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FUNCTIONAL COMPRESSION-EXPANSION FIXED POINT THEOREM

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Abstract. This paper presents a generalization of the fixed point theorems of compression and expansion of functional type. As an application, the existence of a positive solution to a second order conjugate boundary value problem is considered. We conclude with an extension to multivalued maps.

1. Introduction

In this paper we provide a generalization of all the fixed point theorems of compression and expansion involving functionals. The generalization also gives an alternative, simpler argument of the Sun-Zhang fixed point theorem of cone compression-expansion of functional type which applies convex functionals. The use of functionals with the fixed point index to yield positive solutions can be traced back to Leggett and Williams [13]. Several multiple fixed point theorems have employed the use of functionals [4, 5, 6, 7] and most recently they have been used [3, 19] to verify the existence of at least one fixed point. In [3] Anderson and Avery generalized the fixed point theorem of Guo [9] by replacing the norm in places by sublinear functionals, and in [19] Sun and Zhang showed that a certain set was a retract, thus replacing the norm from the argument by a convex functional. In this paper we provide a generalization of all of the compression-expansion arguments that have utilized the norm and/or functionals (including [3, 9, 10, 12, 19]) in verifying the existence of at least one fixed point. Our result does not require sets to be invariant under our operator and yet maintains the freedom gained by using functionals that satisfy either Property A1 or Property A2, properties defined in the next section. We will follow the main result with an application and associated example, and then we will conclude with an extension to multivalued maps.

2. Preliminaries

In this section we will state the definitions that are used in the remainder of the paper.

Definition 2.1. Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a *cone* if it satisfies the following two conditions:

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(i) $x \in P, \lambda \geq 0$ implies $\lambda x \in P$;
(ii) $x \in P, -x \in P$ implies $x = 0$.

Every cone $P \subset E$ induces an ordering in E given by

$$x \leq y \text{ if and only if } y - x \in P.$$

Definition 2.2. An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

Definition 2.3. A map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E if $\alpha : P \to [0, \infty)$ is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $t \in [0, 1]$. Similarly we say the map β is a nonnegative continuous convex functional on a cone P of a real Banach space E if $\beta : P \to [0, \infty)$ is continuous and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)$$

for all $x, y \in P$ and $t \in [0, 1]$. We say the map ψ is a sub-linear functional if

$$\psi(tx) \leq t\psi(x) \text{ for all} x \in P, \ t \in [0, 1].$$

Property A1. Let P be a cone in a real Banach space E and Ω be a bounded open subset of E with $0 \in \Omega$. Then a continuous functional $\beta : P \to [0, \infty)$ is said to satisfy Property A1 if one of the following conditions hold:

(i) β is convex, $\beta(0) = 0$, $\beta(x) \neq 0$ if $x \neq 0$, and $\inf_{x \in P \cap \partial\Omega} \beta(x) > 0$,
(ii) β is sublinear, $\beta(0) = 0$, $\beta(x) \neq 0$ if $x \neq 0$, and $\inf_{x \in P \cap \partial\Omega} \beta(x) > 0$,
(iii) β is concave and unbounded.

Note that if condition (i) of Property A1 is satisfied so is condition (ii), since

$$\beta(tx) = \beta(tx + (1-t)0) \geq t\beta(x) + (1-t)\beta(0) = t\beta(x).$$

Property A2. Let P be a cone in a real Banach space E and Ω be a bounded open subset of E with $0 \in \Omega$. Then a continuous functional $\beta : P \to [0, \infty)$ is said to satisfy Property A2 if one of the following conditions hold:

(i) β is convex, $\beta(0) = 0$ and $\beta(x) \neq 0$ if $x \neq 0$,
(ii) β is sublinear, $\beta(0) = 0$ and $\beta(x) \neq 0$ if $x \neq 0$,
(iii) $\beta(x + y) \geq \beta(x) + \beta(y)$ for all $x, y \in P$, $\beta(0) = 0$, $\beta(x) \neq 0$ if $x \neq 0$.

Note that he assumption $\beta(x+y) \geq \beta(x) + \beta(y)$ for all $x, y \in P$ in condition (iii) could be rephrased as $-\beta$ satisfying the triangle inequality on P. Also note that if condition (i) of Property A2 is satisfied so is condition (ii) (for the same reason as in the remark following Property A1).

Definition 2.4. Let D be a subset of a real Banach space E. If $\rho : E \to D$ is continuous with $\rho(x) = x$ for all $x \in D$, then D is a *retract* of E, and the map ρ is a *retraction*. The *convex hull* of a subset D of a real Banach space X is given by

$$\operatorname{conv}(D) = \Big\{ \sum_{i=1}^{n} \lambda_i x_i : x_i \in D, \ \lambda_i \in [0, 1], \ \sum_{i=1}^{n} \lambda_i = 1, \text{ and } n \in \mathbb{N} \Big\}.$$

The following theorem is due to Dugundji and a proof can be found in [8, p. 44].

Theorem 2.5. *For Banach spaces X and Y, let $D \subset X$ be closed and let $F : D \to Y$ be continuous. Then F has a continuous extension $\tilde{F} : X \to Y$ such that $\tilde{F}(X) \subset \overline{\operatorname{conv}(F(D))}$.*

Corollary 2.6. *Every closed convex set of a Banach space is a retract of that Banach space.*

Note that for any positive real number r and non-negative continuous concave functional α, $Q(\alpha, r) = \{x \in P : r \le \alpha(x)\}$ is a retract of E by Corollary 2.6. Note also, if r is a positive number and if $\alpha : P \to [0, \infty)$ is a uniformly continuous convex functional with $\alpha(0) = 0$ and $\alpha(x) > 0$ for $x \ne 0$, then [19, Theorem 2.1] guarantees that $Q(\alpha, r)$ is a retract of E.

The following theorem, which establishes the existence and uniqueness of the fixed point index, is from [11, pp. 82-86]; an elementary proof can be found in [8, pp. 58 & 238]. The proof of our main result in the next section will invoke the properties of the fixed point index.

Theorem 2.7. *Let X be a retract of a real Banach space E. Then, for every bounded relatively open subset U of X and every completely continuous operator $A : \overline{U} \to X$ which has no fixed points on ∂U (relative to X), there exists an integer $i(A, U, X)$ satisfying the following conditions:*

- (G1) *Normality: $i(A, U, X) = 1$ if $Ax \equiv y_0 \in U$ for any $x \in \overline{U}$;*
- (G2) *Additivity: $i(A, U, X) = i(A, U_1, X) + i(A, U_2, X)$ whenever U_1 and U_2 are disjoint open subsets of U such that A has no fixed points on $\overline{U} - (U_1 \cup U_2)$;*
- (G3) *Homotopy Invariance: $i(H(t, \cdot), U, X)$ is independent of $t \in [0, 1]$ whenever $H : [0, 1] \times \overline{U} \to X$ is completely continuous and $H(t, x) \ne x$ for any $(t, x) \in [0, 1] \times \partial U$;*
- (G4) *Permanence: $i(A, U, X) = i(A, U \cap Y, Y)$ if Y is a retract of X and $A(\overline{U}) \subset Y$;*
- (G5) *Excision: $i(A, U, X) = i(A, U_0, X)$ whenever U_0 is an open subset of U such that A has no fixed points in $\overline{U} - U_0$;*
- (G6) *Solution: If $i(A, U, X) \ne 0$, then A has at least one fixed point in U.*

Moreover, $i(A, U, X)$ is uniquely defined.

3. Fixed Point Theorems

The proof of the following fixed point results can be found in [11, pp. 88-89].

Lemma 3.1. *Let P be a cone in a real Banach space E, Ω a bounded open subset of E with $0 \in \Omega$, and $A : P \cap \overline{\Omega} \to P$ a completely continuous operator. If $Ax \ne \mu x$ for all $x \in P \cap \partial\Omega$ and $\mu \ge 1$, then*

$$i(A, P \cap \Omega, P) = 1.$$

Lemma 3.2. *Let P be a cone in a real Banach space E, Ω a bounded open subset of E, and $A : P \cap \overline{\Omega} \to P$ a completely continuous operator. If*

- (i) $\inf_{x \in P \cap \partial\Omega} \|Ax\| > 0$*; and*
- (ii) *$Ax \ne \nu x$ for all $x \in P \cap \partial\Omega$ and $\nu \in (0, 1]$,*

then $i(A, P \cap \Omega, P) = 0$.

Lemma 3.3. *Let P be a cone in a real Banach space E, Ω a bounded open subset of E with $0 \in \Omega$, and $A : P \cap \overline{\Omega} \to P$ a completely continuous operator. If the* *continuous functional α satisfies Property A1 and $\alpha(Ax) \geq \alpha(x)$ for all $x \in P \cap \partial\Omega$, with $Ax \neq x$, for all $x \in P \cap \partial\Omega$, then*

$$i(A, P \cap \Omega, P) = 0.$$

Proof. Suppose α satisfies Property A1. Then at least one of the conditions (i), (ii) or (iii) of Property A1 is satisfied. We proceed in cases.

Case 1: Condition (i) of Property A1 is satisfied. The result follows from the proof of the following case since if condition (i) of Property A1 is satisfied then condition (ii) of Property A1 is satisfied.

Case 2: Condition (ii) of Property A1 is satisfied. Suppose that α is sublinear and $\alpha(0) = 0$, $\alpha(x) \neq 0$ if $x \neq 0$, and $\inf_{x \in P \cap \partial\Omega} \alpha(x) > 0$.

Claim: $Ax \neq \nu x$ for all $x \in P \cap \partial\Omega$ and $\nu \in (0, 1]$.

Suppose to the contrary that there exists an $x_0 \in P \cap \partial\Omega$ and $\nu_0 \in (0, 1]$ such that

$$Ax_0 = \nu_0 x_0$$

(since $Ax \neq x$ for $x \in P \cap \partial\Omega$ we have that $\nu_0 \neq 1$). Then since $\alpha(x_0) > 0$ we have

$$\alpha(Ax_0) = \alpha(\nu_0 x_0) \leq \nu_0 \alpha(x_0) < \alpha(x_0),$$

which is a contradiction. Also,

$$\inf_{x \in P \cap \partial\Omega} \alpha(Ax) \geq \inf_{x \in P \cap \partial\Omega} \alpha(x) > 0.$$

Now since A is completely continuous $\inf_{x \in P \cap \partial\Omega} \|Ax\| \geq 0$. If $\inf_{x \in P \cap \partial\Omega} \|Ax\| = 0$ then there exists a sequence $x_n \in P \cap \partial\Omega$ with $\|Ax_n\| \to 0$, i.e. $Ax_n \to 0$, as $n \to \infty$ and so α continuous with $\inf_{x \in P \cap \partial\Omega} \alpha(Ax) > 0$ implies

$$0 = \alpha(0) = \alpha(\lim_{n \to \infty} Ax_n) = \lim_{n \to \infty} \alpha(Ax_n) > 0.$$

Therefore, $\inf_{x \in P \cap \partial\Omega} \|Ax\| > 0$ and hence by Lemma 3.2

$$i(A, P \cap \Omega, P) = 0.$$

Case 3: Condition (iii) of Property A1 is satisfied. Let

$$R_1 = \sup_{x \in P \cap \overline{\Omega}} \alpha(Ax) \quad \text{and} \quad R_2 = \sup_{x \in P \cap \overline{\Omega}} \alpha(x)$$

and then define $R = \max\{R_1, R_2\} + 1$. Let $x^* \in P(\alpha, R) = \{x \in P : \alpha(x) \geq R\}$ (which is nonempty since α satisfies condition (iii) of Property A1), and

$$H(t, x) = tAx + (1 - t)x^*.$$

Obviously, $H : [0, 1] \times (P \cap \partial\Omega) \to P$ (note that $P \cap \partial\Omega = \partial(P \cap \Omega)$) is completely continuous.

Claim: $H(t, x) \neq x$ for all $(t, x) \in [0, 1] \times (P \cap \partial\Omega)$.

Suppose to the contrary, that is there is a $(t_0, x_0) \in [0, 1] \times (P \cap \partial\Omega)$ such that $H(t_0, x_0) = x_0$. Note $t_0 \neq 0$ since $\alpha(x^*) \geq R$ and $\alpha(x_0) \leq R_2 < R$. Also since $Ax_0 \neq x_0$ we have that $t_0 \neq 1$. For $t_0 \in (0, 1)$, we have

$$\begin{aligned}
\alpha(x_0) &= \alpha(t_0 Ax_0 + (1 - t_0)x^*) \\
&\geq t_0 \alpha(Ax_0) + (1 - t_0)\alpha(x^*) \\
&> t_0 \alpha(Ax_0) + (1 - t_0)\alpha(Ax_0) \\
&= \alpha(Ax_0)
\end{aligned}$$

which is a contradiction. Thus by the homotopy invariance property

$$i(A, P \cap \Omega, P) = i(x^*, P \cap \Omega, P)$$

and $i(x^*, P \cap \Omega, P) = 0$ since if $i(x^*, P \cap \Omega, P) \neq 0$, then there would be an $x_1 \in P \cap \Omega$ such that $x^* = x_1$ which is a contradiction since $\alpha(x^*) \geq R > \alpha(x_1)$. Thus

$$i(A, P \cap \Omega, P) = 0.$$

Therefore, regardless which of the three conditions of Property A1 is satisfied we have that $i(A, P \cap \Omega, P) = 0$. □

Lemma 3.4. *Let P be a cone in a real Banach space E, Ω a bounded open subset of E with $0 \in \Omega$, and $A : P \cap \overline{\Omega} \to P$ a completely continuous operator. If the continuous functional α satisfies Property A2 and $\alpha(Ax) \leq \alpha(x)$ for all $x \in P \cap \partial\Omega$ with $Ax \neq x$ for all $x \in P \cap \partial\Omega$, then*

$$i(A, P \cap \Omega, P) = 1.$$

Proof. Suppose α satisfies Property A2. Then at least one of the conditions (i), (ii) or (iii) of Property A2 is satisfied. We proceed in cases.
Case 1: Condition (i) of Property A2 is satisfied. The result follows from the proof of the following case since if condition (i) of Property A2 is satisfied then condition (ii) of Property A2 is satisfied.
Case 2: Condition (ii) of Property A2 is satisfied. Suppose that α is sublinear, $\alpha(0) = 0$ and $\alpha(x) \neq 0$ if $x \neq 0$.
Claim: $Ax \neq \lambda x$ for all $x \in P \cap \partial\Omega$ and $\lambda \geq 1$.
Suppose to the contrary; that is, there exists an $x_0 \in \partial\Omega$ and $\lambda_0 \geq 1$ (since $Ax \neq x$ for all $x \in P \cap \partial\Omega$, we have that $\lambda_0 \neq 1$), such that

$$Ax_0 = \lambda_0 x_0.$$

Note, $\alpha(x_0) \neq 0$ and since $\lambda_0 > 1$, we have $0 < \frac{1}{\lambda_0} < 1$, and $x_0 = Ax_0/\lambda_0$. Thus, by the sublinearity of α

$$\alpha(x_0) = \alpha(\frac{Ax_0}{\lambda_0}) \leq \frac{\alpha(Ax_0)}{\lambda_0},$$

and thus

$$\alpha(x_0) < \lambda_0 \alpha(x_0) \leq \alpha(Ax_0),$$

which is a contradiction. Note that $0 \in \Omega$ by assumption. Hence by Lemma 3.1

$$i(A, P \cap \Omega, P) = 1.$$

Case 3: Condition (iii) of Property A2 is satisfied. Let $H(t, x) = tAx$. Obviously, $H : [0, 1] \times (P \cap \overline{\Omega}) \to P$ is completely continuous.
Claim: $H(t, x) \neq x$ for all $(t, x) \in [0, 1] \times (P \cap \partial\Omega)$.
Suppose to the contrary; that is, there is a $(t_0, x_0) \in [0, 1] \times (P \cap \partial\Omega)$ such that $H(t_0, x_0) = x_0$. Note, $t_0 \neq 0$. Also, since $Ax_0 \neq x_0$ we have that $t_0 \neq 1$, and we have

$$Ax_0 = t_0 Ax_0 + (1 - t_0)Ax_0 = x_0 + (1 - t_0)Ax_0.$$

Thus for $t_0 \in (0, 1)$

$$\begin{aligned} \alpha(Ax_0) &= \alpha(x_0 + (1 - t_0)Ax_0) \\ &\geq \alpha(x_0) + \alpha((1 - t_0)Ax_0) \\ &> \alpha(x_0), \end{aligned}$$

since $(1-t_0)Ax_0 \neq 0$ (note, if $(1-t_0)Ax_0 = 0$ then $Ax_0 = x_0$ which contradicts $Ax_0 \neq x_0$), which is a contradiction. Hence, for all $(t,x) \in [0,1] \times (P \cap \partial\Omega)$, $H(t,x) \neq x$, and so by the homotopy invariance property of the fixed point index

$$i(A, P \cap \Omega, P) = i(0, P \cap \Omega, P),$$

and $i(0, P \cap \Omega, P) = 1$ by the solution property of the fixed point index. Therefore

$$i(A, P \cap \Omega, P) = 1.$$

□

Theorem 3.5. *Let Ω_1 and Ω_2 be two bounded open sets in a Banach Space E such that $0 \in \Omega_1$ and $\overline{\Omega_1} \subseteq \Omega_2$ and P is a cone in E. Suppose $A : P \cap (\overline{\Omega_2} - \Omega_1) \to P$ is completely continuous, α and ψ are nonnegative continuous functionals on P, and one of the two conditions:*

- (K1) *α satisfies Property A1 with $\alpha(Ax) \geq \alpha(x)$, for all $x \in P \cap \partial\Omega_1$, and ψ satisfies property* (A2) *with $\psi(Ax) \leq \psi(x)$, for all $x \in P \cap \partial\Omega_2$; or*
- (K2) *α satisfies Property A2 with $\alpha(Ax) \leq \alpha(x)$, for all $x \in P \cap \partial\Omega_1$, and ψ satisfies Property A1 with $\psi(Ax) \geq \psi(x)$, for all $x \in P \cap \partial\Omega_2$,*

is satisfied. Then A has at least one fixed point in $P \cap (\overline{\Omega_2} - \Omega_1)$.

Proof. If there exists an $x \in P \cap \partial(\overline{\Omega_2} - \Omega_1)$ such that $Ax = x$, then there is nothing to prove; thus suppose that $Ax \neq x$ for all x on the boundary of $\overline{\Omega_2} - \Omega_1$. By Dugundji's Theorem (Theorem 2.5), A has a completely continuous extension (which we will also denote by A)

$$A : P \cap \overline{\Omega_2} \to P.$$

Suppose condition (K1) is satisfied; the proof when (K2) is satisfied is nearly identical and will be omitted.

From Lemma 3.3 we have that $i(A, P \cap \Omega_1, P) = 0$, and from Lemma 3.4 we have that $i(A, P \cap \Omega_2, P) = 1$, and since A has no fixed points on $\Omega_2 - (\Omega_1 \cup (\Omega_2 - \overline{\Omega_1}))$, by the additivity property of the fixed point index, we have

$$i(A, P \cap \Omega_2, P) = i(A, P \cap (\Omega_2 - \overline{\Omega_1}), P) + i(A, P \cap \Omega_1, P),$$

and hence $i(A, P \cap (\Omega_2 - \overline{\Omega_1}), P) = 1$. By the solution property of the fixed point index, we have that A has a fixed point in $\Omega_2 - \overline{\Omega_1}$. □

4. Application

In this section, as an application of our main result, Theorem 3.5, we are concerned with the existence of at least one positive solution for the second order boundary value problem,

$$x'' + f(x) = 0, \qquad 0 \leq t \leq 1, \tag{4.1}$$

$$x(0) = 0 = x(1), \tag{4.2}$$

where $f : \mathbb{R} \to [0, \infty)$ is continuous. We look for solutions $x \in C^{(2)}[0,1]$ of (4.1), (4.2) which are both nonnegative and concave on $[0,1]$. We will impose growth conditions on f which ensure the existence of at least one symmetric positive solution of (4.1), (4.2) by applying Theorem 3.5. We will apply Theorem 3.5 to a completely continuous operator whose kernel $G(t,s)$ is the Green's function for

$$-x'' = 0, \tag{4.3}$$

satisfying (4.2). In particular,

$$G(t,s)=\begin{cases} t(1-s), & 0\le t\le s\le 1,\\ s(1-t), & 0\le s\le t\le 1.\end{cases} \tag{4.4}$$

We will make use of various properties of $G(t,s)$ which include

$$\int_0^1 G(t,s)ds=\frac{t(1-t)}{2},\quad 0\le t\le 1; \tag{4.5}$$

$$\int_t^{1-t} G(t,s)ds=\frac{t(1-2t)}{2},\quad 0\le t\le \frac{1}{2}; \tag{4.6}$$

$$\max_{0\le r\le 1}\frac{G(\frac{1}{2},r)}{G(t,r)}=\frac{1}{2t},\quad 0<t\le\frac{1}{2}. \tag{4.7}$$

Let $E=C[0,1]$ be endowed with the maximum norm,

$$\|x\|=\max_{0\le t\le 1}|x(t)|,$$

and define the cone $P\subset E$ by

$$P=\Big\{x\in E: x \text{ is concave, symmetric, nonnegative valued on } [0,1], \text{ and } \min_{t\in[z,1-z]} x(t)\ge 2z\|x\| \text{ for all } z\in[0,1/2]\Big\}.$$

Finally, let the nonnegative continuous functionals α and ψ be defined on the cone P by

$$\alpha(x)=\min_{t\in[1/4,3/4]} x(t)=x(1/4), \tag{4.8}$$

$$\psi(x)=\max_{t\in[0,1]} x(t)=x(1/2). \tag{4.9}$$

We observe here that, for each $x\in P$,

$$\|x\|=x(\frac{1}{2})\le 2x(\frac{1}{4})=2\alpha(x) \tag{4.10}$$

and that $x\in P$ is a solution of (4.1), (4.2) if and only if

$$x(t)=\int_0^1 G(t,s)f(x(s))ds,\quad 0\le t\le 1. \tag{4.11}$$

We now present an application of our main result.

Theorem 4.1. *Suppose there exists positive numbers r and R such that $0<r<R$, and suppose f satisfies the following conditions:*

(i) *$f(x)\ge 16R$ for all $x\in[R,2R]$,*
(ii) *$f(x)\le 8r$ for all $x\in[0,r]$.*

Then, the second order conjugate boundary value problem (4.1), (4.2) has at least one symmetric positive solution x^* such that

$$r\le \max_{t\in[0,1]} x^*(t)\quad \text{and}\quad \min_{t\in[1/2,3/4]} x^*(t)\le R.$$

Proof. Define the completely continuous operator A by

$$Ax(t)=\int_0^1 G(t,s)f(x(s))ds.$$

We seek a fixed point of A which satisfies the conclusion of the theorem. We note first, if $x \in P$, then from properties of $G(t,s)$, $Ax(t) \geq 0$ and $(Ax)''(t) = -f(x(t)) \leq 0$ for $0 \leq t \leq 1$, $Ax(t) \geq 2tAx(\frac{1}{2})$, for $0 \leq t \leq \frac{1}{2}$, $Ax(t) = Ax(1-t)$ for $0 \leq t \leq \frac{1}{2}$, and consequently, $Ax \in P$; that is, $A : P \to P$.

Also for all $x \in P$ we have $\alpha(x) \leq \psi(x)$. Thus if we let

$$\Omega_1 = \{x : \psi(x) < r\} \quad \text{and} \quad \Omega_2 = \{x : \alpha(x) < R\},$$

we have that $0 \in \Omega_1$ and $\overline{\Omega_1} \subseteq \Omega_2$, with Ω_1 and Ω_2 being bounded open subsets of P, since for all $x \in P$, we have $\|x\| \leq 2\alpha(x)$.

Claim 1: If $x \in P \cap \partial\Omega_1$, then $\psi(Ax) \leq \psi(x)$.
To see this, note that

$$\begin{aligned}
\psi(Ax) &= Ax(\frac{1}{2}) \\
&= \int_0^1 G(\frac{1}{2}, s) f(x(s)) ds \\
&\leq 8r \int_0^1 G(\frac{1}{2}, s) ds \\
&= r = \psi(x).
\end{aligned}$$

Claim 2: If $x \in P \cap \partial\Omega_2$, then $\alpha(Ax) \geq \alpha(x)$.
To see this, note that

$$\begin{aligned}
\alpha(Ax) &= Ax(\frac{1}{4}) \\
&= \int_0^1 G(\frac{1}{4}, s) f(x(s)) ds \\
&\geq 16R \int_0^1 G(\frac{1}{4}, s) ds \\
&= R = \alpha(x).
\end{aligned}$$

Clearly ψ satisfies Property A2(i) and α satisfies Property A1(iii) (note α is not convex so does not satisfy the hypothesis of the Sun-Zhang Fixed point theorem) thus the hypothesis (K2) of Theorem 3.5 is satisfied, and therefore A has a fixed point in $\overline{\Omega_2} - \Omega_1$. $\square$

Remark: Similar results can be found applying the Sun-Zhang fixed point theorem (as well as many others), however the arguments in the above result are extremely straightforward and are the result of mapping minimums outward from a boundary and maximums inward from a boundary. Hence, our main result not only provides additional freedom in choosing functionals, but the choice of those functionals can also make for simpler existence arguments.

As an example, we have $f(x) = 8x^2$ which satisfies the hypothesis of Theorem 4.1 with $r = 1$ and $R = 2$.

5. Multi-Valued Generalization

In this section, we extend our main result to multi-valued maps. Let X be a closed, convex subset of some Banach space $E = (E, \|\cdot\|)$. We will consider maps $F : X \to CK(E)$; here $CK(E)$ denotes the family of nonempty convex compact subsets of E. In this section a map $F : X \to CK(E)$ is called completely continuous if it is upper semicontinuous and maps bounded sets to bounded sets. There is a well-defined index for such maps that is unique and satisfies the key properties (all those listed in Theorem 2.7) that we require for our generalization; see Petryshyn [17] for a thorough treatment. The main result of this section compliments the multiple fixed point theorems of Leggett-Williams type that employ functionals [1, 2]. See [15, 16, 18] for examples and techniques of finding fixed points of multivalued maps. The proofs of the following fixed point results can be found in [17].

Lemma 5.1 ([17, p. 505]). *Let P be a cone in a real Banach space E, Ω a bounded open subset of E with $0 \in \Omega$, and $A : P \cap \overline{\Omega} \to CK(E)$ a completely continuous operator. If $\mu x \notin Ax$ for all $x \in P \cap \partial\Omega$ and $\mu \geq 1$, then*

$$i(A, P \cap \Omega, P) = 1.$$

Lemma 5.2 ([17, pp. 506-507]). *Let P be a cone in a real Banach space E, Ω a bounded open subset of E, and $A : P \cap \overline{\Omega} \to CK(E)$ a completely continuous operator. If*

(i) $\inf_{y\in\{Ax : x\in P\cap\partial\Omega\}} \|y\| > 0$*; and*
(ii) *$\nu x \notin Ax$ for all $x \in P \cap \partial\Omega$ and $\nu \in (0, 1]$,*

then $i(A, P \cap \Omega, P) = 0$.

Lemma 5.3. *Let P be a cone in a real Banach space E, Ω a bounded open subset of E with $0 \in \Omega$, and $A : P \cap \overline{\Omega} \to CK(E)$ a completely continuous operator. If the continuous functional α satisfies Property A1,*

$$\alpha(y) \geq \alpha(x)$$

for all $x \in P \cap \partial\Omega$ and for all $y \in Ax$, and

$$x \notin Ax$$

for all $x \in P \cap \partial\Omega$, then

$$i(A, P \cap \Omega, P) = 0.$$

Proof. Suppose α satisfies Property A1. Then at least one of the conditions (i), (ii) or (iii) of Property A1 is satisfied. The proofs when conditions (ii) and (iii) of Property A1 are satisfied are nearly identical to the arguments in the proof of Lemma 3.3 and will therefore be omitted. For completeness we provide the proof when condition (i) of Property A1 is satisfied.
Claim: $\nu x \notin Ax$ for all $x \in P \cap \partial\Omega$ and $\nu \in (0, 1]$.
Suppose to the contrary, that is there is an $x_0 \in P \cap \partial\Omega$ and $\nu \in (0, 1)$ such that $\nu x_0 = y_0 \in Ax_0$ (note we know that $\nu \neq 1$ since $x_0 \notin Ax_0$). Therefore, since α is convex,

$$\begin{aligned} \alpha(y_0) &= \alpha(\nu x_0) \\ &= \alpha(\nu x_0 + (1-\nu)0) \\ &\leq \nu\alpha(x_0) \\ &< \alpha(x_0) \end{aligned}$$

which is a contradiction. Also, since $\alpha(y) \geq \alpha(x)$ for all $x \in P \cap \partial\Omega$ and for all $y \in Ax$, we have that

$$\inf_{y\in\{Ax\ : x\in P\cap\partial\Omega\}} \alpha(y) \geq \inf_{x\in P\cap\partial\Omega} \alpha(x) > 0,$$

and following the argument in Lemma 3.3, we have that

$$\inf_{y\in\{Ax:x\in P\cap\partial\Omega\}}\|y\|>0.$$

Hence, by Lemma 5.1 we have $i(A,P\cap\Omega,P)=0$. $\square$

Lemma 5.4. *Let P be a cone in a real Banach space E, Ω a bounded open subset of E with $0\in\Omega$, and $A:P\cap\overline{\Omega}\to CK(E)$ a completely continuous operator. If the continuous functional α satisfies Property A2,*

$$\alpha(y)\le\alpha(x)$$

for all $x\in P\cap\partial\Omega$ and for all $y\in Ax$, and $x\notin Ax$ for all $x\in P\cap\partial\Omega$, then

$$i(A,P\cap\Omega,P)=1.$$

Proof. Suppose α satisfies Property A2. Then at least one of the conditions (i), (ii) or (iii) of Property A2 is satisfied. The proofs when conditions (ii) and (iii) of Property A2 are satisfied are nearly identical to the arguments in the proof of Lemma 3.4 and will therefore be omitted. For completeness we provide the proof when condition (i) of Property A2 is satisfied.

Claim: $\mu x\notin Ax$ for all $x\in P\cap\partial\Omega$ and $\mu\ge1$.

Suppose to the contrary, that is there is an $x_0\in P\cap\partial\Omega$ and $\mu>1$ such that $\mu x_0=y_0\in Ax_0$ (note we know that $\mu\neq1$ by assumption). Therefore, since α is convex,

$$\begin{aligned}\alpha(x_0)&=\alpha(\frac{y_0}{\mu})\\&=\alpha(\frac{y_0}{\mu}+(1-\frac{1}{\mu})0)\\&\le\frac{\alpha(y_0)}{\mu}\\&<\alpha(y_0),\end{aligned}$$

which is a contradiction. Hence, by Lemma 5.2 we have $i(A,P\cap\Omega,P)=1$. $\square$

Theorem 5.5. *Let Ω_1 and Ω_2 be two bounded open sets in a Banach Space E such that $0\in\Omega_1$ and $\overline{\Omega_1}\subseteq\Omega_2$ and P is a cone in E. Suppose $A:P\cap(\overline{\Omega_2}-\Omega_1)\to CK(E)$ is completely continuous, α and ψ are nonnegative continuous functionals on P, and one of the two conditions:*

- (K1) *α satisfies Property A1 with $\alpha(y)\ge\alpha(x)$ for all $x\in P\cap\partial\Omega_1$ and for all $y\in Ax$ and ψ satisfies Property A2 with $\psi(y)\le\psi(x)$ for all $x\in P\cap\partial\Omega_2$ and for all $y\in Ax$; or*
- (K2) *α satisfies Property A2 with $\alpha(y)\le\alpha(x)$ for all $x\in P\cap\partial\Omega_1$ and for all $y\in Ax$ and ψ satisfies Property A1 with $\psi(y)\ge\psi(x)$ for all $x\in P\cap\partial\Omega_2$ and for all $y\in Ax$*

is satisfied. Then A has at least one fixed point in $P\cap(\overline{\Omega_2}-\Omega_1)$.

Proof. If there exists an $x\in P\cap\partial(\overline{\Omega_2}-\Omega_1)$ such that $x\in Ax$, then there is nothing to prove; thus suppose that $x\notin Ax$ for all x on the boundary of $\overline{\Omega_2}-\Omega_1$. By Ma's extension of Dugundji's Theorem [14], A has a completely continuous extension (which we will also denote by A)

$$A:P\cap\overline{\Omega_2}\to CK(E).$$

Suppose condition $(K1)$ is satisfied; the proof when $(K2)$ is satisfied is nearly identical and will be omitted.

From Lemma 5.3 we have that $i(A, P\cap\Omega_1, P) = 0$, and from Lemma 5.4 we have that $i(A, P\cap\Omega_2, P) = 1$, and since A has no fixed points on $\Omega_2 - (\Omega_1 \cup (\Omega_2 - \overline{\Omega_1}))$ by the additivity property of the fixed point index, we have

$$i(A, P\cap\Omega_2, P) = i(A, P\cap(\Omega_2 - \overline{\Omega_1}), P) + i(A, P\cap\Omega_1, P).$$

Hence $i(A, P\cap(\Omega_2 - \overline{\Omega_1}), P) = 1$, and by the solution property of the fixed point index, we have that A has a fixed point in $\Omega_2 - \overline{\Omega_1}$. □

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