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# QUASILINEAR NONLOCAL INTEGRODIFFERENTIAL EQUATIONS IN BANACH SPACES

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ABSTRACT. In this paper, we study the existence of mild solutions for quasilinear integrodifferential equations with nonlocal conditions in Banach spaces. The results are established by using Hausdorff's measure of noncompactness.

## 1. INTRODUCTION

In this paper, we discuss the existence of mild solution of the following nonlinear integrodifferential equation with nonlocal condition

$$\frac{du(t)}{dt} = A(t,u)u + \int_0^t f(t,s,u(s))ds, \quad t \in [0,b], \tag{1.1}$$

$$u(0) = g(u) + u_0, \tag{1.2}$$

where $f : [0,b] \times [0,b] \times \mathbb{X} \to \mathbb{X}$ and $A : [0,b] \times \mathbb{X} \to \mathbb{X}$ are continuous functions, $g : \mathcal{C}([0,b];\mathbb{X}) \to \mathbb{X}, u_0 \in \mathbb{X}$ and $\mathbb{X}$ is a real Banach space with norm $\|\cdot\|$.

The notion of "nonlocal condition" has been introduced to extend the study of the classical initial value problems; see, e.g. [4, 8, 10, 11, 19]. It is more precise for describing nature phenomena than the classical condition since more information is taken into account, thereby decreasing the negative effects incurred by a possibly erroneous single measurement taken at the initial time. The study of abstract nonlocal initial value problems was initiated by Byszewski,we refer to some of the papers below. Byszewski [6, 7] , Byszewski and Lasmikauthem [9] give the existence and uniqueness of mild solutions and classical solutions when $f$ and $g$ satisfy Lipschitz-type conditions. Subsequently, many authors are devoted to studying of nonlocal problems. See [1, 2, 12, 13, 15, 20] for the references and remarks about the advantage of the nonlocal problems over the classical initial value problems.

This article is motivated by the recent paper of Chandrasekaran [10]. We use some hypotheses in [10], and using the method of Hausdorff's measure of noncompactness, we give the existence of mild solutions of quasilinear integrodifferential

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equations with nonlocal conditions (1.1)–(1.2). Our results improve and extend some corresponding results in [2, 7, 8, 10, 15].

## 2. Preliminaries

Throughout this paper $\mathbb{X}$ will represent a Banach space with norm $\|\cdot\|$. Denoted $\mathcal{C}([0,b];\mathbb{X})$ by the space of $\mathbb{X}$-valued continuous functions on $[0,b]$ with the norm $\|u\| = \sup\{\|u(t)\|, t \in [0,b]\}$ for u $\in \mathcal{C}([0,b];\mathbb{X})$, and denoted $\mathcal{L}(0,b;\mathbb{X})$ by the space of $\mathbb{X}$-valued Bochner integrable functions on $[0,b]$ with the norm $\|u\|_{\mathcal{L}} = \int_0^b \|u(t)\|dt$.

The Hausdroff's measure of noncompactness $\beta_{\mathbb{Y}}$ is defined by $\beta_{\mathbb{Y}}(B) = \inf\{r > 0, B$ can be covered by finite number of balls with radii $r\}$ for bounded set $B$ in a Banach space $\mathbb{Y}$.

**Lemma 2.1** ([3]). *Let $\mathbb{Y}$ be a real Banach space and $B, C \subseteq \mathbb{Y}$ be bounded, with the following properties:*

1. *$B$ is pre-compact if and only if $\beta_{\mathbb{X}}(B) = 0$;*
2. *$\beta_{\mathbb{Y}}(B) = \beta_{\mathbb{Y}}(\overline{B}) = \beta_{\mathbb{Y}}(convB)$, where $\overline{B}$ and* conv $B$ *mean the closure and convex hull of $B$ respectively;*
3. *$\beta_{\mathbb{Y}}(B) \leq \beta_{\mathbb{Y}}(C)$, where $B \subseteq C$;*
4. *$\beta_{\mathbb{Y}}(B + C) \leq \beta_{\mathbb{Y}}(B) + \beta_{\mathbb{Y}}(C)$, where $B + C = \{x + y : x \in B, y \in C\}$;*
5. *$\beta_{\mathbb{Y}}(B \cup C) \leq \max\{\beta_{\mathbb{Y}}(B), \beta_{\mathbb{Y}}(C)\}$;*
6. *$\beta_{\mathbb{Y}}(\lambda B) \leq |\lambda| \beta_{\mathbb{Y}}(B)$ for any $\lambda \in \mathbb{R}$;*
7. *If the map $Q : D(Q) \subseteq \mathbb{Y} \to \mathbb{Z}$ is Lipschitz continuous with constant $k$, then $\beta_{\mathbb{Z}}(QB) \leq k\beta_{\mathbb{Y}}(B)$ for any bounded subset $B \subseteq D(Q)$, where $\mathbb{Z}$ be a Banach space;*
8. *$\beta_{\mathbb{Y}}(B) = \inf\{d_{\mathbb{Y}}(B, C); C \subseteq \mathbb{Y}$ is precompact $\} = \inf\{d_{\mathbb{Y}}(B, C); C \subseteq \mathbb{Y}$ is finite valued$\}$, where $d_{\mathbb{Y}}(B, C)$ means the nonsymmetric (or symmetric) Hausdorff distance between $B$ and $C$ in $\mathbb{Y}$;*
9. *If $\{W_n\}_{n=1}^{+\infty}$ is decreasing sequence of bounded closed nonempty subsets of $\mathbb{Y}$ and $\lim_{n\to\infty} \beta_{\mathbb{Y}}(W_n) = 0$, then $\bigcap_{n=1}^{+\infty} W_n$ is nonempty and compact in $\mathbb{Y}$.*

The map $Q : W \subseteq \mathbb{Y} \to \mathbb{Y}$ is said to be a $\beta_{\mathbb{Y}}$-contraction if there exists a positive constant $k < 1$ such that $\beta_{\mathbb{Y}}(Q(B)) \leq k\beta_{\mathbb{Y}}(B)$ for any bounded closed subset $B \subseteq W$, where $\mathbb{Y}$ is a Bananch space.

**Lemma 2.2** (Darbo-Sadovskii [3]). *If $W \subseteq \mathbb{Y}$ is bounded closed and convex, the continuous map $Q : W \to W$ is a $\beta_{\mathbb{Y}}$-contraction, then the map $Q$ has at least one fixed point in $W$.*

In this paper we denote by $\beta$ the Hausdorff's measure of noncompactness of $\mathbb{X}$ and denote $\beta_{\mathcal{C}}$ by the Hausdorff's measure of noncompactness of $\mathcal{C}([a,b];\mathbb{X})$. To discuss the existence, we need the following Lemmas in this paper.

**Lemma 2.3** ([3]). *If $W \subseteq \mathcal{C}([0,b];\mathbb{X})$ is bounded, then $\beta(W(t)) \leq \beta_{\mathcal{C}}(W)$ for all $t \in [0,b]$, where $W(t) = \{u(t); u \in W\} \subseteq \mathbb{X}$. Furthermore if $W$ is equicontinuous on $[a,b]$, then $\beta(W(t))$ is continuous on $[a,b]$ and $\beta_{\mathcal{C}}(W) = \sup\{\beta(W(t)),\ t \in [a,b]\}$.*

**Lemma 2.4** ([14]). *If $\{u_n\}_{n=1}^{\infty} \subset \mathcal{L}^1(a,b;\mathbb{X})$ is uniformly integrable, then the function $\beta(\{u_n(t)\}_{n=1}^{\infty})$ is measurable and*

$$\beta\Big(\Big\{\int_0^t u_n(s)ds\Big\}_{n=1}^{\infty}\Big) \leq 2\int_0^t \beta\big(\{u_n(s)\}_{n=1}^{\infty}\big)ds. \tag{2.1}$$

**Lemma 2.5** ([3]). *If $W \subseteq \mathcal{C}([0,b];\mathbb{X})$ is bounded and equicontinuous, then $\beta(W(s))$ is continuous and*

$$\beta(\int_0^t W(s)ds) \leq \int_0^t \beta(W(s))ds. \tag{2.2}$$

From [10], we know that for any fixed $u \in \mathcal{C}([0,b];\mathbb{X})$ there exist a unique continuous function $U_u :\ [0,b]\times[0,b] \to B(\mathbb{X})$ defined on $[0,b]\times[0,b]$ such that

$$U_u(t,s) = I + \int_s^t A_u(\omega)U_u(\omega,s)d\omega, \tag{2.3}$$

where $B(\mathbb{X})$ denote the Banach space of bounded linear operators from $\mathbb{X}$ to $\mathbb{X}$ with the norm $\|Q\| = \sup\{\|Qu\| :\ \|u\| = 1\}$, and $I$ stands for the identity operator on $\mathbb{X}$, $A_u(t) = A(t,u(t))$. From (2.3), we have

$$U_u(t,t) = I, \quad U_u(t,s)U_u(s,r) = U_u(t,r), \quad (t,s,r) \in [0,b]\times[0,b]\times[0,b],$$
$$\frac{\partial U_u(t,s)}{\partial t} = A_u(t)U_u(t,s) \quad \text{for almost all } t \in [0,b],\ \forall s \in [0,b].$$

**Definition 2.6.** A continuous function $u(t) \in \mathcal{C}([0,b];\mathbb{X})$ such that

$$u(t) = U_u(t,0)u_0 + U_u(t,0)g(u) + \int_0^t U_u(t,s)\int_0^s f(s,\tau,u(\tau))dsd\tau \tag{2.4}$$

and $u(0) = g(u) + u_0$ is called a mild solution of (1.1)–(1.2).

The evolution family $\{U_u(t,s)\}_{0\leq s\leq t\leq b}$ is said to be equicontinuous if $(t,s) \to \{U_u(t,s)x :\ x \in B\}$ is equicontinuous for $t > 0$ and for all bounded subset $B$ in $\mathbb{X}$.

The following Lemma is obvious.

**Lemma 2.7.** *If the evolution family $\{U_u(t,s)\}_{0\leq s\leq t\leq b}$ is equicontinuous and $\eta \in \mathcal{L}(0,b;\mathbb{R}^+)$, then the set $\{\int_0^t U_u(t-s,s)u(s)ds, \|u(s)\| \leq \eta(s)$ for a.e. $s \in [0,b]\}$ is equicontinuous for $t \in [0,b]$.*

In section 3, we give some existence results when $g$ is compact and $f$ satisfies the conditions with respect to Hauadorff's measure of noncompactness. In section 4, we use the different method to discuss the case when $g$ is Lipschitz continuous and $f$ satisfies the conditions with the Hauadorff's measure of noncompactness.

In this paper, we denote $M = \sup\{\|U_u(t,s)\| : (t,s) \in [0,b]\times[0,b]\}$ for all $u \in \mathbb{X}$. Without loss of generality, we let $u_0 = 0$.

## 3. The existence results for compact $g$

In this section by using the usual techniques of the Hausdorff's measure of noncompactness and its applications in differential equations in Banach spaces (see, e.g. [3, 5, 14]), we give some existence results of the nonlocal problem (1.1)–(1.2).

Here we list the following hypotheses:

(HA) : The evolution family $\{U_u(t,s)\}_{0\leq s\leq t\leq b}$ generated by $A(t,u)$ is equicontinuous, and $\|U_u(t,s)\| \leq M$ for almost all $t,s \in [0,b]$.

(Hg) (1) $g : \mathcal{C}([0,b];\mathbb{X}) \to \mathbb{X}$ is continuous and compact;
(2) There exist $N > 0$ such that $\|g(u)\| \leq N$ for all $u \in \mathcal{C}([0,b];\mathbb{X})$.

(Hf) (1) $f : [0,b]\times[0,b]\times\mathbb{X} \to \mathbb{X}$ satisfies the *Carathéodory-type* condition; i.e., $f(\cdot,\cdot,u)$ is measurable for all $u \in \mathbb{X}$ and $f(t,s,\cdot)$ is continuous for a.e. $t,s \in [a,b]$;

(2) There exist two functions $h : [0, b] \times \mathbb{R}^+ \to \mathbb{R}^+$ and $q : [0, b] \times \mathbb{R}^+ \to \mathbb{R}^+$such that $h(\cdot, r) \in \mathcal{L}(0, b; \mathbb{R}^+)$ for every $r \geq 0$, $h(t, \cdot)$ is continuous and increasing,$q(s) \in \mathcal{L}(0, b; \mathbb{R}^+)$ , and $\|f(t, s, u)\| \leq q(t)h(s, \|u\|)$ for a.e. $t \in [0, b]$, and all $u \in \mathcal{C}([0, b]; \mathbb{X})$, and for all positive constants $K_1, K_2$, the scalar equation
$$m(t) = K_1 + K_2 \int_0^t h(s, m(s))ds,\ t \in [0, b] \tag{3.1}$$
has at least one solution;

(3) There exist $\eta \in \mathcal{L}(0, b; \mathbb{R}^+)$, $\zeta \in \mathcal{L}(0, b; \mathbb{R}^+)$ such that $\beta(f(t, s, D)) \leq \eta(t)\zeta(s)\beta(D)$ for a.e. $t, s \in [0, b]$, and for any bounded subset $D \subset \mathcal{C}([0, b], \mathbb{X})$. Here we let $\int_0^t \eta(s)ds \leq K$

Now, we give an existence result under the above hypotheses.

**Theorem 3.1.** *Assume the hypotheses* (HA), (Hf), (Hg) *are satisfied, then the nonlocal initial value problem* (1.1)–(1.2) *has at least one mild solution.*

*Proof.* Let $m(t)$ be a solution of the scalar equation
$$m(t) = MN + RM \int_0^t h(s, m(s))ds, \tag{3.2}$$
where $R = \int_0^t q(s)ds$. Defined a map $Q : \mathcal{C}([0, b]; \mathbb{X}) \to \mathcal{C}([0, b]; \mathbb{X})$ by
$$(Qu)(t) = U_u(t, 0)g(u) + \int_0^t U_u(t, s) \int_0^s f(s, \tau, u(\tau))d\tau ds, \quad t \in [0, b] \tag{3.3}$$
for all $u \in \mathcal{C}([0, b]; \mathbb{X})$. We can show that $Q$ is continuous by the usual techniques (see, e.g. [16, 17]).

We denote by $W_0 = \{u \in \mathcal{C}([0, b]; \mathbb{X}), \|u(t)\| \leq m(t) \text{ for all } t \in [0, b]\}$. Then $W_0 \subseteq \mathcal{C}([0, b]; \mathbb{X})$ is bounded and convex.

Define $W_1 = \overline{\text{conv}}K(W_0)$, where $\overline{\text{conv}}$ means the closure of the convex hull in $\mathcal{C}([0, b]; \mathbb{X})$. As $U_u(t, s)$ is equicontinuous, $g$ is compact and $W_0 \subseteq \mathcal{C}([0, b]; \mathbb{X})$ is bounded, due to Lemma 2.7 and hypothesis (Hf)(2), $W_1 \subseteq \mathcal{C}([0, b]; \mathbb{X})$ is bounded closed convex nonempty and equicontinuous on $[0, b]$.

For any $u \in Q(W_0)$, we know
$$\begin{aligned} \|u(t)\| &\leq MN + M \int_0^t \int_0^s q(s)h(\tau, m(\tau))d\tau ds \\ &\leq MN + M \int_0^t h(\tau, m(\tau))d\tau \int_0^t q(s)ds \\ &\leq MN + MR \int_0^t h(s, m(s))ds \\ &= m(t) \end{aligned}$$
for $t \in [0, b]$. It follows that $W_1 \subset W_0$.

We define $W_{n+1} = \overline{\text{conv}}Q(W_n)$, for $n = 1, 2, \dots$. Form above we know that $\{W_n\}_{n=1}^{\infty}$ is a decreasing sequence of bounded, closed, convex, equicontinuous on [0,b] and nonempty subsets in $\mathcal{C}([0, b], \mathbb{X})$.

Now for $n \geq 1$ and $t \in [0, b]$, $W_n(t)$ and $Q(W_n(t))$ are bounded subsets of $\mathbb{X}$, hence, for any $\varepsilon > 0$, there is a sequence $\{u_k\}_{k=1}^{\infty} \subset W_n$ such that (see, e.g. [5], pp.125)

$$
\begin{aligned}
\beta(W_{n+1}(t) &= \beta(QW_n(t)) \\
&\le 2\beta(\int_0^t U_u(t,s)\int_0^s f(s,\tau,\{u_k(\tau)\}_{k=1}^\infty)d\tau ds)+\varepsilon \\
&\le 2M\int_0^t \beta(\int_0^s f(s,\tau,\{u_k(\tau)\}_{k=1}^\infty)d\tau)ds+\varepsilon \\
&\le 4M\int_0^t\int_0^s \beta(f(s,\tau,\{u_k(\tau)\}_{k=1}^\infty))d\tau ds+\varepsilon \\
&\le 4M\int_0^t\int_0^s \eta(s)\zeta(\tau)\beta(\{u_k(\tau)\}_{k=1}^\infty)d\tau ds+\varepsilon \\
&\le 4M\int_0^t \zeta(\tau)\beta(W_n(\tau))d\tau\int_0^t\eta(s)ds+\varepsilon \\
&\le 4MK\int_0^t \zeta(s)\beta(W_n(s))ds+\varepsilon.
\end{aligned}
$$

Since $\varepsilon>0$ is arbitrary, it follows from the above inequality that

$$
\beta(QW_{n+1}(t)) \le 4MK\int_0^t \zeta(s)\beta(W_n(s))ds \tag{3.4}
$$

for all $t\in[0,b]$. Because $W_n$ is decreasing for $n$, we define

$$
\alpha(t)=\lim_{n\to\infty}\beta(W_n(t))
$$

for all $t\in[0,b]$. From (3.4), we have

$$
\alpha(t)\le 4MK\int_0^t\zeta(s)\alpha(s)ds
$$

for $t\in[0,b]$, which implies that $\alpha(t)=0$ for all $t\in[0,b]$. By Lemma 2.3, we know that $\lim_{n\to\infty}\beta_{\mathcal{C}}(W_n)=0$. Using Lemma 2.1, we know that $W=\bigcap_{n=1}^\infty W_n$ is convex compact and nonempty in $\mathcal{C}([0,b];\mathbb{X})$ and $Q(W)\subset W$. By the famous Schauder's fixed point theorem, there exists at least one mild solution $u$ of the initial value problem (1.1)–(1.2), where $u\in W$ is a fixed point of the continuous map $Q$. $\square$

**Remark 3.2.** If the function $f$ is compact or Lipschitz continuous (see, e.g. [6, 16, 18]), then (Hf)(3) is automatically satisfied.

In some of the early related results in references and above result, it is supposed that the map $g$ is uniformly bounded. We indicate here that this condition can be released. In fact, if $g$ is compact, then it must be bounded on bounded set. Here we give an existence result under another growth condition of $f$ (see, [11, 20]), when $g$ is not uniformly bounded. Precisely, we replace the hypothesis (Hf)(2) by

(Hf)(2') There exists a function $p\in\mathcal{L}(0,b;\mathbb{R}^+)$ and a increasing function $\psi:\mathbb{R}^+\to\mathbb{R}^+$ such that $\|f(t,s,u)\|\le p(t)\psi(\|u\|)$, for a.e. $t\in[0,b]$, and all $u\in\mathcal{C}([0,b];\mathbb{X})$.

**Theorem 3.3.** *Suppose that* (HA), (Hf)(1), (Hf)(2'), (Hf)(3), (Hg)(1) *are satisfied. Then the equation* (1.1)–(1.2) *has at least one mild solution if*

$$\lim_{k\to\infty}\sup\frac{M}{k}(\varphi(k)+b\psi(k)\int_0^b p(s)ds)<1, \tag{3.5}$$

*where* $\varphi(k)=\sup\{\|g(u)\|,\ \|u\|\le k\}$.

*Proof.* The inequality (3.5) implies that there exists a constant $k>0$ such that

$$M(\varphi(k)+b\psi(k)\int_0^b p(s)ds)<k.$$

Just as in the proof of Theorem 3.1, let $W_0=\{u\in\mathcal{C}([0,b];\mathbb{X}):\|u(t)\|\le k\}$ and $W_1=\overline{\text{conv}}QW_0$. Then for any $u\in W_1$, we have

$$\begin{aligned}\|u(t)\|&\le M\varphi(k)+M\int_0^t\int_0^s p(\tau)\psi(k)d\tau ds\\&\le M\varphi(k)+bM\psi(k)\int_0^b p(s)ds<k\end{aligned}$$

for $t\in[0,b]$. It means that $W_1\subset W_0$. So we can complete the proof similarly to Theorem 3.1 □

## 4. Existence results for Lipschitz $g$

In the previous section, we obtained the existence results when $g$ is compact but without the compactness of $\{U_u(t,s)\}_{0\le s\le t\le b}$ or $f$. In this section, we discuss the equation (1.1)–(1.2) when $g$ is Lipschitz and $f$ is not Lipschitz. Precisely, we replace (Hg)(1) by

(Hg)(1') There exist a constant $L\in(0,\frac{1}{M})$ such that $\|g(u)-g(v)\|\le L\|u-v\|$ for every $u,v\in\mathcal{C}([0,b];\mathbb{X})$.

**Theorem 4.1.** *Let* (HA), (Hg)(1')(2), (Hf) *be satisfied. Then the equation* (1.1)–(1.2) *has at least one mild solution provided that*

$$ML+4MK\int_0^b\zeta(s)ds<1. \tag{4.1}$$

*Proof.* We define $Q_1,Q_2:\mathcal{C}([0,B];\mathbb{X})\to\mathcal{C}([0,B];\mathbb{X})$ by

$$\begin{gathered}(Q_1u)(t)=U_u(t,0)g(u),\\(Q_2u)(t)=\int_0^t U_u(t,s)\int_0^s f(s,\tau,u(\tau))d\tau ds\end{gathered}$$

for $u\in\mathcal{C}([0,B];\mathbb{X})$. Note that $Q_1+Q_2=Q$, as defined in the proof of Theorem 3.1. We define $W_0=\{u\in\mathcal{C}([0,B];\mathbb{X}):\|u(t)\|\le m(t)\ \forall t\in[0,b]\}$, and let $W=\overline{\text{conv}}QW_0$. Then from the proof of Theorem 3.1 we know that $W$ is a bounded closed convex and equicontinuous subset of $\mathcal{C}([0,B];\mathbb{X})$ and $QW\subset W$. We shall prove that $Q$ is $\beta_{\mathcal{C}}$-contraction on $W$. Then Darbo-Sadovskii's fixed point theorem can be used to get a fixed point of $Q$ in $W$, which is a mild solution of (1.1)–(1.2).

First, for every bounded subset $B\subset W$, from the (Hg)(1') and Lemma 2.1 we have

$$\beta_{\mathcal{C}}(Q_1B)=\beta_{\mathcal{C}}(U_B(t,0)g(B))\le M\beta_C(g(B))\le ML\beta_{\mathcal{C}}(B). \tag{4.2}$$

Next, for every bounded subset $B \subset W$, for $t \in [0, b]$ and every $\varepsilon > 0$, there is a sequence $\{u_k\}_{k=1}^{\infty} \subset B$, such that

$$\beta(Q_2 B(t)) \leq 2\beta(\{Q_2 u_k(t)\}_{n=1}^{\infty}) + \varepsilon.$$

Note that $B$ and $Q_2B$ are equicontinuous, we can get from Lemma 2.1, Lemma 2.4, Lemma 2.5 and (Hf)(3) that

$$\begin{aligned}
\beta(Q_2 B(t)) &\leq 2M \int_0^t \beta(\int_0^s f(s, \tau, \{u_k(\tau)\}_{k=1}^{\infty}) d\tau) ds + \varepsilon \\
&\leq 4M \int_0^t \int_0^s \beta(f(s, \tau, \{u_k(\tau)\}_{k=1}^{\infty})) d\tau ds + \varepsilon \\
&\leq 4M \int_0^t \int_0^s \eta(s)\zeta(\tau)\beta(\{u_k(\tau)\}_{k=1}^{\infty}) d\tau ds + \varepsilon \\
&\leq 4M \int_0^t \zeta(\tau)\beta(B(\tau)) d\tau \int_0^t \eta(s) ds + \varepsilon. \\
&\leq 4MK \int_0^t \zeta(\tau)\beta(B(\tau)) d\tau + \varepsilon \\
&\leq 4MK\beta_{\mathcal{C}}(B) \int_0^b \zeta(s) ds + \varepsilon
\end{aligned}$$

Taking supremum in $t \in [0, b]$, we have

$$\beta_{\mathcal{C}}(Q_2 B) \leq 4MK\beta_{\mathcal{C}}(B) \int_0^b \zeta(s) ds + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\beta_{\mathcal{C}}(Q_2 B) \leq 4MK\beta_{\mathcal{C}}(B) \int_0^b \zeta(s) ds \tag{4.3}$$

for any bounded $B \subset W$.

Now, for any subset $B \subset W$, due to Lemma 2.1, (4.2) and (4.3) we have

$$\begin{aligned}
\beta_{\mathcal{C}}(QB) &\leq \beta_{\mathcal{C}}(Q_1 B) + \beta_{\mathcal{C}}(Q_2 B) \\
&\leq (ML + 4MK \int_0^b \zeta(s) ds)\beta_{\mathcal{C}}(B).
\end{aligned}$$

By (4.1) we know that $Q$ is a $\beta_{\mathcal{C}}$-contraction on $W$. By Lemma 2.2, there is a fixed point $u$ of $Q$ in $W$, which is a solution of (1.1)–(1.2). This completes the proof. ☐

Now we give an existence result without the uniform boundedness of $g$.

**Theorem 4.2.** *Suppose that* (HA), (Hf)(1), (Hf)(2'), (Hf)(3), (Hg)(1') *are satisfied. Then the equation* (1.1)–(1.2) *has at least one mild solution if* (4.1) *and the following condition are satisfied*

$$ML + bM \int_0^b p(s) ds \lim_{k \to \infty} \sup \frac{\psi(k)}{k} < 1. \tag{4.4}$$

*Proof.* From (4.4) and the fact that $L < 1$, there exists a constant $k > 0$ such that

$$M(kL + bM \int_0^b p(s) ds \psi(k) + \|g(0)\|) < k.$$

We define $W_0 = \{u \in \mathcal{C}([0,b]);\mathbb{X} : \|u(t)\| \le k,\ \forall t \in [0,b]\}$. Then for every $u \in W_0$, we have

$$\begin{aligned}
\|Qu(t)\| &\le M(\|g(u)\| + \psi(k)\int_0^t\int_0^s p(\tau)d\tau ds) \\
&\le M(\|g(u) - g(0) + g(0)\| + b\psi(k)\int_0^t p(s)ds) \\
&\le M(kL + \|g(0)\| + b\psi(k)\int_0^t p(\tau)d\tau) < k
\end{aligned}$$

for $t \in [0,b]$. This means that $QW_0 \subset W_0$. Define $W = \overline{\text{conv}}QW_0$. The above proof also implies that $QW \subset W$. So we can prove the theorem similar with Theorem 4.1. □

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