}
\end{aligned}$$

for some sufficiently large constant $M > 0$. Condition (3.3) implies that x'' is bounded in $L^2([0,T],\mathbb{R}^n)$. Lemma 2.2 means that x is bounded in $PC^1([0,T],\mathbb{R}^n)$. Applying Schaefer's fixed point theorem, (1.1), (1.2) has at least one solution.

Now we will show that (1.1), (1.2) has a unique solution. If x, y are both solutions of (1.1), (1.2). Then $u = x - y$ satisfies

$$\begin{gathered} u''(t) = f(t, x(t), x'(t)) - f(t, y(t), y'(t)), \quad t \in [0, T] \setminus \{t_1\}, \\ \Delta u(t_1) = I(x(t_1)) - I(y(t_1)), \quad \Delta u'(t_1) = J(x'(t_1)) - J(y'(t_1)) \\ u \in \beta_0. \end{gathered} \tag{3.13}$$

Similar to the above process we have

$$\|u''\|_{L^2}^2 \le \sum_{i=0}^{1} \|a_i\|_0 \big(\frac{2T}{\pi}\big)^{2-i} \|u''\|_{L^2}^2.$$

Condition (3.3) implies that $\|u''\|_{L^2} = 0$. Lemma 2.2 means that $u(t) \equiv 0$ for $t \in [0, T]$. The proof is complete. □

Theorem 3.2. *Suppose that the conditions* (3.1) (3.2) *in Theorem 3.1 hold. Furthermore, we assume that*

$$\|a_1\|_0 < \frac{1}{T}, \quad \sum_{i=0}^{1} \|a_i\|_0 \big(\frac{2T}{\pi}\big)^{2-i} + \frac{(p+q)T^2\Gamma^2(\|a_0\|_0 + 1)}{1 - T\|a_1\|_0} < 1 \tag{3.14}$$

Then problem (1.1), (1.4) *has a unique solution in* Y.

Proof. According to the proof of Theorem 3.1, we have that the function $x \in Y$ satisfies

$$\begin{gathered} x'' - x = \lambda[f(t, x, x') - x], \quad t \in [0, T] \setminus \{t_1\}, \\ \Delta x(t_1) = \lambda I(x(t_1)), \quad \Delta x'(t_1) = \lambda J(x'(t_1)), \\ x'(0) = x'(T) = 0. \end{gathered} \tag{3.15}$$

Multiplying by x and integrating from 0 to T, we have

$$\int_0^T x''(s)x(s)ds = \int_0^T \lambda f(s, x(s), x'(s))x(s)ds + (1-\lambda)\int_0^T x^2(s)ds. \tag{3.16}$$

By boundary condition (1.4) and impulsive condition we have

$$\begin{aligned} \int_0^T x''(s)x(s)ds &= -\Delta(x'(t_1)x(t_1)) + x'(T)x(T) - x'(0)x(0) - \int_0^T [x'(s)]^2 ds \\ &= -\Delta(x'(t_1)x(t_1)) - \|x'\|_{L^2}^2. \end{aligned} \tag{3.17}$$

By (1.4) (3.16) we have

$$\begin{aligned}
\|x'\|_{L^2}^2 &\le |\Delta(x'(t_1)x(t_1))| + \int_0^T |f(s,x(s),x'(s))x(s)|ds \\
&\le |x'(t_1^+)\Delta x(t_1) + x(t_1)\Delta x'(t_1)| \\
&\quad + \int_0^T |f(s,x(s),x'(s)) - f(s,0,0)||x(s)|ds + \|f(t,0,0)\|_{L^2}\|x\|_{L^2} \\
&\le \big|x'(t_1^+)(I(x(t_1)) - I(0)) + x(t_1)(J(x'(t_1)) - J(0))\big| + |x'(t_1^+)I(0)| \\
&\quad + |x(t_1)J(0)| + \sum_{i=0}^{1}\int_0^T |a_i(s)x^{(i)}(s)x(s)|ds + \|f(t,0,0)\|_{L^2}\|x\|_{L^2} \\
&\le (p+q)\|x'\|_{PC}\|x\|_{PC} + \sum_{i=0}^{1}\|a_i\|_0\|x^{(i)}\|_{L^2}\|x\|_{L^2} + \|f(t,0,0)\|_{L^2}\|x\|_{L^2} \\
&\quad + \|x'\|_{PC}|I(0)| + \|x\|_{PC}|J(0)|.
\end{aligned} \tag{3.18}$$

We assume that $\|x'\|_{PC} = |x'(\xi)|$ for $\xi \in [0,T]$. If $\xi \in [0,t_1]$, integrating from 0 to ξ on the both sides of (3.15), we have

$$x'(\xi) = \lambda\int_0^\xi f(s,x(s),x'(s))ds + (1-\lambda)\int_0^\xi x(s)ds. \tag{3.19}$$

If $\xi \in (t_1,T]$, integrating from ξ to T on the both sides of (3.15) we have

$$-x'(\xi) = \lambda\int_\xi^T f(s,x(s),x'(s))ds + (1-\lambda)\int_\xi^T x(s)ds. \tag{3.20}$$

By (3.19) (3.20) and condition (3.1), one has

$$\begin{aligned}
\|x'\|_{PC} &\le \int_0^T |f(s,x(s),x'(s))|ds + \int_0^T |x(s)|ds \\
&\le \int_0^T |f(s,x(s),x'(s)) - f(s,0,0)|ds + \|f(t,0,0)\|_{L^2}T^{1/2} + T\|x\|_{PC} \\
&\le T\sum_{i=0}^{1}\|a_i\|_0\|x^{(i)}\|_{PC} + \|f(t,0,0)\|_{L^2}T^{1/2} + T\|x\|_{PC}.
\end{aligned}$$

So

$$\|x'\|_{PC} \le \frac{T(\|a_0\|+1)}{1-T\|a_1\|_0}\|x\|_{PC} + \frac{\|f(t,0,0)\|_{L^2}T^{1/2}}{1-T\|a_1\|_0}. \tag{3.21}$$

Substituting (3.21) into (3.18), and noticing Lemma 2.1, Lemma 2.2, we obtain that

$$\|x'\|_{L^2}^2 \le \Big\{\frac{(p+q)T^2\Gamma^2(\|a_0\|_0+1)}{1-T\|a_1\|_0} + \sum_{i=0}^{1}\|a_i\|_0\big(\frac{2T}{\pi}\big)^{2-i}\Big\}\|x'\|_{L^2}^2 + N\|x'\|_{L^2}$$

holds for sufficiently large constant $N > 0$. The condition (3.14) means that x' is bounded in $L^2[0,T]$. Lemma 2.2 means that there exists $L_1 > 0$ such that $\|x\|_{PC} \le L_1$. (3.21) means that there exists $L_2 > 0$ such that $\|x'\|_{PC} \le L_2$. So x is bounded in $PC^1([0,T],\mathbb{R}^n)$. Applying Schaefer's fixed point theorem, problem (1.1), (1.4) has at least one solution.

Now we show that (1.1)–(1.4) has a unique solution. If x, y are both solutions, then $u = x - y$ satisfies

$$\begin{gathered} u''(t) = f(t, x(t), x'(t)) - f(t, y(t), y'(t)), \quad t \in [0, T] \setminus \{t_1\}, \\ \Delta u(t_1) = I(x(t_1)) - I(y(t_1)), \quad \Delta u'(t_1) = J(x'(t_1)) - J(y'(t_1)), \\ u'(0) = u'(T) = 0. \end{gathered} \tag{3.22}$$

Similar to the above process, we have

$$\|u'\|_{L^2}^2 \leq \Big[(p+q)\frac{T^2\|a_0\|_0\Gamma^2}{1 - T\|a_1\|_0} + \sum_{i=0}^{1}\|a_i\|_0\big(\frac{2T}{\pi}\big)^{2-i}\Big]\|u'\|_{L^2}^2.$$

The condition (3.14) implies that $\|u'\|_{L^2} = 0$. Lemma 2.2 gives that $u \equiv 0$ for $t \in [0, T]$. So problem (1.1), (1.4) has a unique solution in Y. □

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