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NON-HOMOGENEOUS BOUNDARY-VALUE PROBLEMS OF HIGHER ORDER DIFFERENTIAL EQUATIONS WITH p-LAPLACIAN

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Abstract. We establish sufficient conditions for the existence of positive solutions to five multi-point boundary value problems. These problems have a common equation (in different function domains) and different boundary conditions. It is interesting note that the methods for solving all these problems and most of the reference are based on the Mawhin's coincidence degree theory. First, we present a survey of multi-point boundary-value problems and the motivation of this paper. Then we present the main results which generalize and improve results in the references. We conclude this article with examples of problems that can not solved by methods known so far.

1. Introduction

Multi-point boundary-value problems (BVPs) for differential equations were initialed by Il'in and Moiseev [20] and have received a wide attention because of their potential applications. There are many exciting results concerned with the existence of positive solutions of boundary-value problems of second or higher order differential equations with or without p-Laplacian subjected to the special homogeneous multi-point boundary conditions (BCs); we refer the readers to [1]–[11], [9]–[24] [27]–[47], [49]–[52], [55]–[79]. The methods used for finding positive solutions of these problems at non-resonance cases, or solutions at resonance cases, are critical point theory, fixed point theorems in cones in Banach spaces, fixed point index theory, alternative of Leray-Schauder, upper and lower solution methods with iterative techniques, and so on. There are also several results concerned with the existence of positive solutions of multi-point boundary-value problems for differential equations with non-homogeneous BCs; see for example [12, 13, 25, 26, 48, 53] and the early paper [79]. For the reader's information and to compare our results with the known ones, we now give a simple survey.

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Multi-point boundary-value problems with homogeneous BCs consist of the second order differential equation and the multi-point homogeneous boundary conditions. The second order differential equation is either

$$[\phi(x'(t))]' + f(t, x(t), x'(t)) = 0, \quad t \in (0,1),$$

or one of the following cases

$$\begin{gathered} x''(t) + f(t, x(t), x'(t)) = 0, \quad t \in (0,1), \\ \big[\phi(x'(t))\big]' + f(t, x(t)) = 0, \quad t \in (0,1), \\ x''(t) + f(t, x(t)) = 0, \quad t \in (0,1). \end{gathered}$$

The multi-point homogeneous boundary conditions are either

$$\begin{gathered} x(0) - \sum_{i=1}^{m} \alpha_i x(\xi_i) = x(1) - \sum_{i=1}^{n} \beta_i x(\eta_i) = 0, \\ x'(0) - \sum_{i=1}^{m} \alpha_i x'(\xi_i) = x(1) - \sum_{i=1}^{n} \beta_i x(\eta_i) = 0, \\ x(0) - \sum_{i=1}^{m} \alpha_i x(\xi_i) = x'(1) - \sum_{i=1}^{n} \beta_i x'(\eta_i) = 0, \\ x'(0) - \sum_{i=1}^{m} \alpha_i x'(\xi_i) = x'(1) - \sum_{i=1}^{n} \beta_i x'(\eta_i) = 0, \\ x(0) - \sum_{i=1}^{m} \alpha_i x'(\xi_i) = x(1) - \sum_{i=1}^{n} \beta_i x'(\eta_i) = 0, \\ x(0) - \sum_{i=1}^{m} \alpha_i x'(\xi_i) = x(1) - \sum_{i=1}^{n} \beta_i x(\eta_i) = 0, \\ x(0) - \sum_{i=1}^{m} \alpha_i x'(\xi_i) = x'(1) - \sum_{i=1}^{n} \beta_i x'(\eta_i) = 0, \end{gathered}$$

or their special cases, where $0 < \xi_1 < \dots < \xi_m < 1$ and $0 < \eta_i < \dots < \eta_n < 1$, $\alpha_i, \beta_j \in R$ are constants. These problems were studied extensively in papers [1]–[75] and the references therein.

1. For the second order differential equations, Gupta [16] studied the following multi-point boundary-value problem

$$\begin{gathered} x''(t) = f(t, x(t), x'(t)) + r(t), \quad t \in (0,1), \\ x(0) - \sum_{i=1}^{m} \alpha_i x(\xi_i) = x(1) - \sum_{i=1}^{n} \beta_i x(\eta_i) = 0, \end{gathered} \tag{1.1}$$

and

$$\begin{gathered} x''(t) = f(t, x(t), x'(t)) + r(t), \quad t \in (0,1), \\ x(0) - \sum_{i=1}^{m} \alpha_i x(\xi_i) = x'(1) - \sum_{i=1}^{n} \beta_i x'(\eta_i) = 0, \end{gathered} \tag{1.2}$$

where $0 < \xi_i < \dots < \xi_m < 1$, $0 < \eta_1 < \dots < \eta_n < 1$, $\alpha_i, \beta_i \in \mathbb{R}$ with $(\sum_{i=1}^m \alpha_i \xi_i)(1 - \sum_{i=1}^n \beta_i) \neq (1 - \sum_{i=1}^m \alpha_i)(\sum_{i=1}^n \beta_i \eta_i - 1)$ for (1.1) and with $(1 - \sum_{i=1}^m \alpha_i)(1 - \sum_{i=1}^n \beta_i) \neq 0$ for (1.2). Some existence results for solutions of (1.1) and (1.2) were established in [14]. Liu [36] established the existence results of solutions of (1.1) for the case

$$\sum_{i=1}^{m}\alpha_i\xi_i = 1-\sum_{i=1}^{m}\alpha_i = 1-\sum_{i=1}\beta_i = 1-\sum_{i=1}^{m}\beta_i\xi_i = 0.$$

Liu and Yu [33, 34, 35, 37] studied the existence of solutions of (1.1) and (1.2) at some special cases.

Zhang and Wang [78] studied the multi-point boundary-value problem

$$\begin{gathered} x''(t) = f(t,x(t)), \quad t\in(0,1),\\ x(0)-\sum_{i=1}^{m}\alpha_i x(\xi_i) = x(1)-\sum_{i=1}^{n}\beta_i x(\eta_i) = 0, \end{gathered}\tag{1.3}$$

where $0<\xi_i<\dots<\xi_m<1$, $\alpha_i,\beta_i\in[0,+\infty)$ with $0<\sum_{i=1}^m\alpha_i<1$ and $\sum_{i=1}^m\beta_i<1$. Under certain conditions on f, they established some existence results for positive solutions of (1.3).

Liu in [32], and Liu and Ge in [43] studied the four-point boundary-value problem

$$\begin{gathered} x''(t)+f(t,x(t)) = 0, \quad t\in(0,1),\\ x(0)-\alpha x(\xi) = x(1)-\beta x(\eta) = 0, \end{gathered}\tag{1.4}$$

where $0<\xi,\eta<1$, $\alpha,\beta\geq 0$, f is a nonnegative continuous function. Using the Green's function of its corresponding linear problem, Liu established existence results for at least one or two positive solutions of (1.4).

Ma in [49], and Zhang and Sun in [77] studied the following multi-point boundary-value problem

$$\begin{gathered} x''(t)+a(t)f(x(t)) = 0, \quad t\in(0,1),\\ x(0) = x(1)-\sum_{i=1}^{m}\alpha_i x(\xi_i) = 0, \end{gathered}\tag{1.5}$$

where $0<\xi_i<1$, $\alpha_i\geq 0$ with $\sum_{i=1}^m\alpha_i\xi_i<1$, a and f are nonnegative continuous functions, there is $t_0\in[\xi_m,1]$ so that $a(t_0)>0$. Let

$$\lim_{x\to 0}\frac{f(x)}{x} = l, \quad \lim_{x\to+\infty}\frac{f(x)}{x} = L.$$

It was proved that if $l=0, L=+\infty$ or $l=+\infty, L=0$, then (1.5) has at least one positive solution.

Ma and Castaneda [51] studied the problem

$$\begin{gathered} x''(t)+a(t)f(x(t)) = 0, \quad t\in(0,1),\\ x'(0)-\sum_{i=1}^{m}\alpha_i x'(\xi_i) = x(1)-\sum_{i=1}^{m}\beta_i x(\xi_i) = 0, \end{gathered}\tag{1.6}$$

where $0<\xi_i<\dots<\xi_m<1$, $\alpha_i,\beta_i\geq 0$ with $0<\sum_{i=1}^m\alpha_i<1$ and $0<\sum_{i=1}^m\beta_i<1$ and a and f are nonnegative continuous functions, there is $t_0\in[\xi_m,1]$ so that $a(t_0)>0$. Ma and Castaneda established existence results for positive solutions of (1.6) under the assumptions

$$\lim_{x\to 0}\frac{f(x)}{x} = 0, \quad \lim_{x\to+\infty}\frac{f(x)}{x} = +\infty \text{ or } \lim_{x\to 0}\frac{f(x)}{x} = +\infty, \quad \lim_{x\to+\infty}\frac{f(x)}{x} = 0.$$

2. For second order differential equations with p-Laplacian, Drabek and Takc [8] studied the existence of solutions of the problem

$$\begin{gathered} -(\phi(x'(t))' - \lambda\phi(x) = f(t), \quad t \in (0,T), \\ x(0) = x(T) = 0, \end{gathered} \tag{1.7}$$

In a recent paper [28], the author established multiplicity results for positive solutions of the problems

$$\begin{gathered} \big[\phi_p(x'(t))\big]' + f(t,x(t)) = 0, \quad t \in (0,1), \\ x(0) = \int_0^1 x(s)dh(s), \quad \phi_p(x'(1)) = \int_0^1 \phi_p(x'(s))dg(s), \end{gathered}$$

and

$$\begin{gathered} \big[\phi_p(x'(t))\big]' + f(t,x(t)) = 0, \quad t \in (0,1), \\ \phi_p(x'(0)) = \int_0^1 \phi_p(x'(s))dh(s), \quad x(1) = \int_0^1 x(s)dg(s). \end{gathered}$$

Gupta [17] studied the existence of solutions of the problem

$$\begin{gathered} \big[\phi(x'(t))\big]' + f(t,x(t),x'(t)) + e(t) = 0, \quad t \in (0,1), \\ x(0) - \sum_{i=1}^m \alpha_i x(\xi_i) = x(1) - \sum_{i=1}^m \beta_i x(\xi_i) = 0 \end{gathered} \tag{1.8}$$

by using topological degree and some a priori estimates.

Bai and Fang [6] investigated the following multi-point boundary-value problem

$$\begin{gathered} \big[\phi(x'(t))\big]' + a(t)f(t,x(t)) = 0, \quad t \in (0,1), \\ x(0) = x(1) - \sum_{i=1}^m \beta_i x(\xi_i) = 0, \end{gathered} \tag{1.9}$$

where $0 < \xi_i < \dots < \xi_m < 1$, $\beta_i \ge 0$ with $\sum_{i=1}^m \beta_i\xi_i < 1$, a is continuous and nonnegative and there is $t_0 \in [\xi_m, 1]$ so that $a(t_0) > 0$, f is a continuous nonnegative function. The purpose of [6] is to generalize the results in [49]. Wang and Ge [63], Ji, Feng and Ge [21], Feng, Ge and Jiang [9], Rynne [58] studied the existence of multiple positive solutions of the following more general problem

$$\begin{gathered} \big[\phi(x'(t))\big]' + a(t)f(t,x(t)) = 0, \quad t \in (0,1), \\ x(0) - \sum_{i=1}^m \alpha_i x(\xi_i) = x(1) - \sum_{i=1}^m \beta_i x(\xi_i) = 0 \end{gathered}$$

by using fixed point theorems for operators in cones. Sun, Qu and Ge [62] using the monotone iterative technique established existence results of positive solutions of the problem

$$\begin{gathered} \big[\phi(x'(t))\big]' + a(t)f(t,x(t),x'(t)) = 0, \quad t \in (0,1), \\ x(0) - \sum_{i=1}^m \alpha_i x(\xi_i) = x(1) - \sum_{i=1}^m \beta_i x(\xi_i) = 0. \end{gathered}$$

Bai and Fang [5] studied the problem

$$\begin{gathered}\big[\phi(x'(t))\big]'+f(t,x(t))=0,\quad t\in(0,1),\\ x'(0)-\sum_{i=1}^{m}\alpha_i x'(\xi_i)=x(1)-\sum_{i=1}^{m}\beta_i x(\xi_i)=0,\end{gathered}\tag{1.10}$$

where $0<\xi_i<\dots<\xi_m<1$, $\alpha_i\geq 0,\beta_i\geq 0$ with $0<\sum_{i=1}^{m}\alpha_i<1$ and $0<\sum_{i=1}^{m}\beta_i<1$, f is continuous and nonnegative. The purpose of [5] is to generalize and improve the results in [51]. In paper Ma, Du and Ge [54] studied (1.6) by using the monotone iterative methods. The existence of monotone positive solutions of (1.6) were obtained. Based upon the fixed point theorem due to Avery and Peterson [4], Wang and Ge [64], Sun, Ge and Zhao [61] established existence results of multiple positive solutions of the following problems

$$\begin{gathered}\big[\phi(x'(t))\big]'+a(t)f(t,x(t),x'(t))=0,\quad t\in(0,1),\\ x'(0)-\sum_{i=1}^{m}\alpha_i x'(\xi_i)=x(1)-\sum_{i=1}^{m}\beta_i x(\xi_i)=0\end{gathered}$$

and

$$\begin{gathered}\big[\phi(x'(t))\big]'+a(t)f(t,x(t),x'(t))=0,\quad t\in(0,1),\\ x(0)-\sum_{i=1}^{m}\alpha_i x(\xi_i)=x'(1)-\sum_{i=1}^{m}\beta_i x'(\xi_i)=0.\end{gathered}$$

In [28, 37], the authors studied the existence of solutions of the following BVPs at resonance cases

$$\begin{gathered}x''(t)=f(t,x(t),x'(t))+e(t),\quad 0t\in(0,T),\\ x'(0)=\alpha x'(\xi),\ x'(1)=\sum_{i=1}^{m}\beta_i x'(\xi_i).\end{gathered}\tag{1.11}$$

In a recent paper [11], the authors investigated the existence of solutions of the following problem for p-Laplacian differential equation

$$\begin{gathered}(\phi(x'(t))'=f(t,x(t),x'(t)),\quad t\in(0,T),\\ x'(0)=0,\quad \theta(x'(1))=\sum_{i=1}^{m}\alpha_i\theta(x'(\xi_i)),\end{gathered}\tag{1.12}$$

where θ and ϕ are two odd increasing homeomorphisms from R to R with $\phi(0)=\theta(0)=0$.

In the recent papers [19, 24, 25, 29, 36, 56, 60, 61, 63, 64, 65, 66, 71, 76], the authors studied the existence of multiple positive solutions of (1.8), (1.9), (1.10) or other more general multi-point boundary-value problems, respectively, by using of multiple fixed point theorems in cones in Banach spaces such as the five functionals fixed point theorem [19], the fixed-point index theory [59], the fixed point theorem due to Avery and Peterson, a two-fixed-point theorem [19, 61, 63, 64], Krasnoselskii's fixed point theorem and the contraction mapping principle [22, 29, 56, 60, 71], the Leggett-Williams fixed-point theorem [23, 36], the generalization of polar coordinates [65], using the solution of an implicit functional equation [22, 23].

3. For higher order differential equations, there have been many papers discussed the existence of solutions of multi-point boundary-value problems for third order differential equations [15, 47, 55]. Ma [47] studied the solvability of the problem

$$\begin{gathered} x'''(t) + k^2 x'(t) + g(x(t), x'(t)) = p(t), \quad t \in (0, \pi), \\ x(0) = 0, \quad x'(0) = x'(\pi) = 0, \end{gathered} \tag{1.13}$$

where $k \in N$, g is continuous and bounded, p is continuous. In [15, 55], the authors investigated the solvability of the problem

$$\begin{gathered} x'''(t) + k^2 x'(t) + g(t, x(t), x'(t), x''(t)) = p(t), \\ x'(0) = x'(1) = 0, \quad x(0) = 0, \end{gathered} \tag{1.14}$$

where g and p are continuous, $k \in \mathbb{R}$. It was supposed in [55] that g is bounded and in [15] g satisfies $g(t, u, v, w)v \geq 0$ for $t \in [0, 1]$, $(u, v, w) \in \mathbb{R}^3$,

$$\lim_{|v| \to \infty} \frac{g(t, u, v, w)}{v} < 3\pi^2 \quad \text{uniformly in } t, u, w.$$

The upper and lower solution methods with monotone iterative technique are used to solve multi-point boundary-value problems for third or fourth order differential equations in papers [76] and [66].

In [40], the authors studied the problem

$$\begin{gathered} x^{(n)}(t)) + \lambda f(x(t)) = 0, \quad t \in (0, 1), \\ x^{(i)}(0) = 0, \quad i = 0, \dots, n-3, \\ x^{(n-2)}(0) - \alpha x^{(n-2)}(\eta) = x^{(n-2)}(1) - \beta x^{(n-2)}(\eta) = 0, \end{gathered} \tag{1.15}$$

the existence results for positive solutions of (1.15) were established in [40] in the case that the nonlinearity f changes sign.

The existence of positive solutions of the following two problems:

$$\begin{gathered} x^{(n)}(t)) + \phi(t) f(t, x(t)), \dots, x^{(n-2)}(t)) = 0, \quad t \in (0, 1), \\ x^{(i)}(0) = 0, \quad i = 0, \dots, n-2, \quad x^{(n-1)}(0) = 0, \end{gathered} \tag{1.16}$$

and

$$\begin{gathered} x^{(n)}(t)) + \phi(t) f(t, x(t)), \dots, x^{(n-2)}(t)) = 0, \quad t \in (0, 1), \\ x^{(i)}(0) = 0, \quad i = 0, \dots, n-2, \quad x^{(n-2)}(0) = 0, \end{gathered} \tag{1.17}$$

were studied in [2, 73].

4. For Sturm-Liouville type multi-point boundary conditions, Grossinho [12] studied the problem

$$\begin{gathered} x'''(t) + f(t, x(t), x'(t), x''(t)) = 0, \quad t \in (0, 1), \\ x(0) = 0, \quad ax'(0) - bx''(0) = A, \quad cx'(1) + dx''(1) = B. \end{gathered} \tag{1.18}$$

By using theory of Leray-Schauder degree, it was proved that (1.18) has solutions under the assumptions that there exist super and lower solutions of the corresponding problem.

Agarwal and Wong [3], Qi [57] investigated the solvability of the following problem with Sturm-Liouville type boundary conditions

$$
\begin{gathered}
x^{(n)}(t)=f(t,x(t),\dots,x^{(n-2)}(t)),\quad t\in(0,1),\\
x^{(i)}(0)=0,\quad i=0,\dots,n-3,\\
\alpha x^{(n-2)}(0)-\beta x^{(n-1)}(0)=\gamma x^{(n-2)}(1)+\tau x^{(n-1)}(1)=0,
\end{gathered}
\tag{1.19}
$$

The authors in [24] studied the existence and nonexistence of solutions of a situation more general than (1.18).

Lian and Wong [31] studied the existence of positive solutions of the following BVPs consisting of the p-Laplacian differential equation and Sturm-Liouville boundary conditions

$$
\begin{gathered}
\big[\phi(x^{(n-1)}(t)\big]'+f(t,x(t),\dots,x^{(n-2)}(t))=0,\quad t\in(0,1),\\
x^{(i)}(0)=0,\quad i=0,\dots,n-3,\\
\alpha x^{(n-2)}(0)-\beta x^{(n-1)}(0)=\gamma x^{(n-2)}(1)+\tau x^{(n-1)}(1)=0,
\end{gathered}
\tag{1.20}
$$

In all above mentioned papers, all of the boundary conditions concerned are homogeneous cases. However, in many applications, BVPs are nonhomogeneous BVPc, for example,

$$
\begin{gathered}
y''=\frac{1}{\lambda}(1+y^2)^{\frac12},\quad t\in(a,b),\\
y(a)=a\alpha,\quad y(b)=\beta
\end{gathered}
$$

and

$$
\begin{gathered}
y''=-\frac{(1+y'(t))^2}{2(y(t)-\alpha)},\quad t\in(a,b),\\
y(a)=a\alpha,\quad y(b)=\beta
\end{gathered}
$$

are very well known BVPs, which were proposed in 1690 and 1696, respectively. In 1964, The BVPs studied by Zhidkov and Shirikov in [USSR Computational Mathematics and Mathematical Physics, 4(1964)18-35] and by Lee in [Chemical Engineering Science, 21(1966)183-194] are nonhomogeneous BVPs too.

There are also several papers concerning with the existence of positive solutions of BVPs for differential equations with non-homogeneous BCs. Ma [48] studied existence of positive solutions of the following BVP consisting of second order differential equations and three-point BC

$$
\begin{gathered}
x''(t))+a(t)f(x(t))=0,\quad t\in(0,1),\\
x(0)=0,\quad x(1)-\alpha x(\eta)=b,
\end{gathered}
\tag{1.21}
$$

In a recent paper [25, 26], using lower and upper solutions methods, Kong and Kong established results for solutions and positive solutions of the following two problems

$$
\begin{gathered}
x''(t))+f(t,x(t),x'(t))=0,\quad t\in(0,1),\\
x'(0)-\sum_{i=1}^m\alpha_i x'(\xi_i)=\lambda_1,\quad x(1)-\sum_{i=1}^m\beta_i x(\xi_i)=\lambda_2,
\end{gathered}
\tag{1.22}
$$

and

$$\begin{gathered} x''(t)) + f(t, x(t), x'(t)) = 0, \quad t \in (0,1), \\ x(0) - \sum_{i=1}^{m} \alpha_i x(\xi_i) = \lambda_1, \quad x(1) - \sum_{i=1}^{m} \beta_i x(\xi_i) = \lambda_2, \end{gathered} \tag{1.23}$$

respectively. We note that the boundary conditions in (1.17), (1.20), (1.21) and (1.22) are two-parameter non-homogeneous BCs.

The purpose of this paper is to investigate the more generalized BVPs for higher order differential equation with p-Laplacian subjected to non-homogeneous BCs, in which the nonlinearity f contains $t, x, \dots, x^{(n-1)}$, i.e. the problems

$$\begin{gathered} \big[\phi(x^{(n-1)}(t))\big]' + f(t, x(t), \dots, x^{(n-1)}(t)) = 0, \quad t \in (0,1), \\ x^{(n-2)}(0) - \sum_{i=1}^{m} \alpha_i x^{(n-2)}(\xi_i) = \lambda_1, \\ x^{(n-2)}(1) - \sum_{i=1}^{m} \beta_i x^{(n-2)}(\xi_i) = \lambda_2, \\ x^{(i)}(0) = 0, \quad i = 0, \dots, n-3; \end{gathered} \tag{1.24}$$

$$\begin{gathered} \big[\phi(x^{(n-1)}(t))\big]' + f(t, x(t), \dots, x^{(n-1)}(t)) = 0, \quad t \in (0,1), \\ x^{(n-1)}(0) - \sum_{i=1}^{m} \alpha_i x^{(n-1)}(\xi_i) = \lambda_1, \\ x^{(n-2)}(1) - \sum_{i=1}^{m} \beta_i x^{(n-2)}(\xi_i) = \lambda_2, \\ x^{(i)}(0) = 0, \quad i = 0, \dots, n-3; \end{gathered} \tag{1.25}$$

$$\begin{gathered} \big[\phi(x^{(n-1)}(t))\big]' + f(t, x(t), \dots, x^{(n-1)}(t)) = 0, \quad t \in (0,1), \\ x^{(n-2)}(0) - \sum_{i=1}^{m} \alpha_i x^{(n-2)}(\xi_i) = \lambda_1, \\ x^{(n-1)}(1) - \sum_{i=1}^{m} \beta_i x^{(n-1)}(\xi_i) = \lambda_2, \\ x^{(i)}(0) = 0, \quad i = 0, \dots, n-3; \end{gathered} \tag{1.26}$$

$$\begin{gathered} \big[\phi(x^{(n-1)}(t))\big]' + f(t, x(t), \dots, x^{(n-1)}(t)) = 0, \quad t \in (0,1), \\ x^{(n-2)}(0) - \sum_{i=1}^{m} \alpha_i x^{(n-1)}(\xi_i) = \lambda_1, \\ x^{(n-2)}(1) + \sum_{i=1}^{m} \beta_i x^{(n-1)}(\xi_i) = \lambda_2, \\ x^{(i)}(0) = 0, \quad i = 0, \dots, n-3; \end{gathered} \tag{1.27}$$

and

$$\begin{gathered}
\big[\phi(x^{(n-1)}(t))\big]' + f(t,x(t),\dots,x^{(n-1)}(t)) = 0, \quad t \in (0,1),\\
\phi(x^{(n-1)}(0)) - \sum_{i=1}^{m} \alpha_i \phi(x^{(n-1)}(\xi_i)) = \lambda_1,\\
\theta(x^{(n-1)}(1)) + \sum_{i=1}^{m} \beta_i \theta(x^{(n-1)}(\xi_i)) = \lambda_2,\\
x^{(i)}(0) = 0, \quad i = 0,\dots,n-3;
\end{gathered} \tag{1.28}$$

where $n \geq 2$, $0 < \xi_i < \dots < \xi_m < 1$, $\alpha_i, \beta_i \in \mathbb{R}$, $\lambda_1, \lambda_2 \in \mathbb{R}$, f is continuous, ϕ is called p-Laplacian, $\phi(x) = |x|^{p-2}x$ for $x \neq 0$ and $\phi(0) = 0$ with $p > 1$, its inverse function is denoted by $\phi^{-1}(x)$ with $\phi^{-1}(x) = |x|^{q-2}x$ for $x \neq 0$ and $\phi^{-1}(0) = 0$, where $1/p + 1/q = 1$, θ is an odd increasing homeomorphisms from R to R with $\theta(0) = 0$.

We establish sufficient conditions for the existence of at least one positive solution of (1.24), (1.25), (1.26), (1.27), and at least one solution of (1.28), respectively.

The first motivation of this paper is that it is of significance to investigate the existence of positive solutions of (1.9) and (1.10) since the operators defined in [5, 6, 48, 49] are can not be used; furthermore, it is more interesting to establish existence results for positive solutions of higher order BVPs with non-homogeneous BCs.

The second motivation to study (1.24), (1.25), (1.26), (1.27) and (1.28) comes from the facts that

- (i) (1.24) contains (1.1), (1.3), (1.4), (1.5), (1.7) (1.8), (1.9), (1.13), (1.14), (1.15), (1.17) and (1.23) as special cases;
- (ii) (1.25) contains (1.6), (1.10) and (1.22) as special cases;
- (iii) (1.26) contains (1.2) and (1.16) as special cases;
- (iv) (1.27) contains (1.18) and (1.19) as special cases;
- (v) (1.28) contains (1.11) and (1.12) as special cases.

Furthermore, in most of the known papers, the nonlinearity f only depends on a part of lower derivatives, the problem is that under what conditions problems have solutions when f depends on all lower derivatives, such as in BVPs above, f depends on $x, x', \dots, x^{(n-1)}$.

The third motivation is that there exist several papers discussing the solvability of Sturm-Liouville type boundary-value problems for p-Laplacian differential equations, whereas there is few paper concerned with the solvability of Sturm-Liouville type multi-point boundary-value problems for p-Laplacian differential equations, such as (1.27).

The fourth motivation comes from the challenge to find simple conditions on the function f, for the existence of a solution of (1.28), as the nonlinear homeomorphisms ϕ and θ generating, respectively, the differential operator and the boundary conditions are different. The techniques for studying the existence of positive solutions of multi-point boundary-value problems consisting of the higher-order differential equation with p-Laplacian and non-homogeneous BCs are few.

Additional motivation is that the coincidence degree theory by Mawhin is reported to be an effective approach to the study the existence of periodic solutions of differential equations with or without delays, the existence of solutions of multi-point boundary-value problems at resonance case for differential equations; see for example [33, 35, 37, 39, 45] and the references therein, but there is few paper concerning the existence of positive solutions of non-homogeneous multi-point boundary-value problems for higher order differential equations with p-Laplacian by using the coincidence degree theory.

The following of this paper is organized as follows: the main results and remarks are presented in Section 2, and some examples are given in Section 3. The methods used and the results obtained in this paper are different from those in known papers. Our theorems generalize and improve the known ones.

2. Main Results

In this section, we present the main results in this paper, whose proofs will be done by using the following fixed point theorem due to Mawhin.

Let X and Y be real Banach spaces, $L : D(L) \subset X \to Y$ be a Fredholm operator of index zero, $P : X \to X$, $Q : Y \to Y$ be projectors such that

$$\operatorname{Im} P = \operatorname{Ker} L, \quad \operatorname{Ker} Q = \operatorname{Im} L, \quad X = \operatorname{Ker} L \oplus \operatorname{Ker} P, \quad Y = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

It follows that

$$L|_{D(L)\cap \operatorname{Ker} P} : D(L) \cap \operatorname{Ker} P \to \operatorname{Im} L$$

is invertible, we denote the inverse of that map by K_p.

If Ω is an open bounded subset of X, $D(L)\cap\overline{\Omega} \neq \emptyset$, the map $N : X \to Y$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N : \overline{\Omega} \to X$ is compact.

Lemma 2.1 ([10]). *Let L be a Fredholm operator of index zero and let N be L-compact on Ω. Assume that the following conditions are satisfied:*

- (i) *$Lx \neq \lambda Nx$ for every $(x, \lambda) \in [(D(L) \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0, 1)$;*
- (ii) *$Nx \notin \operatorname{Im} L$ for every $x \in \operatorname{Ker} L \cap \partial\Omega$;*
- (iii) *$\deg(\wedge QN\big|_{\operatorname{Ker} L}, \ \Omega \cap \operatorname{Ker} L, 0) \neq 0$, where $\wedge : Y/\operatorname{Im} L \to \operatorname{Ker} L$ is an isomorphism.*

Then the equation $Lx = Nx$ has at least one solution in $D(L) \cap \overline{\Omega}$.

In this paper, we choose $X = C^{n-2}[0, 1] \times C^0[0, 1]$ with the norm

$$\|(x, y)\| = \max\{\|x\|_\infty, \dots, \|x^{(n-2)}\|_\infty, \|y\|_\infty\},$$

and $Y = C^0[0, 1] \times C^0[0, 1] \times R^2$ with the norm

$$\|(x, y, a, b)\| = \max\{\|x\|_\infty, \|y\|_\infty, |a|, |b|\},$$

then X and Y are real Banach spaces. Let

$$D(L) = \big\{(x_1, x_2) \in C^{n-1}[0, 1] \times C^1[0, 1] : x_1^{(i)}(0) = 0,\ i = 0, \dots, n-3\big\}.$$

Now we prove an important lemma. Then we will establish existence results for positive solutions of (1.24), (1.25), (1.26), (1.27) and (1.28) in sub-section 2.1, 2.2, 2.3, 2.4 and 2.5, respectively.

Lemma 2.2. *$\sum_{i=1}^m a_i^\sigma \le K_\sigma^{m-1}(\sum_{i=1}^m a_i)^\sigma$ for all $a_i \ge 0$ and $\sigma > 0$, where K_σ is defined by $K_\sigma = 1$ for $\sigma \ge 1$ and $K_\sigma = 2$ for $\sigma \in (0, 1)$.*

Proof. **Case 1.** $m = 2$. Without loss of generality, suppose $a_1 \ge a_2$. Let $g(x) = K_\sigma(1 + x)^\sigma - (1 + x^\sigma), x \in [1, +\infty)$, then

$$g(1) = K_\sigma 2^\sigma - 2 = \begin{cases} 2^\sigma - 2 \ge 0, & \sigma \ge 1, \\ 2^{\sigma+1} - 2 \ge 0, & \sigma \in (0, 1) \end{cases}$$

and for $x \in [1, \infty)$, we get

$$g'(x) = \sigma x^{\sigma-1}[K_\sigma(1+1/x)^{\sigma-1} - 1] \geq \begin{cases} 0, & \sigma \geq 1, \\ \sigma x^{\sigma-1}[2(1+1/1)^{0-1} - 1] = 0, & \sigma \in (0,1). \end{cases}$$

We get that $g(x) \geq g(1)$ for all $x \geq 1$ and so $1+x^\sigma \leq K_\sigma(1+x)^\sigma$ for all $x \in [1, +\infty)$. Hence $a_1^\sigma + a_2^\sigma = a_2^\sigma[1 + (a_1/a_2)^\sigma] \leq K_\sigma a_2^\sigma[1 + a_1/a_2]^\sigma = K_\sigma(a_1 + a_2)^\sigma$.
Case 2. $m > 2$. It is easy to see that

$$\begin{aligned}
\sum_{i=1}^m a_i^\sigma &= a_1^\sigma + a_2^\sigma + \sum_{i=3}^m a_i^\sigma \\
&\leq K_\sigma(a_1 + a_2)^\sigma + \sum_{i=3}^m a_i^\sigma \\
&\leq K_\sigma\Big((a_1 + a_2)^\sigma + \sum_{i=3}^m a_i^\sigma\Big) \\
&\leq K_\sigma\Big((a_1 + a_2)^\sigma + a_3^\sigma + \sum_{i=4}^m a_i^\sigma\Big) \\
&\leq K_\sigma\Big(K_\sigma(a_1 + a_2 + a_3)^\sigma + \sum_{i=4}^m a_i^\sigma\Big) \\
&\leq K_\sigma^2\Big((a_1 + a_2 + a_3)^\sigma + \sum_{i=4}^m a_i^\sigma\Big) \\
&\leq \dots \\
&\leq K_\sigma^{m-1}\Big(\sum_{i=1}^m a_i\Big)^\sigma.
\end{aligned}$$

The proof is complete. □

Remark 2.3. It is easy to see that

$$\sum_{i=1}^m \phi(a_i) \leq K_{p-1}^{m-1}\phi(\sum_{i=1}^m a_i), \quad \sum_{i=1}^m \phi^{-1}(a_i) \leq K_{q-1}^{m-1}\phi^{-1}(\sum_{i=1}^m a_i).$$

2.1. **Positive solutions of Problem** (1.24)**.** Let

$$f^*(t, x_0, \dots, x_{n-1}) = f(t, \overline{x_0}, \dots, \overline{x_{n-2}}, x_{n-1}), (t, x_0, \dots, x_{n-1}) \in [0,1] \times \mathbb{R}^n,$$

where $\overline{x} = \max\{0, x\}$. The following assumptions, which will be used in the proofs of all lemmas in this sub-section, are supposed.

- (H1) $f : [0,1] \times [0, +\infty)^{n-1} \times R \to [0, +\infty)$ is continuous with $f(t, 0, \dots, 0) \not\equiv 0$ on each sub-interval of [0,1];
- (H2) $\lambda_1, \lambda_2 \geq 0$, $\alpha_i \geq 0$, $\beta_i \geq 0$ satisfy $0 < \sum_{i=1}^m \alpha_i < 1$, $\quad 0 < \sum_{i=1}^m \beta_i < 1$ and $\lambda_1/(1 - \sum_{i=1}^m \alpha_i) = \lambda_2/(1 - \sum_{i=1}^m \beta_i)$;
- (H3) there exist continuous nonnegative functions a, b_i and c so that

$$|f(t, x_0, \dots, x_{n-2}, x_{n-1})| \leq a(t) + \sum_{i=0}^{n-2} b_i(t)\phi(|x_i|) + c(t)\phi(|x_{n-1}|),$$

for $(t, x_0, \dots, x_{n-1}) \in [0,1] \times R^n$;

(H4) The following inequality holds

$$K_{q-1}^{m-1}\phi\Big(1+\frac{\sum_{i=1}^m\alpha_i\xi_i}{1-\sum_{i=1}^m\alpha_i}\Big)\Big[\sum_{i=0}^{n-3}\phi\big(\frac{1}{(n-2-i)!}\big)\int_0^1 b_i(s)ds + \int_0^1 b_{n-2}(s)ds\Big]+\int_0^1 c(s)ds<1.$$

We consider the problem

$$\begin{gathered}\big[\phi(x^{(n-1)}(t))\big]'+f^*(t,x(t),\dots,x^{(n-1)}(t))=0,\quad t\in(0,1),\\ x^{(i)}(0)=0,\quad i=0,\dots,n-3,\\ x^{(n-2)}(0)-\sum_{i=1}^m\alpha_ix^{(n-2)}(\xi_i)=\lambda_1,\\ x^{(n-2)}(1))-\sum_{i=1}^m\beta_ix^{(n-2)}(\xi_i)=\lambda_2.\end{gathered}\tag{2.1}$$

Lemma 2.4. *If* (H1)–(H2) *hold and x is a solution of* (2.1)*, then $x(t)>0$ for all $t\in(0,1)$, and x is a positive solution of* (1.24).

Proof. (H1) implies that $[\phi(x^{(n-1)}(t)]'=-f^*(t,x(t),\dots,x^{(n-1)}(t))\le 0$ ($\not\equiv 0$), and then $x^{(n-1)}(t)$ is decreasing and so $x^{(n-2)}$ is concave on [0,1], thus

$$\min_{t\in[0,1]}x^{(n-2)}(t)=\min\{x^{(n-2)}(0),x^{(n-2)}(1)\}.$$

Together with the boundary conditions in (29) and (H2), we get that

$$x^{(n-2)}(0)=\sum_{i=1}^m\alpha_ix^{(n-2)}(\xi_i)+\lambda_1\ge\sum_{i=1}^m\alpha_i\min\{x^{(n-2)}(0),x^{(n-2)}(1)\},\tag{2.2}$$

and

$$x^{(n-2)}(1)=\sum_{i=1}^m\beta_ix^{(n-2)}(\xi_i)+\lambda_2\ge\sum_{i=1}^m\beta_i\min\{x^{(n-2)}(0),x^{(n-2)}(1)\}.$$

Without loss of generality, assume that $\sum_{i=1}^m\alpha_i\ge\sum_{i=1}^m\beta_i$.

If $\min\{x^{(n-2)}(0),x^{(n-2)}(1)\}<0$, then

$$x^{(n-2)}(1)\ge\sum_{i=1}^m\beta_i\min\{x^{(n-2)}(0),x^{(n-2)}(1)\}\ge\sum_{i=1}^m\alpha_i\min\{x^{(n-2)}(0),x^{(n-2)}(1)\}.$$

Together with (30), we have

$$\min\{x^{(n-2)}(0),x^{(n-2)}(1)\}\ge\sum_{i=1}^m\alpha_i\min\{x^{(n-2)}(0),x^{(n-2)}(1)\}.$$

Hence $\min\{x^{(n-2)}(0),x^{(n-2)}(1)\}\ge 0$. It follows that $\min\{x^{(n-2)}(0),x^{(n-2)}(1)\}\ge 0$. So (H1) implies that $x^{(n-2)}(t)>0$ for all $t\in(0,1)$. Then from the boundary conditions, we get $x^{(i)}(t)>0$ for all $t\in(0,1)$ and $i=0,\dots,n-3$. Then $f^*(t,x(t),\dots,x^{(n-1)}(t))=f(t,x(t),\dots,x^{(n-1)}(t))$. Thus x is a positive solution of (1.24). The proof is complete. $\square$

Lemma 2.5. *If* (H1)–(H2) *hold and x is a solutions of* (2.1)*, then there exists $\xi\in[0,1]$ such that $x^{(n-1)}(\xi)=0$.*

Proof. In fact, if $x^{(n-1)}(t) > 0$ for all $t \in [0,1]$, then

$$x^{(n-2)}(0) = \sum_{i=1}^{m} \alpha_i x^{(n-2)}(\xi_i) + \lambda_1 > \sum_{i=1}^{m} \alpha_i x^{(n-2)}(0) + \lambda_1,$$

then $x^{(n-2)}(0) > \lambda_1/(1 - \sum_{i=1}^{m} \alpha_i)$, it follows that $x^{(n-2)}(1) > \lambda_1/(1 - \sum_{i=1}^{m} \alpha_i)$. On the other hand,

$$x^{(n-2)}(1) = \sum_{i=1}^{m} \beta_i x^{(n-2)}(\xi_i) + \lambda_2 < \sum_{i=1}^{m} \beta_i x^{(n-2)}(1) + \lambda_2,$$

thus

$$x^{(n-2)}(1) < \lambda_2/(1 - \sum_{i=1}^{m} \beta_i) = \lambda_1/(1 - \sum_{i=1}^{m} \alpha_i) < x^{(n-2)}(1),$$

a contradiction. if $x^{(n-1)}(t) < 0$ for all $t \in [0,1]$, then

$$x^{(n-2)}(1) = \sum_{i=1}^{m} \beta_i x^{(n-2)}(\xi_i) + \lambda_2 > \sum_{i=1}^{m} \beta_i x^{(n-2)}(1) + \lambda_2,$$

then $x^{(n-2)}(1) > \lambda_2/(1 - \sum_{i=1}^{m} \beta_i)$, it follows that $x^{(n-2)}(0) > \lambda_2/(1 - \sum_{i=1}^{m} \beta_i)$. On the other hand,

$$x^{(n-2)}(0) = \sum_{i=1}^{m} \alpha_i x^{(n-2)}(\xi_i) + \lambda_1 < \sum_{i=1}^{m} \alpha_i x^{(n-2)}(0) + \lambda_1,$$

thus

$$x^{(n-2)}(0) < \lambda_1/(1 - \sum_{i=1}^{m} \alpha_i) = \lambda_2/(1 - \sum_{i=1}^{m} \beta_i) < x^{(n-2)}(0),$$

contradiction too. Hence there is $\xi \in [0,1]$ so that $x^{(n-1)}(\xi) = 0$. The proof is complete. □

Lemma 2.6. *If (x_1, x_2) is a solution of the problem*

$$\begin{gathered}
x_1^{(n-1)}(t) = \phi^{-1}(x_2(t)), \quad t \in [0,1], \\
x_2'(t) = -f^*(t, x_1(t), \dots, x_1^{(n-2)}(t), \phi^{-1}(x_2(t))), \quad t \in [0,1], \\
x_1^{(n-2)}(0) - \sum_{i=1}^{m} \alpha_i x_1^{(n-2)}(\xi_i) = \lambda_1, \\
x_1^{(n-2)}(1) - \sum_{i=1}^{n} \beta_i x_1^{(n-2)}(\xi_i) = \lambda_2, \\
x_1^{(i)}(0) = 0, \quad i = 0, \dots, n-3,
\end{gathered} \tag{2.3}$$

then x_1 is a solution of (2.1).

The proof of the above lemma is simple; os it is omitted. Define the operators

$$L(x_1, x_2) = \Big(x_1^{(n-1)}, x_2', x_1^{(n-2)}(0) - \sum_{i=1}^{m} \alpha_i x_1^{(n-2)}(\xi_i), x_1^{(n-2)}(1) - \sum_{i=1}^{n} \beta_i x_1^{(n-2)}(\xi_i)\Big),$$

$(x_1, x_2) \in X \cap D(L)$;

$$N(x_1, x_2) = (\phi^{-1}(x_2), -f^*(t, x_1, \dots, x^{(n-2)}, \phi^{-1}(x_2)), \lambda_1, \lambda_2), \quad (x_1, x_2) \in X.$$

Under the assumptions (H1)–(H2), it is easy to show the following results:

(i) $\operatorname{Ker} L = \{(0, c) : c \in \mathbb{R}\}$ and

$$\begin{aligned}\operatorname{Im} L = \Big\{(y_1, y_2, a, b) :& \frac{1}{1-\sum_{i=1}^m \alpha_i}(\sum_{i=1}^m \alpha_i \int_0^{\xi_i} y_1(s)ds + a)\\ &+ \frac{1}{1-\sum_{i=1}^m \beta_i}(\int_0^1 y_1(s)ds - \sum_{i=1}^m \beta_i \int_0^{\xi_i} y_1(s)ds - b) = 0\Big\}\end{aligned}$$

(ii) L is a Fredholm operator of index zero;

(iii) There exist projectors $P : X \to X$ and $Q : Y \to Y$ such that $\operatorname{Ker} L = \operatorname{Im} P$ and $\operatorname{Ker} Q = \operatorname{Im} L$. Furthermore, let $\Omega \subset X$ be an open bounded subset with $\overline{\Omega} \cap D(L) \neq \emptyset$, then N is L-compact on $\overline{\Omega}$;

(iv) $x = (x_1, x_2)$ is a solution of (2.3) if and only if x is a solution of the operator equation $Lx = Nx$ in $D(L)$.

We present the projectors P and Q as follows: $P(x_1, x_2) = (0, x_2(0))$ for all $x = (x_1, x_2) \in X$ and

$$\begin{aligned}Q(y_1, y_2, a, b) = \Big(\frac{1}{\Delta}\Big[&\frac{1}{1-\sum_{i=1}^m \alpha_i}\Big(\sum_{i=1}^m \alpha_i \int_0^{\xi_i} y_1(s)ds + a\Big)\\ &+ \frac{1}{1-\sum_{i=1}^m \beta_i}\Big(\int_0^1 y_1(s)ds - \sum_{i=1}^m \beta_i \int_0^{\xi_i} y_1(s)ds - b\Big)\Big], 0, 0, 0\Big),\end{aligned}$$

where

$$\Delta = \frac{1}{1-\sum_{i=1}^m \alpha_i}\sum_{i=1}^m \alpha_i\xi_i + \frac{1}{1-\sum_{i=1}^m \beta_i}\Big(1 - \sum_{i=1}^m \beta_i\xi_i\Big).$$

The generalized inverse of $L : D(L) \cap \operatorname{Ker} P \to \operatorname{Im} L$ is defined by

$$\begin{aligned}K_P(y_1, y_2, a, b) = \Big(&\int_0^t \frac{(t-s)^{n-2}}{(n-2)!} y_1(s)ds\\ &+ \frac{t^{n-2}}{(n-2)!}\frac{1}{1-\sum_{i=1}^m \alpha_i}\Big(\sum_{i=1}^m \alpha_i \int_0^{\xi_i} y_1(s)ds + a\Big), \int_0^t y_2(s)ds\Big),\end{aligned}$$

the isomorphism $\wedge : Y/\operatorname{Im} L \to \operatorname{Ker} L$ is defined by $\wedge(c, 0, 0, 0) = (0, c)$.

Lemma 2.7. *Suppose that (H1)-(H4) hold, and let*

$$\Omega_0 = \{(x_1, x_2) \in D(L) \setminus \operatorname{Ker} L : L(x_1, x_2) = \lambda N(x_1, x_2) \text{ for some } \lambda \in (0, 1)\}.$$

Then Ω_0 is bounded.

Proof. For $(x_1, x_2) \in \Omega_0$, we get $L(x_1, x_2) = \lambda N(x_1, x_2)$. Then

$$\begin{gathered}x_1^{(n-1)}(t) = \lambda\phi^{-1}(x_2(t)), \quad t \in [0, 1],\\ x_2'(t) = -\lambda f^*(t, x_1(t), \dots, x_1^{(n-2)}(t), \phi^{-1}(x_2(t))), \quad t \in [0, 1],\\ x_1^{(n-2)}(0) - \sum_{i=1}^m \alpha_i x_1^{(n-2)}(\xi_i) = \lambda\lambda_1,\\ x_1^{(n-2)}(1) - \sum_{i=1}^m \beta_i x_1^{(n-2)}(\xi_i) = \lambda\lambda_2,\\ x_1^{(i)}(0) = 0, \quad i = 0, \dots, n-3,\end{gathered}$$

where $\lambda \in (0,1)$. If (x_1,x_2) is a solution of $L(x_1,x_2)=\lambda N(x_1,x_2)$ and $(x_1,x_2)\not\equiv (0,c)$, it follows from Lemma 2.5 that there is $\xi\in[0,1]$ so that $x_2(\xi)=0$. Then (H3) implies

$$\begin{aligned}
|x_2(t)| &= \big| -\lambda\int_\xi^t f^*(s,x_1(s),\dots,x_1^{(n-2)}(s),\phi^{-1}(x_2(s)))ds\big| \\
&\le \int_0^1 |f^*(s,x_1(s),\dots,x_1^{(n-2)}(s),\phi^{-1}(x_2(s)))|ds \\
&\le \int_0^1 (a(s)+\sum_{i=0}^{n-2} b_i(s)\phi(|x_1^{(i)}(s)|)+c(s)|x_2(s)|)ds,
\end{aligned}$$

$$\begin{aligned}
|x_1^{(n-2)}(0)| &= \frac{1}{1-\sum_{i=1}^m\alpha_i}\big|x_1^{(n-2)}(0)-\sum_{i=1}^m\alpha_i x_1^{(n-2)}(0)\big| \\
&\le \frac{1}{1-\sum_{i=1}^m\alpha_i}\Big(\sum_{i=1}^m \alpha_i|x_1^{(n-2)}(0)-x_1^{(n-2)}(\xi_i)|+\lambda_1\Big) \\
&\le \frac{1}{1-\sum_{i=1}^m\alpha_i}\Big(\sum_{i=1}^m \alpha_i\xi_1^{(i)}|x_1^{(n-1)}(\theta_i)|+\lambda_1\Big), \quad \theta_i\in[0,\xi_i], \\
&\le \frac{1}{1-\sum_{i=1}^m\alpha_i}\Big(\sum_{i=1}^m \alpha_i\xi_i\phi^{-1}(\|x_2\|_\infty)+\lambda_1\Big).
\end{aligned}$$

Then Lemma 2.2; i.e., Remark 2.3, implies

$$\begin{aligned}
|x_1^{(n-2)}(t)| &\le |x_1^{(n-2)}(0)| + \big|\int_0^t x_1^{(n-1)}(s)ds\big| \\
&\le \big(1+\frac{\sum_{i=1}^m\alpha_i\xi_i}{1-\sum_{i=1}^m\alpha_i}\big)\phi^{-1}(\|x_2\|_\infty)+\frac{\lambda_1}{1-\sum_{i=1}^m\alpha_i} \\
&\le K_{q-1}\Big[\phi\big(1+\frac{\sum_{i=1}^m\alpha_i\xi_i}{1-\sum_{i=1}^m\alpha_i}\big)\|x_2\|_\infty+\phi\big(\frac{\lambda_1}{1-\sum_{i=1}^m\alpha_i}\big)\Big].
\end{aligned}$$

Similarly, for $i=0,\dots,n-3$, we get

$$\begin{aligned}
|x_1^{(i)}(t)| &\le \big|x^{(i)}(0)+\int_0^t\frac{(t-s)^{n-3-i}}{(n-3-i)!}x^{(n-2)}(s)ds\big| \\
&\le \big|\int_0^t \frac{(t-s)^{n-i-3}}{(n-i-3)!}ds\big|\|x^{(n-2)}\|_\infty \\
&\le \frac{1}{(n-2-i)!}\|x_1^{(n-2)}\|_\infty \\
&\le \frac{1}{(n-2-i)!}\big(1+\frac{\sum_{i=1}^m\alpha_i\xi_i}{1-\sum_{i=1}^m\alpha_i}\big)\phi^{-1}(\|x_2\|_\infty)+\frac{1}{(n-2-i)!}\frac{\lambda_1}{1-\sum_{i=1}^m\alpha_i} \\
&\le K_{q-1}\Big[\phi\big(\frac{1}{(n-2-i)!}\big)\phi\big(1+\frac{\sum_{i=1}^m\alpha_i\xi_i}{1-\sum_{i=1}^m\alpha_i}\big)\|x_2\|_\infty+\phi\big(\frac{1}{(n-2-i)!}\big) \\
&\quad\times\big(\frac{\lambda_1}{1-\sum_{i=1}^m\alpha_i}\big)\Big].
\end{aligned}$$

It follows that

$$
\begin{aligned}
&|x_2(t)| \\
&\le \int_0^1 a(s)ds + \int_0^1 b_{n-2}(s)ds\phi\Big(\big(1+\frac{\sum_{i=1}^m \alpha_i\xi_i}{1-\sum_{i=1}^m \alpha_i}\big)\phi^{-1}(\|x_2\|_\infty) + \frac{\lambda_1}{1-\sum_{i=1}^m \alpha_i}\Big) \\
&\quad + \sum_{i=0}^{n-3}\int_0^1 b_i(s)ds\phi\Big(\frac{1}{(n-2-i)!}\big(1+\frac{\sum_{i=1}^m \alpha_i\xi_i}{1-\sum_{i=1}^m \alpha_i}\big)\phi^{-1}(\|x_2\|_\infty) \\
&\quad + \frac{1}{(n-2-i)!}\frac{\lambda_1}{1-\sum_{i=1}^m \alpha_i}\Big) + \int_0^1 c(s)ds\|x_2\|_\infty \\
&\le \int_0^1 a(s)ds + \phi(K_{q-1})\sum_{i=0}^{n-3}\int_0^1 b_i(s)ds\phi\big(\frac{1}{(n-2-i)!}\big)\phi\big(1+\frac{\sum_{i=1}^m \alpha_i\xi_i}{1-\sum_{i=1}^m \alpha_i}\big)\|x_2\|_\infty \\
&\quad + \phi(K_{q-1})\int_0^1 b_{n-2}(s)ds\phi\big(1+\frac{\sum_{i=1}^m \alpha_i\xi_i}{1-\sum_{i=1}^m \alpha_i}\big)\|x_2\|_\infty + \int_0^1 c(s)ds\|x_2\|_\infty \\
&\quad + \phi(K_{q-1})\sum_{i=0}^{n-3}\int_0^1 b_i(s)ds\phi\big(\frac{1}{(n-2-i)!}\frac{\lambda_1}{1-\sum_{i=1}^m \alpha_i}\big) \\
&\quad + \phi(K_{q-1})\int_0^1 b_{n-2}(s)ds\phi\big(\frac{\lambda_1}{1-\sum_{i=1}^m \alpha_i}\big).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\|x_2\|_\infty \\
&\le \int_0^1 a(s)ds + \phi(K_{q-1})\sum_{i=0}^{n-3}\int_0^1 b_i(s)ds\phi\big(\frac{1}{(n-2-i)!}\big)\phi\big(1+\frac{\sum_{i=1}^m \alpha_i\xi_i}{1-\sum_{i=1}^m \alpha_i}\big)\|x_2\|_\infty \\
&\quad + \phi(K_{q-1})\int_0^1 b_{n-2}(s)ds\phi\big(1+\frac{\sum_{i=1}^m \alpha_i\xi_i}{1-\sum_{i=1}^m \alpha_i}\big)\|x_2\|_\infty + \int_0^1 c(s)ds\|x_2\|_\infty \\
&\quad + \phi(K_{q-1})\sum_{i=0}^{n-3}\int_0^1 b_i(s)ds\phi\big(\frac{1}{(n-2-i)!}\frac{\lambda_1}{1-\sum_{i=1}^m \alpha_i}\big) \\
&\quad + \phi(K_{q-1})\int_0^1 b_{n-2}(s)ds\phi\big(\frac{\lambda_1}{1-\sum_{i=1}^m \alpha_i}\big).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\Big[1-\phi(K_{q-1})\sum_{i=0}^{n-3}\phi\big(\frac{1}{(n-2-i)!}\big)\int_0^1 b_i(s)ds\phi\big(1+\frac{\sum_{i=1}^m \alpha_i\xi_i}{1-\sum_{i=1}^m \alpha_i}\big) \\
&-\phi(K_{q-1})\int_0^1 b_{n-2}(s)ds\phi\big(1+\frac{\sum_{i=1}^m \alpha_i\xi_i}{1-\sum_{i=1}^m \alpha_i}\big) - \int_0^1 c(s)ds\Big]\|x_2\|_\infty \\
&\le \int_0^1 a(s)ds + \phi(K_{q-1})\sum_{i=0}^{n-3}\int_0^1 b_i(s)ds\phi\big(\frac{1}{(n-2-i)!}\frac{\lambda_1}{1-\sum_{i=1}^m \alpha_i}\big) \\
&\quad + \phi(K_{q-1})\int_0^1 b_{n-2}(s)ds\phi\big(\frac{\lambda_1}{1-\sum_{i=1}^m \alpha_i}\big).
\end{aligned}
$$

It follow from (H4) that there is a constant $M > 0$ so that $\|x_2\|_\infty \le M$. Since $|x_1^{(i)}(t)| \le \frac{1}{(n-3-i)!}\|x_1^{(n-2)}\|_\infty$ and $|x_1^{(n-2)}(t)| \le (1 + \frac{\sum_{i=1}^m \alpha_i\xi_i}{1-\sum_{i=1}^m \alpha_i})\phi^{-1}(\|x_2\|_\infty) + \frac{\lambda_1}{1-\sum_{i=1}^m \alpha_i}$, there exist constants $M_i > 0$ so that $\|x_1^{(i)}\|_\infty \le M_i$ for all $i = 0, \dots, n-2$. Then Ω_0 is bounded. The proof is complete. □

Lemma 2.8. *Suppose that (H2) holds. Then there exists a constant $M_1' > 0$ such that for each $x = (0, c) \in \operatorname{Ker} L$, if $N(0, c) \in \operatorname{Im} L$, we get that $|c| \le M_1'$.*

Proof. For each $x = (0, c) \in \operatorname{Ker} L$, if $N(0, c) \in \operatorname{Im} L$, we get

$$\big(\phi^{-1}(c), -f^*(t, 0, \dots, 0, \phi^{-1}(c)), \lambda_1, \lambda_2\big) \in \operatorname{Im} L.$$

Then

$$\begin{aligned}&\frac{1}{1-\sum_{i=1}^m \alpha_i}\Big(\sum_{i=1}^m \alpha_i \int_0^{\xi_i} \phi^{-1}(c)ds + \lambda_1\Big)\\ &+ \frac{1}{1-\sum_{i=1}^m \beta_i}\Big(\int_0^1 \phi^{-1}(c)ds - \sum_{i=1}^m \beta_i \int_0^{\xi_i} \phi^{-1}(c)ds - \lambda_2\Big) = 0.\end{aligned}$$

It follows that

$$\phi^{-1}(c) = \Big(\frac{\sum_{i=1}^m \alpha_i\xi_i}{1-\sum_{i=1}^m \alpha_i} + \frac{1-\sum_{i=1}^m \beta_i\xi_i}{1-\sum_{i=1}^m \beta_i}\Big)^{-1}\Big(\frac{\lambda_2}{1-\sum_{i=1}^m \beta_i} + \frac{\lambda_1}{1-\sum_{i=1}^m \alpha_i}\Big).$$

So there exists a constant $M_1' > 0$ such that $|c| \le M_1'$. The proof is complete. □

Lemma 2.9. *Suppose that* (H2) *holds. Then there exists a constant $M_2' > 0$ such that for each $x = (0, c) \in \operatorname{Ker} L$, if $\lambda \wedge^{-1} (0, c) + (1-\lambda)\operatorname{sgn}(\Delta)QN(0, c) = 0$, then $|c| \le M_2'$.*

Proof. For each $x = (0, c) \in \operatorname{Ker} L$, if $\lambda \wedge^{-1} (0, c) + (1-\lambda)\operatorname{sgn}(\Delta)QN(0, c) = 0$, we get

$$\begin{aligned}\lambda c = -(1-\lambda)\operatorname{sgn}(\Delta)\frac{1}{\Delta}\Big[&\Big(\frac{\sum_{i=1}^m \alpha_i\xi_i}{1-\sum_{i=1}^m \alpha_i} + \frac{1-\sum_{i=1}^m \beta_i\xi_i}{1-\sum_{i=1}^m \beta_i}\Big)\phi^{-1}(c)\\ &+ \frac{\lambda_1}{1-\sum_{i=1}^m \alpha_i} + \frac{\lambda_2}{1-\sum_{i=1}^m \beta_i}\Big].\end{aligned}$$

Thus

$$\begin{aligned}\lambda c^2 = -(1-\lambda)\operatorname{sgn}(\Delta)\frac{1}{\Delta}\Big[&\Big(\frac{\sum_{i=1}^m \alpha_i\xi_i}{1-\sum_{i=1}^m \alpha_i} + \frac{1-\sum_{i=1}^m \beta_i\xi_i}{1-\sum_{i=1}^m \beta_i}\Big)\phi^{-1}(c)c\\ &+ \Big(\frac{\lambda_1}{1-\sum_{i=1}^m \alpha_i} + \frac{\lambda_2}{1-\sum_{i=1}^m \beta_i}\Big)c\Big].\end{aligned}$$

If $\lambda = 1$, then $c = 0$. If $\lambda \in [0, 1)$, since

$$q > 1, \quad \frac{\sum_{i=1}^m \alpha_i\xi_i}{1-\sum_{i=1}^m \alpha_i} + \frac{1-\sum_{i=1}^m \beta_i\xi_i}{1-\sum_{i=1}^m \beta_i} > 0,$$

one sees, for sufficiently large $|c|$, that

$$\begin{aligned}\lambda c^2 = -(1-\lambda)\operatorname{sgn}(\Delta)\frac{1}{\Delta}\Big[&\Big(\frac{\sum_{i=1}^m \alpha_i\xi_i}{1-\sum_{i=1}^m \alpha_i} + \frac{1-\sum_{i=1}^m \beta_i\xi_i}{1-\sum_{i=1}^m \beta_i}\Big)|c|^q\\ &+ \Big(\frac{\lambda_1}{1-\sum_{i=1}^m \alpha_i} + \frac{\lambda_2}{1-\sum_{i=1}^m \beta_i}\Big)c\Big] < 0\end{aligned}$$

a contradiction. So there exists a constant $M_2' > 0$ such that $|c| \le M_2'$. The proof is complete. □

Theorem 2.10. *Suppose* (H1)–(H4) *hold. Then* (1.24) *has at least one positive solution.*

Proof. Let $\Omega \supseteq \Omega_0$ be a bounded open subset of X centered at zero with its diameter greater than $\max\{M_1', M_2'\}$. It follows from Lemmas 2.7, 2.8, 2.9 that $Lx \ne \lambda Nx$ for all $(x,\lambda) \in [(D(L) \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0,1)$; $Nx \notin \operatorname{Im} L$ for every $x \in \operatorname{Ker} L \cap \partial\Omega$; $\deg(\wedge QN\big|_{\operatorname{Ker} L}, \ \Omega \cap \operatorname{Ker} L, 0) \ne 0$.

Since (H1) holds, L be a Fredholm operator of index zero and N be L-compact on Ω. It follows from Lemma 2.1 that $Lx = Nx$ has at least one solution $x = (x_1, x_2)$. Then x_1 is a solution of (2.1). We note that $x_1^{(i)}(t) \ge 0$ for $t \in [0,1]$ and $i = 0, \dots, n-2$, so

$$f^*\big(t, x_1(t), \dots, x_1^{(n-2)}(t), \phi^{-1}(x_2(t))\big) = f\big(t, x_1(t), \dots, x_1^{(n-2)}(t), \phi^{-1}(x_2(t))\big).$$

Hence x_1 is a positive solution of (1.24). The proof is complete. □

Remark 2.11. The operator defined in [6] can not be used, so we follow a different method. Theorem 2.10 also generalizes and improves the results in [8, 14, 32, 40, 49].

2.2. **Positive solutions of Problem** (1.25)**.** Let

$$f^*(t, x_0, \dots, x_{n-1}) = f(t, \overline{x_0}, \dots, \overline{x_{n-2}}, \underline{x_{n-1}}), \quad (t, x_0, \dots, x_{n-1}) \in [0,1] \times \mathbb{R}^n,$$

and $\overline{x} = \max\{0, x\}$ and $\underline{y} = \min\{0, y\}$. We consider the problem

$$\begin{gathered} \big[\phi(x^{(n-1)}(t))\big]' + f^*(t, x(t), \dots, x^{(n-1)}(t)) = 0, \quad t \in (0,1), \\ x^{(n-1)}(0) - \sum_{i=1}^m \alpha_i x^{(n-1)}(\xi_i) = \lambda_1, \\ x^{(n-2)}(1) - \sum_{i=1}^m \beta_i x^{(n-2)}(\xi_i) = \lambda_2, \\ x^{(i)}(0) = 0, \quad i = 0, \dots, n-3, \end{gathered} \tag{2.4}$$

Suppose (H3) and the following assumptions, which will be used in the proof of all lemmas in this sub-section.

- (H5) $f : [0,1] \times [0,+\infty)^{n-1} \times (-\infty, 0] \to [0,+\infty)$ is continuous and $f(t,0,\dots,0) \not\equiv 0$ on each sub-interval of [0,1];
- (H6) $\lambda_1 \le 0$, $\lambda_2 \ge 0$, $\alpha_i, \beta_i \ge 0$ with $\sum_{i=1}^m \phi(\alpha_i) < 1/\phi(K_{q-1}^m)$, $\sum_{i=1}^m \alpha_i < 1$ and $\sum_{i=1}^m \beta_i < 1$.
- (H7) The following inequality holds

$$\begin{aligned} &\Big(1 + \frac{\phi(K_{q-1}^m)}{1 - \phi(K_{q-1}^m)\sum_{i=1}^m \phi(\alpha_i)} \sum_{i=1}^m \phi(\alpha_i)\xi_i\Big) \\ &\times \Big[\|b_{n-2}\|_\infty \phi(K_{q-1}) \phi\Big(1 + \frac{1}{1 - \sum_{i=1}^m \beta_i} \sum_{i=1}^m \beta_i (1-\xi_i)\Big) \\ &+ \sum_{i=0}^{n-3} \|b_i\|_\infty \phi(K_{q-1}) \phi\big(\frac{1}{(n-2-i)!}\big) \phi\Big(1 + \frac{\sum_{i=1}^m \beta_i(1-\xi_i)}{1 - \sum_{i=1}^m \beta_i}\big) + \|c\|_\infty\Big] < 1. \end{aligned}$$

Lemma 2.12. *If (x_1, x_2) is a solution of the problem*

$$
\begin{gathered}
x_1^{(n-1)}(t) = \phi^{-1}(x_2(t)), \quad t \in [0,1], \\
x_2'(t) = -f^*(t, x_1(t), \dots, x_1^{(n-2)}(t), \phi^{-1}(x_2(t))), \quad t \in [0,1], \\
\phi^{-1}(x_2(0)) - \sum_{i=1}^m \alpha_i \phi^{-1}(x_2(\xi_i)) = \lambda_1, \\
x_1^{(n-2)}(1) - \sum_{i=1}^n \beta_i x_1^{(n-2)}(\xi_i) = \lambda_2, \\
x_1^{(i)}(0) = 0, \quad i = 0, \dots, n-3,
\end{gathered}
\tag{2.5}
$$

then x_1 is a solution of (2.4).

The proof of the above lemma is simple and is omitted.

Lemma 2.13. *If (H5)–(H6) hold and x is a solution of (2.4), then $x(t) > 0$ for all $t \in (0,1)$, and x is a positive solution of (1.25).*

Proof. Firstly, since $[\phi(x^{(n-1)}(t)]' = -f^*(t, x(t), \dots, x^{(n-1)}(t)) \le 0$ and $\alpha_i \ge 0, \sum_{i=1}^m \alpha_i < 1$ and $\lambda_1 \le 0$, we have, using (1.6), that

$$
x^{(n-1)}(0) = \sum_{i=1}^m \alpha_i x^{(n-1)}(\xi_i) + \lambda_1 \le \sum_{i=1}^m \alpha_i x^{(n-1)}(0).
$$

Hence $x^{(n-1)}(0) \le 0$ and (H6). We get $x^{(n-1)}(t) \le 0$ for all $t \in [0,1]$.

Since $x^{(n-1)}(t) \le 0$ for all $t \in [0,1]$, we get $x^{(n-2)}(1) = \sum_{i=1}^m \beta_i x^{(n-2)}(\xi_i) + \lambda_2 \ge \sum_{i=1}^m \beta_i x^{(n-2)}(1)$. So one gets $x^{(n-2)}(1) \ge 0$. Thus we get $x^{(n-2)}(t) > 0$ for all $t \in (0,1)$ since $x^{(n-1)}(t) \le 0$ for all $t \in [0,1]$. It follows from the boundary conditions that $x^{(i)}(t) > 0$ for all $t \in (0,1)$, $i = 0, \dots, n-3$. Then $f^*(t, x(t), \dots, x^{(n-1)}(t)) = f(t, x(t), \dots, x^{(n-1)}(t))$. Thus x is a positive solution of (1.25). The proof is complete. ◻

Let $\lambda \in (0,1)$, consider the problem

$$
\begin{gathered}
x_1^{(n-1)}(t) = \lambda \phi^{-1}(x_2(t)), \quad t \in [0,1], \\
x_2'(t) = -\lambda f^*(t, x_1(t), \dots, x_1^{(n-2)}(t), \phi^{-1}(x_2(t))), \quad t \in [0,1], \\
0 = \lambda(\phi^{-1}(x_2(0)) - \sum_{i=1}^m \alpha_i \phi^{-1}(x_2(\xi_i)) - \lambda_1), \\
x_1^{(n-2)}(1) - \sum_{i=1}^n \beta_i x_1^{(n-2)}(\xi_i) = \lambda \lambda_2, \\
x_1^{(i)}(0) = 0, \quad i = 0, \dots, n-3.
\end{gathered}
\tag{2.6}
$$

Lemma 2.14. *Suppose* (H5)–(H6) *hold. If (x_1, x_2) is a solution of (2.6), then*

$$
\begin{aligned}
&\|x_2\|_\infty \\
&\le \|x_2'\|_\infty \Big(1 + \frac{\phi(K_{q-1}^m)}{1 - \phi(K_{q-1}^m) \sum_{i=1}^m \phi(\alpha_i)} \sum_{i=1}^m \phi(\alpha_i) \xi_i \Big) + \frac{\phi(K_{q-1}^m)|\lambda_1|}{1 - \phi(K_{q-1}^m) \sum_{i=1}^m \phi(\alpha_i)}.
\end{aligned}
$$

Proof. Since (H5)–(H6) and (2.6) imply that $x_2'(t) \le 0$ for all $t \in [0,1]$. Similar to the discussion of Lemma 2.13, $\lambda_1 \le 0$ and using (2.6), we have

$$\phi^{-1}(x_2(0)) = \sum_{i=1}^m \alpha_i \phi^{-1}(x_2(\xi_i)) + \lambda_1 \le \sum_{i=1}^m \alpha_i \phi^{-1}(x_2(\xi_i)) \le \sum_{i=1}^m \alpha_i \phi^{-1}(x_2(0)).$$

This together with (H6), one sees that $x_2(0) \le 0$. Then $x_2(t) \le 0$ for all $t \in [0,1]$. It follows from

$$\phi^{-1}(x_2(0)) - \sum_{i=1}^m \alpha_i \phi^{-1}(x_2(\xi_i)) - \lambda_1 = 0$$

and Lemma 2.2 that

$$-\phi^{-1}(x_2(0)) \le K_{q-1}^m \phi^{-1}(-\sum_{i=1}^m \phi(\alpha_i) x_2(\xi_i) - \phi(\lambda_1)).$$

Hence we get $-x_2(0) \le -\phi(K_{q-1}^m)\Big(\sum_{i=1}^m \phi(\alpha_i) x_2(\xi_i) - \phi(\lambda_1)\Big)$. Thus, from (H6), we see, there is $\eta_i \in [0,\xi_i]$, that

$$\begin{aligned}
-x_2(0) &= \frac{1}{1-\phi(K_{q-1}^m)\sum_{i=1}^m \phi(\alpha_i)}\Big(-x_2(0) + \phi(K_{q-1}^m)\sum_{i=1}^m \phi(\alpha_i) x_2(0)\Big)\\
&\le \frac{1}{1-\phi(K_{q-1}^m)\sum_{i=1}^m \phi(\alpha_i)}\Big(-\phi(K_{q-1}^m)\sum_{i=1}^m \phi(\alpha_i) x_2(\xi_i)\\
&\quad + \phi(K_{q-1}^m)\sum_{i=1}^m \phi(\alpha_i) x_2(0) + \phi(K_{q-1}^m|\lambda_1|)\Big)\\
&= \frac{\phi(K_{q-1}^m)}{1-\phi(K_{q-1}^m)\sum_{i=1}^m \phi(\alpha_i)}\sum_{i=1}^m \phi(\alpha_i)\xi_i[-x_2'(\eta_i)] + \frac{\phi(K_{q-1}^m|\lambda_1|)}{1-\phi(K_{q-1}^m)\sum_{i=1}^m \phi(\alpha_i)}\\
&\le \|x_2'\|_\infty \frac{\phi(K_{q-1}^m)}{1-\phi(K_{q-1}^m)\sum_{i=1}^m \phi(\alpha_i)}\sum_{i=1}^m \phi(\alpha_i)\xi_i + \frac{\phi(K_{q-1}^m|\lambda_1|)}{1-\phi(K_{q-1}^m)\sum_{i=1}^m \phi(\alpha_i)}.
\end{aligned}$$

Hence we get

$$\begin{aligned}
|x_2(t)| &\le |x_2(t) - x_2(0)| + |x_2(0)|\\
&\le \|x_2'\|_\infty + \|x_2'\|_\infty \frac{\phi(K_{q-1}^m)}{1-\phi(K_{q-1}^m)\sum_{i=1}^m \phi(\alpha_i)}\sum_{i=1}^m \phi(\alpha_i)\xi_i\\
&\quad + \frac{\phi(K_{q-1}^m|\lambda_1|)}{1-\phi(K_{q-1}^m)\sum_{i=1}^m \phi(\alpha_i)}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\|x_2\|_\infty\\
&\le \|x_2'\|_\infty \Big(1 + \frac{\phi(K_{q-1}^m)}{1-\phi(K_{q-1}^m)\sum_{i=1}^m \phi(\alpha_i)}\sum_{i=1}^m \phi(\alpha_i)\xi_i\Big) + \frac{\phi(K_{q-1}^m|\lambda_1|)}{1-\phi(K_{q-1}^m)\sum_{i=1}^m \phi(\alpha_i)}.
\end{aligned}$$

□

Lemma 2.15. *Suppose* (H5)–(H6) *hold. If* (x_1, x_2) *is a solution of* (2.6)*, then*

$$\|x_1^{(n-2)}\|_\infty \le \phi^{-1}(\|x_2\|_\infty)\Big(1 + \frac{1}{1-\sum_{i=1}^m \beta_i}\sum_{i=1}^m \beta_i(1-\xi_i)\Big) + \frac{\lambda_2}{1-\sum_{i=1}^m \beta_i}.$$

Proof. In fact,

$$\begin{aligned}
|x_1^{(n-2)}(1)| &= \frac{1}{1-\sum_{i=1}^m \beta_i}\Big|x_1^{(n-2)}(1) - \sum_{i=1}^m \beta_i x_1^{(n-2)}(1)\Big| \\
&\le \frac{1}{1-\sum_{i=1}^m \beta_i}\sum_{i=1}^m \beta_i(1-\xi_i)\|x_1^{(n-1)}\|_\infty + \frac{\lambda_2}{1-\sum_{i=1}^m \beta_i} \\
&\le \frac{1}{1-\sum_{i=1}^m \beta_i}\sum_{i=1}^m \beta_i(1-\xi_i)\phi^{-1}(\|x_2\|_\infty) + \frac{\lambda_2}{1-\sum_{i=1}^m \beta_i}.
\end{aligned}$$

Then we get

$$\begin{aligned}
|x_1^{(n-2)}(t)| &\le |x_1^{(n-2)}(t) - x_1^{(n-2)}(1)| + |x_1^{(n-2)}(1)| \\
&\le \frac{1}{1-\sum_{i=1}^m \beta_i}\sum_{i=1}^m \beta_i(1-\xi_i)\phi^{-1}(\|x_2\|_\infty) + \frac{\lambda_2}{1-\sum_{i=1}^m \beta_i} + \|x_1^{(n-1)}\|_\infty \\
&\le \frac{1}{1-\sum_{i=1}^m \beta_i}\sum_{i=1}^m \beta_i(1-\xi_i)\phi^{-1}(\|x_2\|_\infty) + \frac{\lambda_2}{1-\sum_{i=1}^m \beta_i} + \phi(\|x_2\|_\infty).
\end{aligned}$$

Then

$$\|x_1^{(n-2)}\|_\infty \le \phi^{-1}(\|x_2\|_\infty)\Big(1 + \frac{1}{1-\sum_{i=1}^m \beta_i}\sum_{i=1}^m \beta_i(1-\xi_i)\Big) + \frac{\lambda_2}{1-\sum_{i=1}^m \beta_i}.$$

and for $i = 0, \dots, n-3$,

$$\begin{aligned}
\|x_1^{(i)}\|_\infty &\le \frac{1}{(n-i-2)!}\|x_1^{(n-2)}\|_\infty \\
&\le \frac{1}{(n-i-2)!}\Big[\phi^{-1}(\|x_2\|_\infty)\Big(1 + \frac{1}{1-\sum_{i=1}^m \beta_i}\sum_{i=1}^m \beta_i(1-\xi_i)\Big) \\
&\quad + \frac{\lambda_2}{1-\sum_{i=1}^m \beta_i}\Big].
\end{aligned}$$

Define the operators

$$\begin{aligned}
L(x_1, x_2) &= ((x_1^{(n-1)}, x_2', 0, x_1^{(n-2)}(1) - \sum_{i=1}^n \beta_i x_1^{(n-2)}(\xi_i)), \quad (x_1, x_2) \in X \cap D(L), \\
N(x_1, x_2) &= (\phi^{-1}(x_2), -f^*(t, x_1, \phi^{-1}(x_2), \phi^{-1}(x_2(0)) \\
&\quad - \sum_{i=1}^m \alpha_i \phi^{-1}(x_2(\xi_i)) - \lambda_1, \lambda_2), \quad (x_1, x_2) \in X.
\end{aligned}$$

Suppose (H5)–(H6) hold. It is easy to show the following results:

(i) $\operatorname{Ker} L = \{(0, c) : c \in \mathbb{R}\}$ and $\operatorname{Im} L = \{(y_1, y_2, a, b) : a = 0\}$;
(ii) L is a Fredholm operator of index zero;

(iii) There are projectors $P : X \to X$ and $Q : Y \to Y$ such that $\operatorname{Ker} L = \operatorname{Im} P$ and $\operatorname{Ker} Q = \operatorname{Im} L$. Furthermore, let $\Omega \subset X$ be an open bounded subset with $\overline{\Omega} \cap D(L) \neq \emptyset$, then N is L-compact on $\overline{\Omega}$;

(iv) $x = (x_1, x_2)$ is a solution of (2.5) if and only if x is a solution of the operator equation $Lx = Nx$ in $D(L)$.

We present the projectors P and Q as follows: $P(x_1, x_2) = (0, x_2(0))$ for all $x = (x_1, x_2) \in X$ and $Q(y_1, y_2, a, b) = (0, 0, a, 0)$. The generalized inverse of $L : D(L) \cap \operatorname{Ker} P \to \operatorname{Im} L$ is defined by

$$\begin{aligned} K_P(y_1, y_2, a, b) = \Big(\int_0^t \frac{(t-s)^{n-2}}{(n-2)!} y_1(s)ds - \frac{t^{n-2}}{(n-2)!} \frac{1}{1 - \sum_{i=1}^m \beta_i} \Big(\int_0^1 y_1(s)ds \\ - \sum_{i=1}^m \beta_i \int_0^{\xi_i} y_1(s)ds + b \Big), \int_0^t y_2(s)ds \Big), \end{aligned}$$

the isomorphism $\wedge : Y/\operatorname{Im} L \to \operatorname{Ker} L$ is defined by $\wedge(0, 0, c, 0) = (0, c)$. □

Lemma 2.16. *Suppose* (H3), (H5)–(H7) *hold. Then the set*

$$\Omega_0 = \big\{ (x_1, x_2) \in D(L) \setminus \operatorname{Ker} L : L(x_1, x_2) = \lambda N(x_1, x_2) \text{ for some } \lambda \in (0, 1) \big\}$$

is bounded.

Proof. It follows from (2.6), (H3), Lemmas 2.2, 2.14 and 2.15 that

$$\begin{aligned}
&\|x_2\|_\infty \\
&\leq \|x_2'\|_\infty \Big(1 + \frac{\phi(K_{q-1}^m)}{1 - \phi(K_{q-1}^m) \sum_{i=1}^m \phi(\alpha_i)} \sum_{i=1}^m \phi(\alpha_i)\xi_i \Big) + \frac{\phi(K_{q-1}^m |\lambda_1|)}{1 - \phi(K_{q-1}^m) \sum_{i=1}^m \phi(\alpha_i)} \\
&\leq \max_{t\in[0,1]} |f^*(t, x_1(t), \dots, x_1^{(n-2)}(t), \phi^{-1}(x_2(t))| \\
&\quad \times \Big(1 + \frac{\phi(K_{q-1}^m)}{1 - \phi(K_{q-1}^m) \sum_{i=1}^m \phi(\alpha_i)} \sum_{i=1}^m \phi(\alpha_i)\xi_i \Big) + \frac{\phi(K_{q-1}^m |\lambda_1|)}{1 - \phi(K_{q-1}^m) \sum_{i=1}^m \phi(\alpha_i)} \\
&\leq \Big(1 + \frac{\phi(K_{q-1}^m)}{1 - \phi(K_{q-1}^m) \sum_{i=1}^m \phi(\alpha_i)} \sum_{i=1}^m \phi(\alpha_i)\xi_i \Big) \\
&\quad \times \Big(\|a\|_\infty + \sum_{i=0}^{n-2} \|b_i\|_\infty \phi(\|x_1^{(i)}\|_\infty) + \|c\|_\infty \phi(\phi^{-1}(\|x_2\|_\infty)) \Big) \\
&\quad + \frac{\phi(K_{q-1}^m |\lambda_1|)}{1 - \phi(K_{q-1}^m) \sum_{i=1}^m \phi(\alpha_i)} \\
&\leq \Big(1 + \frac{\phi(K_{q-1}^m)}{1 - \phi(K_{q-1}^m) \sum_{i=1}^m \phi(\alpha_i)} \sum_{i=1}^m \phi(\alpha_i)\xi_i \Big) \\
&\quad \times \Big\{ \|a\|_\infty + \|b_{n-2}\|_\infty \phi\Big(\phi^{-1}(\|x_2\|_\infty) \big(1 + \frac{1}{1 - \sum_{i=1}^m \beta_i} \sum_{i=1}^m \beta_i (1 - \xi_i) \big) \\
&\quad + \frac{\lambda_2}{1 - \sum_{i=1}^m \beta_i} \Big) \\
&\quad + \sum_{i=0}^{n-3} \|b_i\|_\infty \phi\Big(\frac{1}{(n-2-i)!} \Big[\phi^{-1}(\|x_2\|_\infty) \big(1 + \frac{1}{1 - \sum_{i=1}^m \beta_i} \sum_{i=1}^m \beta_i (1 - \xi_i) \big)
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\lambda_2}{1-\sum_{i=1}^m \beta_i}\Big]\Big) + \|c\|_\infty \|x_2\|_\infty\Big\} + \frac{\phi(K_{q-1}^m|\lambda_1|)}{1-\phi(K_{q-1}^m)\sum_{i=1}^m \phi(\alpha_i)}\\
&\le \Big(1+\frac{\phi(K_{q-1}^m)}{1-\phi(K_{q-1}^m)\sum_{i=1}^m \phi(\alpha_i)}\sum_{i=1}^m \phi(\alpha_i)\xi_i\Big)\\
&\quad\times\Big[\|a\|_\infty + \Big(\|b_{n-2}\|_\infty \phi(K_{q-1})\phi\big(1+\frac{1}{1-\sum_{i=1}^m\beta_i}\sum_{i=1}^m \beta_i(1-\xi_i)\big) + \|c\|_\infty\Big)\|x_2\|_\infty\\
&\quad+\|b_{n-2}\|_\infty\phi(K_{q-1})\phi\big(\frac{\lambda_2}{1-\sum_{i=1}^m\beta_i}\big) + \sum_{i=0}^{n-3}\|b_i\|_\infty \phi(K_{q-1})\phi\big(\frac{1}{(n-2-i)!}\big)\|x_2\|_\infty\\
&\quad\times\phi\Big(1+\frac{1}{1-\sum_{i=1}^m\beta_i}\sum_{i=1}^m \beta_i(1-\xi_i)\Big)\\
&\quad+\sum_{i=0}^{n-3}\|b_i\|_\infty\phi\big(\frac{1}{(n-2-i)!}\big)\phi(K_{q-1})\phi\big(\frac{\lambda_2}{1-\sum_{i=1}^m\beta_i}\big)\Big]\\
&\quad+\frac{\phi(K_{q-1}^m|\lambda_1|)}{1-\phi(K_{q-1}^m)\sum_{i=1}^m \phi(\alpha_i)}.
\end{aligned}
$$

We get

$$
\begin{aligned}
&\Big\{1-\Big(1+\frac{\phi(K_{q-1}^m)}{1-\phi(K_{q-1}^m)\sum_{i=1}^m \phi(\alpha_i)}\sum_{i=1}^m \phi(\alpha_i)\xi_i\Big)\\
&\times\Big[\|b_{n-2}\|_\infty\phi(K_{q-1})\phi\Big(1+\frac{1}{1-\sum_{i=1}^m\beta_i}\sum_{i=1}^m \beta_i(1-\xi_i)\Big)\\
&+\sum_{i=0}^{n-3}\|b_i\|_\infty\phi(K_{q-1})\phi\big(\frac{1}{(n-2-i)!}\big)\phi\Big(1+\frac{1}{1-\sum_{i=1}^m\beta_i}\sum_{i=1}^m \beta_i(1-\xi_i)\Big)\\
&+\|c\|_\infty\Big]\Big\}\|x_2\|_\infty\\
&\le\Big(1+\frac{\phi(K_{q-1}^m)}{1-\phi(K_{q-1}^m)\sum_{i=1}^m \phi(\alpha_i)}\sum_{i=1}^m \phi(\alpha_i)\xi_i\Big)\Big(\|a\|_\infty+\phi(K_{q-1})\phi\big(\frac{\lambda_2}{1-\sum_{i=1}^m\beta_i}\big)\\
&\quad\times\|b_{n-2}\|_\infty+\sum_{i=0}^{n-3}\|b_i\|_\infty\phi\big(\frac{1}{(n-2-i)!}\big)\phi(K_{q-1})\phi\big(\frac{\lambda_2}{1-\sum_{i=1}^m\beta_i}\big)\Big)\\
&\quad+\frac{\phi(K_{q-1}^m|\lambda_1|)}{1-\phi(K_{q-1}^m)\sum_{i=1}^m \phi(\alpha_i)}.
\end{aligned}
$$

It follows from (H7) that there is a constant $M>0$ so that $\|x_2\|_\infty \le M$. Thus, from Lemma 2.15,

$$
\|x_1^{(n-2)}\|_\infty \le \phi^{-1}(M)\Big(1+\frac{1}{1-\sum_{i=1}^m\beta_i}\sum_{i=1}^m \beta_i(1-\xi_i)\Big)+\frac{\lambda_2}{1-\sum_{i=1}^m\beta_i} =: M_{n-2},
$$

and there exist constants $M_i>0$ such that $\|x_1^{(i)}\|_\infty \le M_i$ for $i=0,\dots,n-3$. So Ω_0 is bounded. The proof is complete. □

Lemma 2.17. *Suppose* (H5)–(H7) *hold. If* $(0,c)\in \operatorname{Ker} L$ *and* $N(0,c)\in \operatorname{Im} L$, *then there exists a constant* $M_1'>0$ *such that* $|c|\le M_1'$.

Proof. If $(0,c)\in \operatorname{Ker} L$ and $N(0,c)\in \operatorname{Im} L$, we get $\phi^{-1}(c)-\sum_{i=1}^m \alpha_i\phi^{-1}(c)=\lambda_1$. Then (H6) implies that there is $M_1'>0$ such that $|c|\le M_1'$. □

Lemma 2.18. *Suppose* (H5)–(H7) *hold. If* $(0,c)\in \operatorname{Ker} L$ *with* $\lambda\wedge^{-1}(0,c)+(1-\lambda)QN(0,c)=0$*, then there exists a constant* $M_2'>0$ *such that* $|c|\le M_2'$*.*

Proof. If $(0,c)\in \operatorname{Ker} L$ with $\lambda\wedge^{-1}(0,c)+(1-\lambda)QN(0,c)=0$, then

$$\lambda c=-(1-\lambda)\Big(\phi^{-1}(c)-\sum_{i=1}^m \alpha_i\phi^{-1}(c)-\lambda_1\Big).$$

It follows that

$$\lambda c^2=-c\phi^{-1}(c)(1-\lambda)\Big(1-\sum_{i=1}^m \alpha_i-\frac{\lambda_1}{\phi^{-1}(c)}\Big).$$

It is easy to see that there is $M_2'>0$ so that $|c|\le M_2'$. □

Theorem 2.19. *Suppose* (H3), (H5)–(H7) *hold. Then* (1.25) *has at least one positive solution.*

Proof. Let $\Omega\supseteq\Omega_0$ be a bounded open subset of X centered at zero with its diameter greater than $\max\{M_1',M_2'\}$. Then Lemmas 2.16, 2.17 and 2.18 imply that $Lx\ne\lambda Nx$ for all $(x,\lambda)\in[(D(L)\setminus \operatorname{Ker} L)\cap\partial\Omega]\times(0,1)$; $Nx\notin \operatorname{Im} L$ for every $x\in \operatorname{Ker} L\cap\partial\Omega$; $\deg(\wedge QN\big|_{\operatorname{Ker} L},\Omega\cap \operatorname{Ker} L,0)\ne 0$.

Since (H5) holds, we let L be a Fredholm operator of index zero and N be L-compact on Ω. It follows from Lemma 2.1 that $Lx=Nx$ has at least one solution $x=(x_1,x_2)$. Then x_1 is a solution of (2.4). We note that $x_1^{(i)}(t)\ge 0$ for $t\in[0,1]$ and $i=0,\dots,n-2$, and $x_2(t)\le 0$ for all $t\in[0,1]$, so

$$f^*\big(t,x_1(t),\dots,x_1^{(n-2)}(t),\phi^{-1}(x_2(t))\big)=f\big(t,x_1(t),\dots,x_1^{(n-2)}(t),\phi^{-1}(x_2(t))\big).$$

Hence x_1 is a positive solution of (1.25). □

Remark 2.20. The operator defined in [5] can not be used, so follow a different method. Theorem 2.19 generalizes and improves the theorems in [5, 14, 26, 51].

2.3. **Positive solutions of Problem** (1.26)**.** Let

$$f^*(t,x_0,\dots,x_{n-1})=f(t,\overline{x_0},\dots,\overline{x_{n-2}},\overline{x_{n-1}}),\quad (t,x_0,\dots,x_{n-1})\in[0,1]\times\mathbb{R}^n,$$

and $\overline{x}=\max\{0,x\}$. We consider the problem

$$\begin{gathered}\big[\phi(x^{(n-1)}(t))\big]'+f^*(t,x(t),\dots,x^{(n-1)}(t))=0,\quad t\in(0,1),\\ x^{(n-2)}(0)-\sum_{i=1}^m \alpha_i x^{(n-2)}(\xi_i)=\lambda_1,\\ x^{(n-1)}(1)-\sum_{i=1}^m \beta_i x^{(n-1)}(\xi_i)=\lambda_2,\\ x^{(i)}(0)=0,\quad i=0,\dots,n-3,\end{gathered}\tag{2.7}$$

Problem (2.3) can be transformed into

$$
\begin{gathered}
x_1^{(n-1)}(t) = \phi^{-1}(x_2(t)), \quad t \in [0,1], \\
x_2'(t) = -f^*(t, x_1(t), \dots, x_1^{(n-2)}(t), \phi^{-1}(x_2(t)), \quad t \in [0,1], \\
x_1^{(n-2)}(0) - \sum_{i=1}^{m} \alpha_i x_1^{(n-2)}(\xi_i) = \lambda_1, \\
0 = \phi^{-1}(x_2(1)) - \sum_{i=1}^{n} \beta_i \phi^{-1}(x_2(\xi_i)) - \lambda_2, \\
x_1^{(i)}(0) = 0, \quad i = 0, \dots, n-3.
\end{gathered}
\tag{2.8}
$$

Suppose that (H3) and the following assumptions hold.

- (H8) $f : [0,1] \times [0,+\infty)^n \to [0,+\infty)$ is continuous with $f(t,0,\dots,0) \not\equiv 0$ on each sub-interval of [0,1];
- (H9) $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, $\alpha_i, \beta_i \geq 0$ with $\sum_{i=1}^m \phi(\beta_i) < 1/\phi(K_{q-1}^m)$, $\sum_{i=1}^m \alpha_i < 1$ and $\sum_{i=1}^m \beta_i < 1$.
- (H10) The following inequality holds

$$
\begin{aligned}
&\Big(1 + \frac{\phi(K_{q-1}^m)}{1 - \phi(K_{q-1}^m)\sum_{i=1}^m \phi(\beta_i)} \sum_{i=1}^m \phi(\beta_i)(1-\xi_i)\Big) \\
&\Big[\|b_{n-2}\|_\infty \phi(K_{q-1}) \phi\Big(1 + \frac{1}{1 - \sum_{i=1}^m \alpha_i} \sum_{i=1}^m \alpha_i \xi_i\Big) \\
&+ \sum_{i=0}^{n-3} \|b_i\|_\infty \phi(K_{q-1}) \phi\big(\frac{1}{(n-2-i)!}\big) \phi\big(\frac{\sum_{i=1}^m \alpha_i \xi_i}{1 - \sum_{i=1}^m \alpha_i}\big) + \|c\|_\infty\Big] < 1.
\end{aligned}
$$

Define the operators

$$
\begin{aligned}
L(x_1, x_2) &= \Big(x_1^{(n-1)}, x_2', x_1^{(n-2)}(0) - \sum_{i=1}^{n} \alpha_i x_1^{(n-2)}(\xi_i), 0\Big), \quad (x_1, x_2) \in X \cap D(L), \\
N(x_1, x_2) &= \Big(\phi^{-1}(x_2), -f^*(t, x_1(t), \dots, x_1^{(n-2)}(t)), \lambda_1, \phi^{-1}(x_2(1)) \\
&\quad - \sum_{i=1}^{m} \beta_i \phi^{-1}(x_2(\xi_i) - \lambda_2)\Big)
\end{aligned}
$$

for $(x_1, x_2) \in X$. It is easy to show the following results:

- (i) $\operatorname{Ker} L = \{(0,c) : c \in \mathbb{R}\}$ and $\operatorname{Im} L = \{(y_1, y_2, a, b) : b = 0\}$;
- (ii) L is a Fredholm operator of index zero;
- (iii) There are projectors $P : X \to X$ and $Q : Y \to Y$ such that $\operatorname{Ker} L = \operatorname{Im} P$ and $\operatorname{Ker} Q = \operatorname{Im} L$. Furthermore, let $\Omega \subset X$ be an open bounded subset with $\overline{\Omega} \cap D(L) \neq \emptyset$, then N is L-compact on $\overline{\Omega}$;
- (iv) $x = (x_1, x_2)$ is a solution of (2.8) if and only if x is a solution of the operator equation $Lx = Nx$ in $D(L)$.

We define the projectors P and Q as follows: $P(x_1, x_2) = (0, x_2(0))$ for all $x = (x_1, x_2) \in X$ and $Q(y_1, y_2, a, b) = (0, 0, 0, b)$. The generalized inverse of L is defined by

$$\begin{aligned} K_P(y_1,y_2,a,b) = \Big(& \int_0^t \frac{(t-s)^{n-2}}{(n-2)!} y_1(s)ds \\ & + \frac{t^{n-2}}{(n-2)!} \frac{1}{1-\sum_{i=1}^m \alpha_i} \Big(\sum_{i=1}^m \alpha_i \int_0^{\xi_i} y_1(s)ds + a \Big), \int_0^t y_2(s)ds \Big), \end{aligned}$$

the isomorphism $\wedge : Y/\operatorname{Im} L \to \operatorname{Ker} L$ is defined by $\wedge(0,0,0,b) = (0,b)$.

Similar to the lemmas in sub-section 2.2, it is easy to prove the following Lemmas.

Lemma 2.21. *If (x_1,x_2) is a solution of problem* (2.8)*, then x_1 is a solution of* (2.7).

The proof is easy; it is omitted.

Lemma 2.22. *Suppose that* (H8)–(H9) *hold. If x is a solution of the problem* (2.7)*, then $x(t) > 0$ for all $t \in (0,1)$.*

The proof is similar to that of Lemma 2.13; it is omitted.

Let $\lambda \in (0,1)$, consider the problem

$$\begin{gathered} x_1^{(n-1)}(t) = \lambda \phi^{-1}(x_2(t)), \quad t \in [0,1], \\ x_2'(t) = -\lambda f^*(t, x_1(t), \dots, x_1^{(n-2)}(t), \phi^{-1}(x_2(t))), \quad t \in [0,1], \\ 0 = \lambda(\phi^{-1}(x_2(1)) - \sum_{i=1}^m \beta_i \phi^{-1}(x_2(\xi_i)) - \lambda_2), \\ x_1^{(n-2)}(0) - \sum_{i=1}^n \alpha_i x_1^{(n-2)}(\xi_i) = \lambda\lambda_1, \\ x_1^{(i)}(0) = 0, \quad i = 0, \dots, n-3. \end{gathered} \tag{2.9}$$

Lemma 2.23. *Suppose that* (H8)–(H9) *hold. If (x_1,x_2) is a solution of* (2.9)*, then*

$$\begin{aligned} \|x_2\|_\infty \le \|x_2'\|_\infty \Big(1 + \frac{\phi(K_{q-1}^m)}{1-\phi(K_{q-1}^m)\sum_{i=1}^m \phi(\beta_i)} \sum_{i=1}^m \phi(\beta_i)(1-\xi_i) \Big) \\ + \frac{\phi(K_{q-1}^m \lambda_2)}{1-\phi(K_{q-1}^m)\sum_{i=1}^m \phi(\beta_i)}. \end{aligned}$$

Lemma 2.24. *Suppose that* (H8)–(H9) *hold. If (x_1,x_2) is a solution of* (2.5)*, then*

$$\|x_1^{(n-2)}\|_\infty \le \phi^{-1}(\|x_2\|_\infty) \Big(1 + \frac{1}{1-\sum_{i=1}^m \alpha_i} \sum_{i=1}^m \alpha_i \xi_i \Big) + \frac{\lambda_1}{1-\sum_{i=1}^m \alpha_i},$$

and for $i = 0, \dots, n-3$,

$$\begin{aligned} \|x_1^{(i)}\|_\infty &\le \frac{1}{(n-i-3)!} \|x_1^{(n-2)}\|_\infty \\ &\le \frac{1}{(n-i-3)!} (\phi^{-1}(\|x_2\|_\infty) \big(1 + \frac{1}{1-\sum_{i=1}^m \alpha_i} \sum_{i=1}^m \alpha_i (1-\xi_i) \big) \\ &\quad + \frac{\lambda_1}{1-\sum_{i=1}^m \alpha_i}). \end{aligned}$$

Similar to Theorem 2.19, we obtain the following theorem.

Theorem 2.25. *Suppose* (H3), (H8)–(H10) *hold. Then* (1.26) *has at least one positive solution.*

We remark that Theorem 2.25 generalizes the theorems in [2, 14, 73].

2.4. **Solutions of Problem** (1.27)**.** We consider (1.27), (H1), (H3) and the following assumptions are supposed in this sub-section.

- (H11) $\alpha_i, \beta_i \geq 0$ for all $i = 1, \dots, m$ and $\lambda_1, \lambda_2 \in \mathbb{R}$;
- (H12) The following inequality holds

$$\begin{aligned} &\phi(K_{q-1})\Big[\sum_{i=0}^{n-3} \phi\big(\frac{1 + \sum_{i=1}^{m} \alpha_i}{(n-2-i)!}\big) \int_0^1 b_i(s)ds \\ &+ \phi\big(1 + \sum_{i=1}^{m} \alpha_i\big) \int_0^1 b_{n-2}(s)ds\Big] + \int_0^1 c(s)ds < 1\,. \end{aligned}$$

Let $x_1 = x$ and $x_2 = \phi(x_1)$, then (1.24) is transformed into

$$\begin{gathered} x_1^{(n-1)}(t) = \phi^{-1}(x_2(t)), \quad t \in [0,1], \\ x_2'(t) = -f(t, x_1(t), \dots, x_1^{(n-2)}(t), \phi^{-1}(x_2(t))), \quad t \in [0,1], \\ x_1^{(n-2)}(0) = \sum_{i=1}^{m} \alpha_i \phi^{-1}(x_2(\xi_i)) + \lambda_1, \\ x_1^{(n-2)}(1) = -\sum_{i=1}^{n} \beta_i \phi^{-1}(x_2(\xi_i)) + \lambda_2, \\ x_1^{(i)}(0) = 0, \quad i = 0 \dots, n-3, \end{gathered} \tag{2.10}$$

Suppose $\lambda \in (0,1)$, we consider the problem

$$\begin{gathered} x_1^{(n-1)}(t) = \lambda\phi^{-1}(x_2(t)), \quad t \in [0,1], \\ x_2'(t) = -\lambda f(t, x_1(t), \dots, x_1^{(n-2)}(t), \phi^{-1}(x_2(t))), \quad t \in [0,1], \\ x_1^{(n-2)}(0) = \lambda(\sum_{i=1}^{m} \alpha_i \phi^{-1}(x_2(\xi_i)) + \lambda_1), \\ x_1^{(n-2)}(1) = \lambda(-\sum_{i=1}^{n} \beta_i \phi^{-1}(x_2(\xi_i)) + \lambda_2), \\ x_1^{(i)}(0) = 0, \quad i - 0 \dots, n-3, \end{gathered} \tag{2.11}$$

Define the operators

$$\begin{aligned} L(x_1, x_2) &= \big(x_1^{(n-1)}, x_2', x_1^{(n-2)}(0), x_1^{(n-2)}(1)\big), \quad (x_1, x_2) \in X \cap D(L), \\ N(x_1, x_2) &= \Big(\phi^{-1}(x_2), -f^*(t, x_1(t), \dots, x_1^{(n-2)}(t), \phi^{-1}(x_2(t))), \\ &\quad \sum_{i=1}^{m} \alpha_i \phi^{-1}(x_2(\xi_i)) + \lambda_1, -\sum_{i=1}^{n} \beta_i \phi^{-1}(x_2(\xi_i)) + \lambda_2\Big), \end{aligned}$$

for $(x_1, x_2) \in X$. Suppose (H11)–(H12) hold. It is easy to show the following results:

(i) $\operatorname{Ker} L = \{(0, c) : c \in \mathbb{R}\}$ and $\operatorname{Im} L = \{(y_1, y_2, a, b) : \int_0^1 y_1(s)ds = b - a\}$;

(ii) L is a Fredholm operator of index zero;
(iii) There are projectors $P : X \to X$ and $Q : Y \to Y$ such that $\operatorname{Ker} L = \operatorname{Im} P$ and $\operatorname{Ker} Q = \operatorname{Im} L$. Furthermore, let $\Omega \subset X$ be an open bounded subset with $\overline{\Omega} \cap D(L) \neq \emptyset$, then N is L-compact on $\overline{\Omega}$;
(iv) $x = (x_1, x_2)$ is a solution of (2.11) if and only if x is a solution of the operator equation $Lx = \lambda Nx$ in $D(L)$.

We define the projectors P and Q as follows: $P(x_1, x_2) = (0, x_2(0))$ for all $x = (x_1, x_2) \in X$ and $Q(y_1, y_2, a, b) = (\int_0^1 y_1(s)ds - b + a, 0, 0, 0)$. The generalized inverse of L is

$$K_P(y_1, y_2, a, b) = \Big(\frac{a}{(n-2)!}t^{n-2} + \int_0^t \frac{(t-s)^{n-2}}{(n-2)!} y_1(s)ds, \int_0^t y_2(s)ds\Big),$$

the isomorphism $\wedge : Y/\operatorname{Im} L \to \operatorname{Ker} L$ is defined by $\wedge(c, 0, 0, 0) = (0, c)$.

Similar to Lemmas in sub-section 2.2, it is easy to prove the following Lemmas.

Lemma 2.26. *Suppose* (H11), (H12) *hold. If* $x = (x_1, x_2)$ *is a solution of* (2.11), *then there exists* $\xi \in [0, 1]$ *such that*

$$|\phi^{-1}(x^{(n-1)}(\xi))| \le M =: \begin{cases} \frac{|\lambda_1-\lambda_2|}{\sum_{i=1}^m \alpha_i + \sum_{i=1}^m \beta_i}, & \sum_{i=1}^m \alpha_i + \sum_{i=1}^m \beta_i \neq 0, \\ |\lambda_1 - \lambda_2|, & \sum_{i=1}^m \alpha_i + \sum_{i=1}^m \beta_i = 0. \end{cases}$$

Proof. **Case 1.** $\sum_{i=1}^m \alpha_i + \sum_{i=1}^m \beta_i = 0$. Then $\alpha_i = \beta_i = 0$. In this case, $x_1^{(n-2)}(0) = \lambda\lambda_1$ and $x_1^{(n-2)}(1) = \lambda\lambda_2$, it is easy to see that there is $\xi \in [0, 1]$ so that $|x^{(n-1)}(\xi)| = \lambda|\lambda_1 - \lambda_2|$. Then

$$\lambda|\phi^{-1}(x_2(\xi))| = \lambda|\lambda_1 - \lambda_2|.$$

So $|\phi^{-1}(x_2(\xi))| = |\lambda_1 - \lambda_2|$.

Case 2. $\sum_{i=1}^m \alpha_i + \sum_{i=1}^m \beta_i \neq 0$. In this case, if $|\phi^{-1}(x_2(t))| > M$ for all $t \in (0, 1)$, then $x_1^{(n-2)}(0) < x_1^{(n-2)}(1)$.

If $\lambda_1 \ge \lambda_2$, from the boundary conditions, using (H11), we obtain

$$\begin{aligned} x_1^{(n-2)}(0) &> \lambda \sum_{i=1}^m \alpha_i M + \lambda\lambda_1 \\ &\ge \lambda \sum_{i=1}^m \alpha_i \frac{-\lambda_1 + \lambda_2}{\sum_{i=1}^m \alpha_i + \sum_{i=1}^m \beta_i} + \lambda\lambda_1 \\ &= \lambda \frac{\lambda_2 \sum_{i=1}^m \alpha_i + \lambda_1 \sum_{i=1}^m \beta_i}{\sum_{i=1}^m \alpha_i + \sum_{i=1}^m \beta_i}. \end{aligned}$$

On the other hand,

$$x_1^{(n-2)}(1) < -\lambda \sum_{i=1}^m \beta_i M + \lambda\lambda_2 \le \lambda \sum_{i=1}^m \beta_i \frac{\lambda_1 - \lambda_2}{\sum_{i=1}^m \alpha_i + \sum_{i=1}^m \beta_i} + \lambda\lambda_2 < x_1^{(n-2)}(0),$$

a contradiction. Similar to above discussion, if $x^{(n-1)}(t) < -M$ for all $t \in (0, 1)$, we can get a contradiction. Then there is $\xi \in (0, 1)$ so that $|\phi^{-1}(x_2(\xi))| \le M$.

If $\lambda_1 < \lambda_2$, we can get that there is $\xi \in (0, 1)$ so that $|\phi^{-1}(x_2(\xi))| \le M$. $\square$

Lemma 2.27. *Suppose* (H11)–(H12) *hold. If* (x_1, x_2) *is a solution of* (2.11), *then*

$$\|x_1^{(n-2)}\|_\infty \le \big(1 + \sum_{i=1}^m \alpha_i\big)\phi^{-1}(\|x_2\|_\infty) + |\lambda_1|.$$

Proof. From the boundary conditions, we get $|x_1^{(n-2)}(0)| \le \sum_{i=1}^m \alpha_i \phi^{-1}(\|x_2\|_\infty) + |\lambda_1|$. So we get

$$|x_1^{(n-2)}(t)| \le |x_1^{(n-2)}(t) - x_1^{(n-2)}(0)| + |x_1^{(n-2)}(0)| \le \big(1 + \sum_{i=1}^m \alpha_i\big)\phi^{-1}(\|x_2\|_\infty) + |\lambda_1|.$$

This completes the proof. We also get, for $i = 0, \dots, n-3$, that

$$|x_1^{(i)}(t)| \le \frac{1}{(n-2-i)!}\big[\big(1 + \sum_{i=1}^m \alpha_i\big)\phi^{-1}(\|x_2\|_\infty) + |\lambda_1|\big].$$

□

Lemma 2.28. *Let* $\Omega_0 = \{x \in D(L) \setminus \operatorname{Ker} L : Lx = \lambda Nx$ *for* $\lambda \in (0,1)\}$. *Then* Ω_0 *is bounded.*

Proof. In fact, if $x \in \Omega_0$, we get (2.11). It follows from Lemma 2.26 that there is $\xi \in (0,1)$ so that $\phi^{-1}(|x_2(\xi)|) \le M$. Thus using (H3) and Lemma 2.2 we get

$$\begin{aligned}
|x_2(t)| &\le \phi(M) + \lambda\Big|\int_\xi^t f(t, x_1(t), \dots, x_1^{(n-2)}(t), \phi^{-1}(x_2(t)))dt\Big| \\
&\le \phi(M) + \int_0^1 |f(t, x_1(t), \dots, x_1^{(n-2)}(t), \phi^{-1}(x_2(t)))|dt \\
&\le \phi(M) + \int_0^1 a(s)ds + \sum_{i=0}^{n-2}\int_0^1 b_i(s)\phi(|x_1^{(i)}(s)|)ds + \int_0^1 c(s)|x_2(s)|ds \\
&\le \phi(M) + \int_0^1 a(s)ds + \sum_{i=0}^{n-3}\int_0^1 b_i(s)ds \\
&\quad \times \phi\Big(\frac{1}{(n-2-i)!}\big(1 + \sum_{i=1}^m \alpha_i\big)\phi^{-1}(\|x_2\|_\infty) + \frac{1}{(n-2-i)!}|\lambda_1|\Big) \\
&\quad + \int_0^1 b_{n-2}(s)ds\phi\Big(\big(1 + \sum_{i=1}^m \alpha_i\big)\phi^{-1}(\|x_2\|_\infty) + |\lambda_1|\Big) + \int_0^1 c(s)ds\|x_2\|_\infty \\
&\le \phi(M) + \int_0^1 a(s)ds + \sum_{i=0}^{n-3}\int_0^1 b_i(s)ds \\
&\quad \times \phi(K_{q-1})\phi\big(\frac{1}{(n-2-i)!}\big)\phi\big(1 + \sum_{i=1}^m \alpha_i\big)\|x_2\|_\infty + \phi\Big(\frac{1}{(n-2-i)!}|\lambda_1|\Big) \\
&\quad + \int_0^1 b_{n-2}(s)ds\phi\big(1 + \sum_{i=1}^m \alpha_i\big)\|x_2\|_\infty + \phi(|\lambda_1|) + \int_0^1 c(s)ds\|x_2\|_\infty.
\end{aligned}$$

Then

$$\begin{aligned}
\|x_2\|_\infty &\le \phi(M) + \int_0^1 a(s)ds + \sum_{i=0}^{n-3}\int_0^1 b_i(s)ds\phi(K_{q-1})\phi\big(\frac{1}{(n-2-i)!}\big) \\
&\quad \times \phi\big(1 + \sum_{i=1}^m \alpha_i\big)\|x_2\|_\infty + \phi\big(\frac{1}{(n-2-i)!}|\lambda_1|\big)
\end{aligned}$$

$$+\int_0^1 b_{n-2}(s)ds\phi\big(1+\sum_{i=1}^m \alpha_i\big)\|x_2\|_\infty + \phi(|\lambda_1|) + \int_0^1 c(s)ds\|x_2\|_\infty .$$

Hence

$$\begin{aligned}&\Big[1-\phi(K_{q-1})(\sum_{i=0}^{n-3}\phi\Big(\frac{1+\sum_{i=1}^m \alpha_i}{(n-2-i)!}\Big)\int_0^1 b_i(s)ds + \phi\big(1+\sum_{i=1}^m \alpha_i\big)\int_0^1 b_{n-2}(s)ds)\\ &+\int_0^1 c(s)ds\Big]\|x_2\|_\infty\\ &\le \phi(M)+\int_0^1 a(s)ds + \phi\big(\frac{1}{(n-2-i)!}|\lambda_1|\big)+\phi(|\lambda_1|).\end{aligned}$$

From (H12), we get that there is $A>0$ so that $\|x_2\|_\infty \le A$. Hence

$$\|x_1^{(n-2)}\|_\infty \le \big(1+\sum_{i=1}^m \alpha_i\big)\phi^{-1}(A)+|\lambda_1|.$$

And for $i=0,\dots,n-3$, we get

$$\|x_1^{(i)}\|_\infty \le \frac{1}{(n-2-i)!}\big[\big(1+\sum_{i=1}^m \alpha_i\big)\phi^{-1}(A)+|\lambda_1|\big].$$

The above inequalities imply that Ω_0 is bounded. □

Lemma 2.29. *Let $\Omega_1=\{x\in \operatorname{Ker} L : Nx \in \operatorname{Im} L\}$. Then Ω_1 is bounded.*

Proof. In fact, $(0,c)\in\Omega_1$, then $(0,c)\in \operatorname{Ker} L$ and $N(0,c)\in \operatorname{Im} L$, then we get

$$\phi^{-1}(c) = -\sum_{i=1}^m \alpha_i\phi^{-1}(c) - \sum_{i=1}^m \beta_i\phi^{-1}(c)+\lambda_2-\lambda_1.$$

Hence there is $M_1>0$ so that $|c|\le M_1$. □

Lemma 2.30. *Let $\Omega_2=\{x\in \operatorname{Ker} L : \lambda\wedge^{-1}x+(1-\lambda)QNx=0\}$. Then Ω_2 is bounded.*

Proof. In fact, if $(0,c)\in\Omega_2$, then

$$\lambda c = -(1-\lambda)\Big(\sum_{i=1}^m \alpha_i\phi^{-1}(c)+\sum_{i=1}^m \beta_i\phi^{-1}(c)-\lambda_2+\lambda_1\Big).$$

Thus

$$\lambda c^2 = -(1-\lambda)c\phi^{-1}(c)\Big(\sum_{i=1}^m \alpha_i+\sum_{i=1}^m \beta_i-\frac{\lambda_2-\lambda_1}{\phi^{-1}(c)}\Big).$$

Hence there is $M_1>0$ so that $|c|\le M_1$. □

The following theorem has proof similar to that of Theorem 2.10; its proof is omitted.

Theorem 2.31. *Suppose* (H1), (H3), (H11), (H12) *hold. Then* (1.27) *has at least one solution.*

Remark 2.32. Consider the problems

$$
\begin{gathered}
\big[\phi(x^{(n-1)}(t))\big]' + f(t, x(t), \dots, x^{(n-1)}(t)) = 0, \quad t \in (0,1), \\
x^{(n-2)}(0) - \alpha x^{(n-1)}(0) = \lambda_1, \\
x^{(n-2)}(1) + \beta x^{(n-1)}(1) = \lambda_2, \\
x^{(i)}(0) = 0, \quad i = 0, \dots, n-3,
\end{gathered} \tag{2.12}
$$

where $\alpha \geq 0, \beta \geq 0$, $\lambda_1 \geq 0, \lambda_2 \geq 0$, and f is nonnegative and continuous. If $x(t)$ is a solution of (2.12), then $x^{(n-1)}$ is decreasing on $[0,1]$.
Case 1. $x^{(n-1)}(0) \geq 0$ and $x^{(n-1)}(1) \geq 0$; At this case, we see that $x^{(n-2)}(t)$ is increasing on $[0,1]$. It follows from

$$
x^{(n-2)}(0) = \alpha x^{(n-1)}(0) + \lambda_1 \geq 0
$$

that $x^{(n-2)}(t) > 0$ for all $t \in (0,1)$. Then $x(t)$ is a positive solution of (2.12).
Case 2. $x^{(n-1)}(0) \geq 0$ and $x^{(n-1)}(1) \leq 0$; At this case, one sees that

$$
x^{(n-2)}(1) = -\beta x^{(n-1)}(1) + \lambda_2 \geq 0
$$

and

$$
x^{(n-2)}(0) = -\alpha x^{(n-1)}(0) + \lambda_2 \geq 0.
$$

It follows from $x^{(n)}(t) \leq 0$ that $x^{(n-2)}(t) \geq 0$ for all $t \in [0,1]$. Then x is a positive solution of (2.12).
Case 2. $x^{(n-1)}(0) \leq 0$ and $x^{(n-1)}(1) \leq 0$; At this case, one sees that $x^{(n-2)}(t)$ is decreasing on $[0,1]$. It follows from

$$
x^{(n-2)}(1) = -\beta x^{(n-1)}(1) + \lambda_2 \geq 0
$$

that $x^{(n-2)}(t) > 0$ for all $t \in (0,1)$. Then $x(t)$ is a positive solution of (2.12).

We can establish similar results for the existence of positive solutions of (2.12) and the details are omitted.

Remark 2.33. Consider the problems

$$
\begin{gathered}
\big[\phi(x^{(n-1)}(t))\big]' + f(t, x(t), \dots, x^{(n-1)}(t)) = 0, \quad t \in (0,1), \\
x^{(n-2)}(0) - \alpha x^{(n-1)}(0) = \lambda_1, \\
x^{(n-2)}(1) + \sum_{i=1}^{m} \beta_i x^{(n-1)}(\xi_i) = \lambda_2, \\
x^{(i)}(0) = 0, \quad i = 0, \dots, n-3,
\end{gathered} \tag{2.13}
$$

where $\alpha \geq 0, \beta_i \geq 0$, $\lambda_1 \geq 0, \lambda_2 \geq 0$, and f is nonnegative and continuous. (2.13) need not has positive solution. It is easy to show that the problem

$$
\begin{gathered}
x'' + 8 + 6t = 0, \quad t \in (0,1), \\
x(0) - x'(0) = 1, \\
x(1) + \theta x'(\frac{1}{8}) = 0
\end{gathered}
$$

has no positive solution since the solution of the above problem is

$$
x(t) = -4t^2 - t^3 + \frac{4 + \frac{67}{64}\theta}{2+\theta} t + \frac{4 + \frac{67}{64}\theta}{2+\theta} + 1.
$$

Then

$$x(1) = -5 + 2\frac{4 + \frac{67}{64}\theta}{2+\theta} + 1 = -4 + 2\frac{4 + \frac{67}{64}\theta}{2+\theta} < 0 \quad \text{if } \theta > 0.$$

2.5. **Solutions of** (1.28)**.** We consider (1.28), assuming (H3) and the following conditions:

(H13) there are nonnegative numbers α, θ_i and L so that $|f(t, x_0, \dots, x_{n-1})| \geq \alpha\phi(|x_{n-2}|) - \sum_{i=0}^{n-3} \theta_i \phi(|x_i|) - \theta_{n-1}\phi(|x_{n-1}|) - L$;

(H14)

$$\lim_{|a|\to+\infty} \frac{|f(t, \frac{t^{n-2}}{(n-2)!}a, \dots, a, \phi(\frac{\lambda_1}{1-\sum_{i=1}^m \alpha_i}))|}{\phi(|a|)} = \mu$$

and the following three inequalities hold:

$$\phi^{-1}(K_{p-1}^{n-1}) \sum_{i=0}^{n-3} \phi(\frac{\theta_i}{\alpha})\phi\big(\frac{1}{(n-2-i)!}\big) < 1,$$
$$\mu + \sum_{i=0}^{n-3} \theta_i \phi\big(\frac{1}{(n-2-i)!}\big) < \alpha,$$

and

$$\begin{aligned}
&(1 + \frac{\sum_{i=1}^m \alpha_i\xi_i}{1 - \sum_{i=1}^m \alpha_i})(\sum_{i=0}^{n-3} \|b_i\|_\infty \phi\big(\frac{1}{(n-2-i)!}\big) + \|b_{n-2}\|_\infty) \\
&\times \phi\big(\frac{1}{(1-\phi^{-1}(K_{p-1}^{n-1}))\sum_{i=1}^m \phi^{-1}(\frac{\beta_i}{\alpha})} \frac{1}{(n-2-i)!}\big)\phi(K_{p-1}) \\
&\quad \times \phi\big(1 + \phi^{-1}(K_{p-1})\big)\phi^{-1}(\frac{\beta_{n-1}}{\alpha}) + \big(1 + \frac{\sum_{i=1}^m \alpha_i\xi_i}{1-\sum_{i=1}^m \alpha_i}\big)\|c\|_\infty < 1.
\end{aligned}$$

(H15) $\phi\big(\theta^{-1}(\frac{\lambda_2}{1-\sum_{i=1}^m \beta_i})\big) = \frac{\lambda_1}{1-\sum_{i=1}^m \alpha_i}$;

(H16) $\lambda_1, \lambda_2 \in \mathbb{R}, \alpha_i \geq 0, \beta_i \geq 0$ for $i = 1, \dots, m$ with $\sum_{i=1}^m \alpha_i < 1$ and $\sum_{i=1}^m \beta_i < 1$.

Let $x_1 = x$ and $x_2 = \phi(x_1)$, then (1.28) is transformed into

$$\begin{gathered}
x_1^{(n-1)}(t) = \phi^{-1}(x_2(t)), \quad t \in [0,1], \\
x_2'(t) = -f(t, x_1(t), \dots, x_1^{(n-2)}(t), \phi^{-1}(x_2(t))), \quad t \in [0,1], \\
0 = x_2(0) - \sum_{i=1}^m \alpha_i x_2(\xi_i) - \lambda_1, \\
0 = \theta(\phi^{-1}(x_2(1))) - \sum_{i=1}^m \beta_i \theta(\phi^{-1}(x_2(\xi_i))) - \lambda_2, \\
x_1^{(i)}(0) = 0, \quad i = 0 \dots, n-3,
\end{gathered} \tag{2.14}$$

Suppose $\lambda \in (0,1)$, we consider the following problem

$$\begin{gathered}
x_1^{(n-1)}(t) = \lambda \phi^{-1}(x_2(t)), \quad t \in [0,1], \\
x_2'(t) = -\lambda f(t, x_1(t), \dots, x_1^{(n-2)}(t), \phi^{-1}(x_2(t))), \quad t \in [0,1], \\
0 = \lambda(x_2(0) - \sum_{i=1}^m \alpha_i x_2(\xi_i) - \lambda_1), \\
0 = \lambda(\theta(\phi^{-1}(x_2(1))) - \sum_{i=1}^m \beta_i \theta(\phi^{-1}(x_2(\xi_i))) - \lambda_2), \\
x_1^{(i)}(0) = 0, \quad i = 0 \dots, n-3,
\end{gathered} \tag{2.15}$$

Define the operators

$$L(x_1, x_2) = (x_1^{(n-1)}, x_2', 0, 0), \quad (x_1, x_2) \in X \cap D(L),$$

$$N(x_1, x_2) = \begin{pmatrix} \phi^{-1}(x_2) \\ -f^*(t, x_1, \dots, x_1^{(n-2)}(t), \phi^{-1}(x_2)) \\ x_2(0) - \sum_{i=1}^m \alpha_i x_2(\xi_i) - \lambda_1 \\ \theta(\phi^{-1}(x_2(1))) - \sum_{i=1}^n \beta_i \theta(\phi^{-1}(x_2(\xi_i))) - \lambda_2 \end{pmatrix}^T, \quad (x_1, x_2) \in X.$$

Suppose that (H14), (H15), (H16) hold. It is easy to show the following results:

(i) $\operatorname{Ker} L = \{(\frac{t^{n-2}}{(n-2)!} a, b) : a, b \in \mathbb{R}\}$ and $\operatorname{Im} L = \{(y_1, y_2, a, b) : a = b = 0\}$;
(ii) L is a Fredholm operator of index zero;
(iii) There are projectors $P : X \to X$ and $Q : Y \to Y$ such that $\operatorname{Ker} L = \operatorname{Im} P$ and $\operatorname{Ker} Q = \operatorname{Im} L$. Furthermore, let $\Omega \subset X$ be an open bounded subset with $\overline{\Omega} \cap D(L) \neq \emptyset$, then N is L-compact on $\overline{\Omega}$;
(iv) $x = (x_1, x_2)$ is a solution of (2.14) if and only if x is a solution of the operator equation $Lx = Nx$ in $D(L)$.

We present the projectors P and Q as follows: $P(x_1, x_2) = (\frac{t^{n-2}}{(n-2)!} x_1^{(n-2)}(0), x_2(0))$ for all $x = (x_1, x_2) \in X$ and $Q(y_1, y_2, a, b) = (0, 0, a, b)$. The generalized inverse of L is

$$K_P(y_1, y_2, a, b) = (\int_0^t \frac{(t-s)^{n-2}}{(n-2)!} y_1(s) ds, \int_0^t y_2(s) ds),$$

the isomorphism $\wedge : Y/\operatorname{Im} L \to \operatorname{Ker} L$ is defined by $\wedge(0, 0, a, b) = (\frac{t^{n-2}}{(n-2)!} a, b)$.

Lemma 2.34. *If (x_1, x_2) is a solution of problem (42), then x_1 is a solution of* (1.28).

Lemma 2.35. *Suppose that $(H_{14}), (H_{15}), (H_{16})$ hold. If (x_1, x_2) is a solution of problem* (2.15)*, then there is a $\xi \in (0,1)$ so that $x_2'(\xi) = 0$.*

Proof. In fact, if $x_2'(t) > 0$ for all $t \in (0,1)$, we get

$$x_2(0) = \sum_{i=1}^m \alpha_i x_2(\xi_i) + \lambda_1 > \sum_{i=1}^m \alpha_i x_2(0) + \lambda_1.$$

So $x_2(0) > \lambda_1/(1 - \sum_{i=1}^m \alpha_i)$. It follows from $x_2'(t) > 0$ that $x_2(1) > \lambda_1/(1 - \sum_{i=1}^m \alpha_i)$. On the other hand,

$$\theta(\phi^{-1}(x_2(1))) = \sum_{i=1}^m \beta_i \theta(\phi^{-1}(x_2(\xi_i))) + \lambda_2 < \sum_{i=1}^m \beta_i \theta(\phi^{-1}(x_2(1))) + \lambda_2.$$

Then $\theta(\phi^{-1}(x_2(1))) < \lambda_2/(1-\sum_{i=1}^m \beta_i)$. So we get $x_2(1) < \phi(\theta^{-1}(\lambda_2/(1-\sum_{i=1}^m \beta_i))) = \lambda_1/(1-\sum_{i=1}^m \alpha_i) < x_2(1)$, a contradiction. If $x_2(t) < 0$ for all $t \in (0,1)$, the same contradiction can be derived. So there is $\xi \in [0,1]$ such that $x_2(\xi) = 0$. □

Lemma 2.36. *Suppose that* (H14), (H15), (H16) *hold. If* (x_1, x_2) *is a solution of problem* (2.15), *then*

$$|x_2(t)| \le \Big(1 + \frac{\sum_{i=1}^m \alpha_i \xi_i}{1-\sum_{i=1}^m \alpha_i}\Big)\|x_2'\|_\infty + \frac{|\lambda_1|}{1-\sum_{i=1}^m \alpha_i}.$$

Proof. In fact,

$$\begin{aligned}
|x_2(0)| &= \frac{1}{1-\sum_{i=1}^m \alpha_i}\Big|x_2(0) - \sum_{i=1}^m \alpha_i x_2(0)\Big| \\
&\le \frac{1}{1-\sum_{i=1}^m \alpha_i}\Big(\sum_{i=1}^m \alpha_i |x_2(\xi_i) - x_2(0)| + |\lambda_1|\Big) \\
&\le \frac{1}{1-\sum_{i=1}^m \alpha_i}\Big(\sum_{i=1}^m \alpha_i \xi_i \|x_2'\|_\infty + |\lambda_1|\Big).
\end{aligned}$$

Hence

$$|x_2(t)| \le |x_2(t) - x_2(0)| + |x_2(0)| \le \big(1 + \frac{\sum_{i=1}^m \alpha_i \xi_i}{1-\sum_{i=1}^m \alpha_i}\big)\|x_2'\|_\infty + \frac{|\lambda_1|}{1-\sum_{i=1}^m \alpha_i}.$$

□

Lemma 2.37. *Suppose that* (H3), (H13)-(H16) *hold. Let* $\Omega_0 = \{(x_1, x_2) \in D(L) \setminus \operatorname{Ker} L : L(x_1, x_2) = \lambda N(x_1, x_2)$ *for some* $\lambda \in (0,1)\}$. *Then* Ω_0 *is bounded.*

Proof. In fact, if $(x_1, x_2) \in \Omega_0$, we get (2.15). It follows from Lemmas 2.34, 2.35 and 2.36 that

$$|x_2(t)| \le \Big(1 + \frac{\sum_{i=1}^m \alpha_i \xi_i}{1-\sum_{i=1}^m \alpha_i}\Big)\|x_2'\|_\infty + \frac{|\lambda_1|}{1-\sum_{i=1}^m \alpha_i}$$

and there is $\xi \in [0,1]$ so that $x_2(\xi) = 0$. Then

$$\begin{aligned}
|x_2(t)| \le{}& \big(1 + \frac{\sum_{i=1}^m \alpha_i \xi_i}{1-\sum_{i=1}^m \alpha_i}\big) \max_{t\in[0,1]} \big|f(t, x_1(t), \dots, x_1^{(n-2)}(t), \phi^{-1}(x_2(t)))\big| \\
&+ \frac{|\lambda_1|}{1-\sum_{i=1}^m \alpha_i}
\end{aligned}$$

and $f(\xi, x_1(\xi), \dots, x_1^{(n-2)}(\xi), \phi^{-1}(x_2(\xi))) = 0$. Then (H13) implies

$$\begin{aligned}
\phi(|x_1^{(n-2)}(\xi)|) &\le \frac{1}{\alpha}\sum_{i=0}^{n-3} \theta_i \phi(|x_1^{(i)}(\xi)|) + \frac{\theta_{n-1}}{\alpha}|x_2(\xi)| + \frac{L}{\alpha} \\
&\le \frac{1}{\alpha}\sum_{i=0}^{n-3} \theta_i \phi\Big(\frac{1}{(n-2-i)!}\|x_1^{(n-2)}\|_\infty\Big) + \frac{\theta_{n-1}}{\alpha}\|x_2\|_\infty + \frac{L}{\alpha}.
\end{aligned}$$

So from Lemma 2.2 we have

$$|x_1^{(n-2)}(\xi)| \le \phi^{-1}\Big(\frac{1}{\alpha}\sum_{i=0}^{n-3} \theta_i \phi\big(\frac{1}{(n-2-i)!}\|x_1^{(n-2)}\|_\infty\big) + \frac{\theta_{n-1}}{\alpha}\|x_2\|_\infty + \frac{L}{\alpha}\Big)$$

$$
\begin{aligned}
&\le \phi^{-1}\Big[K_{p-1}^{n-1}\phi\Big(\sum_{i=0}^{n-3}\phi^{-1}(\theta_i/\alpha)\frac{1}{(n-2-i)!}\|x_1^{(n-2)}\|_\infty \\
&\quad +\phi^{-1}(\theta_{n-1}/\alpha)\phi^{-1}(\|x_2\|_\infty)+\phi^{-1}(L/\alpha)\Big)\Big] \\
&\le \phi^{-1}(K_{p-1}^{n-1})\Big(\sum_{i=0}^{n-3}\phi^{-1}(\theta_i/\alpha)\frac{1}{(n-2-i)!}\|x_1^{(n-2)}\|_\infty \\
&\quad +\phi^{-1}(\theta_{n-1}/\alpha)\phi^{-1}(\|x_2\|_\infty)+\phi^{-1}(L/\alpha)\Big).
\end{aligned}
$$

So

$$
\begin{aligned}
|x_1^{(n-2)}(t)| &\le |x_1^{(n-2)}(t)-x_1^{(n-2)}(\xi)|+|x_1^{(n-2)}(\xi)| \\
&\le \phi^{-1}(\|x_2\|_\infty)+\phi^{-1}(K_{p-1}^{n-1})\Big(\sum_{i=0}^{n-3}\phi^{-1}(\theta_i/\alpha)\frac{1}{(n-2-i)!}\|x_1^{(n-2)}\|_\infty \\
&\quad +\phi^{-1}\big(\frac{\theta_{n-1}}{\alpha}\big)\phi^{-1}(\|x_2\|_\infty)+\phi^{-1}(L/\alpha)\Big) \\
&\le \phi^{-1}(K_{p-1}^{n-1})\sum_{i=0}^{n-3}\phi^{-1}(\theta_i/\alpha)\phi\big(\frac{1}{(n-2-i)!}\big)\|x_1^{(n-2)}\|_\infty \\
&\quad +\Big(1+\phi^{-1}(K_{p-1}^{n-1})\phi^{-1}\big(\frac{\theta_{n-1}}{\alpha}\big)\Big)\phi^{-1}(\|x_2\|_\infty)+\phi^{-1}(K_{p-1}^{n-1})\phi^{-1}(L/\alpha).
\end{aligned}
$$

We get, from (H13) and

$$
\phi^{-1}(K_{p-1}^{n-1})\sum_{i=0}^{n-3}\phi\big(\frac{\theta_i}{\alpha}\big)\phi\big(\frac{1}{(n-2-i)!}\big)<1,
$$

that

$$
\begin{aligned}
\|x_1^{(n-2)}\|_\infty &\le \Big(1-\phi^{-1}(K_{q-1}^{n-1})\sum_{i=0}^{n-3}\phi^{-1}(\theta_i/\alpha)\phi\big(\frac{1}{(n-2-i)!}\big)\Big)^{-1} \\
&\quad\times\Big[(1+\phi^{-1}(K_{q-1}^{n-1})\phi^{-1}(\theta_{n-1}|/\alpha))\phi^{-1}(\|x_2\|_\infty) \\
&\quad +\phi^{-1}(K_{q-1}^{n-1})\phi^{-1}(L/\alpha)\Big].
\end{aligned}
$$

It follows from Lemma 2.2 that

$$
\begin{aligned}
&\phi(\|x_1^{(n-2)}\|_\infty) \\
&\le \phi\Big(\frac{1}{(1-\phi^{-1}(K_{p-1}^{n-1}))\sum_{i=0}^{n-3}\phi^{-1}(\theta_i/\alpha)\phi\big(\frac{1}{(n-2-i)!}\big)}\Big) \\
&\quad\times\phi\Big((1+\phi^{-1}(K_{p-1}^{n-1})\phi^{-1}(\theta_{n-1}/\alpha))\phi^{-1}(\|x_2\|_\infty)+\phi^{-1}(K_{p-1}^{n-1})\phi^{-1}\big(\frac{L}{\alpha}\big)\Big) \\
&\le \phi\Big(\frac{1}{(1-\phi^{-1}(K_{p-1}^{n-1}))\sum_{i=0}^{n-3}\phi^{-1}(\theta_i/\alpha)\phi\big(\frac{1}{(n-2-i)!}\big)}\Big) \\
&\quad\times\phi(K_{p-1})\Big[\phi(1+\phi^{-1}(K_{p-1}^{n-1}))\phi^{-1}\big(\frac{\theta_{n-1}}{\alpha}\big)\|x_2\|_\infty+\frac{K_{p-1}L}{\alpha}\Big].
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
|x_2(t)| &\le \Big(1+\frac{\sum_{i=1}^m \alpha_i\xi_i}{1-\sum_{i=1}^m \alpha_i}\Big)\\
&\quad\times \max_{t\in[0,1]} \big|f(t,x_1(t),\dots,x_1^{(n-2)}(t),\phi^{-1}(x_2(t)))\big| + \frac{|\lambda_1|}{1-\sum_{i=1}^m \alpha_i}\\
&\le \big(1+\frac{\sum_{i=1}^m \alpha_i\xi_i}{1-\sum_{i=1}^m \alpha_i}\Big)\Big(\|a\|_\infty + \sum_{i=0}^{n-2}\|b_i\|_\infty \phi(\|x_1^{(i)}\|_\infty) + \|c\|_\infty\|x_2\|_\infty\Big)\\
&\quad + \frac{|\lambda_1|}{1-\sum_{i=1}^m \alpha_i}\\
&\le \Big(1+\frac{\sum_{i=1}^m \alpha_i\xi_i}{1-\sum_{i=1}^m \alpha_i}\Big)\Big(\|a\|_\infty + \sum_{i=0}^{n-3}\|b_i\|_\infty \phi(1/(n-2-i)!)\phi(\|x_1^{(n-2)}\|_\infty)\\
&\quad + \|b_{n-2}\|_\infty\phi(\|x_1^{(n-2)}\|_\infty) + \|c\|_\infty\|x_2\|_\infty\Big) + \frac{|\lambda_1|}{1-\sum_{i=1}^m \alpha_i}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\|x_2\|_\infty &\le \Big(1+\frac{\sum_{i=1}^m \alpha_i\xi_i}{1-\sum_{i=1}^m \alpha_i}\Big)\Big\{\|a\|_\infty + \Big(\sum_{i=0}^{n-3}\|b_i\|_\infty\phi(1/(n-2-i)!) + \|b_{n-2}\|_\infty\Big)\\
&\quad\times \phi\Big(\frac{1}{(1-\phi^{-1}(K_{p-1}))\sum_{i=0}^{n-3}\phi^{-1}(\theta_i/\alpha)}\phi\big(\frac{1}{(n-2-i)!}\big)\Big)\\
&\quad\times \phi(K_{p-1})\Big[\phi(1+\phi^{-1}(K_{p-1}^{n-1}))\phi^{-1}\big(\frac{\theta_{n-1}}{\alpha}\big)\|x_2\|_\infty + \frac{K_{p-1}^{n-1}L}{\alpha}\Big]\\
&\quad + \|c\|_\infty\|x_2\|_\infty\Big\} + \frac{|\lambda_1|}{1-\sum_{i=1}^m \alpha_i}.
\end{aligned}
$$

We get

$$
\begin{aligned}
&\Big[1-\Big(1+\frac{\sum_{i=1}^m \alpha_i\xi_i}{1-\sum_{i=1}^m \alpha_i}\Big)\Big(\sum_{i=0}^{n-3}\|b_i\|_\infty\phi\big(\frac{1}{(n-2-i)!}\big) + \|b_{n-2}\|_\infty\Big)\\
&\quad\times \phi\Big(\frac{1}{(1-\phi^{-1}(K_{p-1}))\sum_{i=0}^{n-3}\phi^{-1}(\theta_i/\alpha)\phi\big(\frac{1}{(n-2-i)!}\big)}\Big)\phi(K_{p-1})\\
&\quad\times \phi\big(1+\phi^{-1}(K_{p-1}^{n-1})\big)\phi^{-1}\big(\frac{\theta_{n-1}}{\alpha}\big) - \big(1+\frac{\sum_{i=1}^m \alpha_i\xi_i}{1-\sum_{i=1}^m \alpha_i}\big)\|c\|_\infty\Big]\|x_2\|_\infty\\
&\le \Big(1+\frac{\sum_{i=1}^m \alpha_i\xi_i}{1-\sum_{i=1}^m \alpha_i}\Big)\|a\|_\infty + \frac{|\lambda_1|}{1-\sum_{i=1}^m \alpha_i}.
\end{aligned}
$$

It follows that there is a constant $M>0$ so that $\|x_2\|_\infty \le M$. Hence

$$
\begin{aligned}
\|x_1^{(n-2)}\|_\infty &\le \Big(1-\phi^{-1}(K_{p-1})\sum_{i=0}^{n-3}\phi^{-1}(\theta_i/\alpha)\frac{1}{(n-2-i)!}\Big)^{-1}\\
&\quad\times \big[(1+\phi^{-1}(K_{p-1})\phi^{-1}(\theta_{n-1}/\alpha))\phi^{-1}(M) + \phi^{-1}(K_{p-1})\phi^{-1}(L/\alpha)\big]\\
&=: M_{n-2},
\end{aligned}
$$

and for $i = 0, \dots, n-3$, we get

$$\|x_1^{(i)}\|_\infty \le \frac{1}{(n-2-i)!}\|x_1^{(n-2)}\|_\infty \le \frac{1}{(n-2-i)!}M_{n-2} =: M_i.$$

Hence Ω_0 is bounded. □

Lemma 2.38. *Suppose that* (H3), (H14), (H15), (H16) *hold. Let* $\Omega_1 = \{x \in \operatorname{Ker} L,\ Nx \in \operatorname{Im} L\}$. *Then* Ω_1 *is bounded.*

Proof. In fact, if $x = (\frac{t^{n-2}}{(n-2)!}a, b) \in \operatorname{Ker} L$ and $Nx \in \operatorname{Im} L$, then we get

$$b - \sum_{i=1}^m \alpha_i b - \lambda_1 = 0, \quad \theta(\phi^{-1}(b)) - \sum_{i=1}^m \beta_i \theta(\phi^{-1}(b)) - \lambda_2 = 0.$$

So we have $b = \lambda_1/(1 - \sum_{i=1}^m \alpha_i)$. From (H14), choose $\epsilon > 0$ so that

$$\sum_{i=0}^{n-3} \theta_i \phi\big(\frac{1}{(n-2-i)!}\big) + (\mu + \epsilon) < \alpha,$$

and then there is a $\delta > 0$ so that

$$\big|f\big(t, \frac{t^{n-2}}{(n-2)!}x, \dots, x, \phi^{-1}(\lambda_1/(1 - \sum_{i=1}^m \alpha_i))\big)\big| < (\mu+\epsilon)\phi(|x|), \quad |x| > \delta.$$

Let

$$A = \max_{t\in[0,1], |x|\le\delta} \big|f\big(t, \frac{t^{n-2}}{(n-2)!}x, \dots, x, \phi^{-1}(\lambda_1/(1 - \sum_{i=1}^m \alpha_i))\big)\big|.$$

Then one sees that

$$\begin{aligned}
(\mu+\epsilon)\phi(|a|) + A &\ge \big|f\big(t, \frac{t^{n-2}}{(n-2)!}a, \dots, a, \phi^{-1}(\lambda_1/(1 - \sum_{i=1}^m \alpha_i))\big)\big| \\
&\ge \alpha\phi(|a|) - \sum_{i=0}^{n-3} \theta_i \phi\big(\frac{t^{n-2-i}}{(n-2-i)!}|a|\big) - \theta_{n-1}|\lambda_1|/\big(1 - \sum_{i=1}^m \alpha_i\big) - L.
\end{aligned}$$

So

$$\begin{aligned}
\theta_{n-1}|\lambda_1|/(1 - \sum_{i=1}^m \alpha_i) + L &\ge \phi(|a|)\Big(\alpha - \sum_{i=0}^{n-3} \theta_i\phi\big(\frac{t^{n-2-i}}{(n-2-i)!}\big) - (\mu+\epsilon)\Big) \\
&\ge \phi(|a|)\Big(\alpha - \sum_{i=0}^{n-3} \theta_i\phi\big(\frac{1}{(n-2-i)!}\big) - (\mu+\epsilon)\Big)
\end{aligned}$$

Then there is $M_1' > 0$ so that $|a| \le M_1'$. Hence $|a|, |b|$ are bounded. Then Ω_1 is bounded. □

Lemma 2.39. *Suppose that* (H3), (H14), (H15), (H16) *hold. Then the set* $\Omega_2 = \{x \in \operatorname{Ker} L,\ \lambda \wedge^{-1} x + (1-\lambda)QNx = 0,\ \lambda \in [0,1]\}$ *is bounded.*

Proof. In fact, if Ω_2 is unbounded, then there are sequences $\{\lambda_n \in [0,1]\}$ and $\{x_n = (\frac{t^{n-2}}{(n-2)!}a_n, b_n)\}$ such that

$$\lambda_n(0,0,a_n,b_n) + (1-\lambda_n)\Big(0,0,b_n - \sum_{i=1}^m \alpha_i b_n - \lambda_1, \theta(\phi^{-1}(b_n)) - \sum_{i=1}^m \beta_i\theta(\phi^{-1}(b_n)) - \lambda_2\Big)$$

and either $|b_n| \to +\infty$ as n tends to infinity or $\{b_n\}$ is bounded and $|a_n| \to +\infty$ as n tends to infinity. It follows that

$$\lambda_n a_n = -(1-\lambda_n)(b_n - \sum_{i=1}^m \alpha_i b_n - \lambda_1), \tag{2.16}$$

$$\lambda_n b_n = -(1-\lambda_n)(\theta(\phi^{-1}(b_n)) - \sum_{i=1}^m \beta_i \theta(\phi^{-1}(b_n)) - \lambda_2). \tag{2.17}$$

Then

$$\lambda_n b_n^2 = -(1-\lambda_n)\theta(\phi^{-1}(b_n))b_n\big[(1-\sum_{i=1}^m \beta_i) - \frac{\lambda_2}{\theta(\phi^{-1}(b_n))}\big]$$

implies that there is a constant $B > 0$ so that $|b_n| \le B$ since $\phi^{-1}(b_n))b_n > 0$. Thus we get that $|a_n| \to +\infty$ as n tends to infinity. It follows from (2.16) that $\lambda_n \to 0$ as n tends to infinity. Thus (2.17) implies that

$$b_n \to b_0 = \phi(\theta^{-1}(\lambda_2/(1-\sum_{i=1}^m \beta_i))) = \lambda_1/(1-\sum_{i=1}^m \alpha_i).$$

Then

$$\begin{aligned}
(\mu+\epsilon)\phi(|a_n|) + A &\ge \big|f\big(t, \frac{t^{n-2}}{(n-2)!}a_n, \dots, a_n, \phi^{-1}(\lambda_1/(1-\sum_{i=1}^m \alpha_i))\big)\big| \\
&\ge \alpha\phi(|a_n|) - \sum_{i=0}^{n-3}\theta_i\phi\big(\frac{t^{n-2-i}}{(n-2-i)!}\phi(|a_n|)\big) \\
&\quad - \theta_{n-1}|\lambda_1|/(1-\sum_{i=1}^m \alpha_i) - L.
\end{aligned}$$

So

$$\begin{aligned}
\theta_{n-1}|\lambda_1|/(1-\sum_{i=1}^m \alpha_i) + L &\ge \phi(|a_n|)(\alpha - \sum_{i=0}^{n-3}\theta_i\phi(\frac{t^{n-2-i}}{(n-2-i)!}) - (\mu+\epsilon)) \\
&\ge \phi(|a_n|)(\alpha - \sum_{i=0}^{n-3}\theta_i\phi(\frac{1}{(n-2-i)!}) - (\mu+\epsilon)).
\end{aligned}$$

It follows from

$$\sum_{i=0}^{n-3}\theta_i\phi(\frac{1}{(n-2-i)!}) + (\mu+\epsilon) < \alpha$$

that there is a constant $C > 0$ so that $|a_n| \le C$, a contradiction. Hence Ω_2 is bounded. $\square$

Theorem 2.40. *Suppose that* (H3), (H13)–(H16) *hold. Then* (1.28) *has at least one solution.*

Proof. Let $\Omega \supseteq \Omega_0 \cup \Omega_1 \cup \Omega_2$ be a bounded open subset of X centered at zero Then $Lx \ne \lambda Nx$ for all $(x,\lambda) \in [(D(L) \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0,1)$; $Nx \notin \operatorname{Im} L$ for every $x \in \operatorname{Ker} L \cap \partial\Omega$; $\deg(\wedge QN\big|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) \ne 0$. It follows from Lemma 2.1 that $Lx = Nx$ has at least one solution $x = (x_1, x_2)$. Then x_1 is a solution of (2.14). Hence x_1 is a solution of (1.28). $\square$

We remark that Theorem 2.40 generalizes the results in [11, 36].

3. Examples

Now, we present some examples to illustrate the main results. These BVPs can not be solved by known results.

Example 3.1. Consider the problem

$$\begin{aligned} &x''(t) + b(t)|x'(t)| + c(t)x(t) + r(t) = 0, \quad t \in (0,1), \\ &x(0) = \frac{1}{2}x(1/4) + 2, \quad x(1) = \frac{1}{2}x(\frac{1}{2}) + 2, \end{aligned} \tag{3.1}$$

where b, c and r are nonnegative continuous functions. Corresponding to (1.24), it is easy to find that (H1), (H2), (H3) hold. We find from Theorem 2.10 that if $\frac{5}{4}\int_0^1 c(s)ds + \int_0^1 b(s)ds < 1$, then (3.1) has at least one positive solution for each $r \in C[0,1]$ with $r(t) \geq 0$ and $\not\equiv 0$ on each subinterval of [0,1].

Example 3.2. Consider the problem

$$\begin{aligned} &(\phi_3(x'))' + a(t)\phi_3(x) + b(t)\phi_3(|x'|) + r(t) = 0, \quad t \in (0,1), \\ &x(0) = \frac{1}{2}x(1/2) + 6, \quad x(1) = \frac{1}{4}x(1/4) + \frac{1}{3}x(1/2) + 7, \end{aligned} \tag{3.2}$$

where a, b and r are nonnegative continuous functions. We find $p = 3$ and $q = 3/2$. Then by application of Theorem 2.10, (3.2) has at least one positive solution if $\phi_3(2)\phi_3(\frac{3}{2})\int_0^1 a(s)ds + \int_0^1 b(s)ds < 1$ for each $r \in C[0,1]$ with $r(t) \geq 0$ and $\not\equiv 0$ on each subinterval of [0,1].

Example 3.3. Consider the problem

$$\begin{aligned} &(\phi_3(x'))' + a(t)\phi_3(x) + b(t)\phi_3(|x'|) + r(t) = 0, \quad t \in (0,1), \\ &x'(0) = \frac{1}{2}x'(1/2) - 3, \quad x(1) = \frac{1}{4}x(1/4) + \frac{1}{3}x(1/2) + 4, \end{aligned} \tag{3.3}$$

where a, b and r are nonnegative continuous functions. We find $p = 3, q = 3/2, m = 2$. Then by application of Theorem 2.19, (3.3) has at least one positive solution if

$$\Big(1 + \frac{\phi_3(4)}{1 - \phi_3(4)\phi_3(1/2)}\frac{\phi_3(1/2)}{2}\Big)\Big[\|b\|_\infty + \phi_3(2)\phi_3(1 + \frac{12}{5}\frac{7}{48})\|a\|_\infty\Big] < 1$$

for each $r \in C[0,1]$ with $r(t) \geq 0$ and $\not\equiv 0$ on each subinterval of [0,1].

Example 3.4. Consider the problem

$$\begin{aligned} &(\phi_3(x'))' + a(t)\phi_3(x) + b(t)\phi_3(|x'|) + r(t) = 0, \quad t \in (0,1), \\ &x(0) = \frac{1}{2}x(1/2) + \frac{1}{3}x(3/4) + 6, \quad x'(1) = \frac{1}{4}x(1/2) + \frac{1}{3}x'(3/4) + 7, \end{aligned} \tag{3.4}$$

where a, b and r are nonnegative continuous functions. Then by application of Theorem 2.25, (3.4) has at least one positive solution if

$$\begin{aligned} &\Big(1 + \frac{\phi(4)}{1 - \phi(4)(\phi(1/2) + \phi(1/3))}(\phi(1/4)\frac{1}{2} + \phi(1/3)\frac{1}{4})\Big) \\ &\times \Big[\|b\|_\infty + \phi(2)\phi_3(41/13)\|a\|_\infty\Big] < 1 \end{aligned}$$

for each $r \in C[0,1]$ with $r(t) \geq 0$ and $\not\equiv 0$ on each subinterval of [0,1].

Example 3.5. Consider the problem

$$\begin{aligned} x''(t) + a(t)|x'(t)| + b(t)x(t) + r(t) = 0, \quad t \in (0,1), \\ x(0) = \frac{1}{2}x'(0) + 6, \quad x(1) = 4x'(1) + 7, \end{aligned} \tag{3.5}$$

where a, b and r are nonnegative continuous functions. Then by Theorem 2.25, (3.5) has at least one positive solution if

$$\frac{3}{2}\int_0^1 b(t)dt + \int_0^1 a(t)dt < 1$$

for each $r \in C[0,1]$ with $r(t) \geq 0$ and $\not\equiv 0$ on each subinterval of [0,1].

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