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GROWTH RATE AND EXISTENCE OF SOLUTIONS TO DIRICHLET PROBLEMS FOR PRESCRIBED MEAN CURVATURE EQUATIONS ON UNBOUNDED DOMAINS

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Abstract. We prove growth rate estimates and existence of solutions to Dirichlet problems for prescribed mean curvature equation on unbounded domains inside the complement of a cone or a parabola like region in $\mathbb{R}^n$ $(n \geq 2)$. The existence results are proved using a modified Perron's method by which a subsolution is a solution to the minimal surface equation, while the role played by a supersolution is replaced by estimates on the uniform C^0 bounds on the liftings of subfunctions on compact sets.

1. Introduction and main results

Let Ω be an unbounded domain with $C^{2,\gamma}$ $(0 < \gamma < 1)$ boundary in $\mathbb{R}^n$ $(n \geq 2)$, ϕ be a C^0 function on $\partial\Omega$, and Λ be a C^1 function on $\overline{\Omega}$, we consider the Dirichlet problem for the prescribed mean curvature equation on Ω (here the summation convention is used):

$$((1 + |Du|^2)\delta_{ij} - D_iuD_ju)D_{ij}u = n\Lambda(1 + |Du|^2)^{3/2} \quad \text{on } \Omega; \tag{1.1}$$

$$u = \phi \quad \text{on } \partial\Omega. \tag{1.2}$$

In this paper, we investigate the conditions from which we can derive growth estimates and existence of solutions u for (1.1)-(1.2).

When Ω is a bounded domain, Serrin proved in [13] that (1.1)-(1.2) has a solution in $C^0(\overline{\Omega}) \cap C^2(\Omega)$ as long as one can get C^0 estimates and the mean curvature H' on the boundary $\partial\Omega$ with respect to the inner normal satisfying $H' \geq \frac{n}{n-1}|\Lambda|$ on $\partial\Omega$. Furthermore, a counterexample is given [13, page 480] to show that for some functions Λ, (1.1)-(1.2) do not have a C^2 solution (the only thing that did not work out in the example is the C^0 estimate).

When $n = 2$, Ω is a strip and Λ is a constant H, there have been a lot of interest in investigating the solutions of (1.1)-(1.2). Finn [4] showed that the solvability of (1.1) in Ω implies that the width of Ω will be less than $\frac{1}{|H|}$. When the width of a strip Ω is $1/|H|$, the half cylinder of radius $1/(2|H|)$ is a graph with constant mean curvature H in the strip. Collin [2] and Wang [14] showed independently that there

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are graphs with constant mean curvature H on the strip Ω with width $1/|H|$ other than the half cylinder. When $\Lambda = H$ and Ω is an unbounded convex domain on a plane, Lopez [10] proved that the necessary and sufficient condition for (1.1) to have solutions with zero boundary value is that Ω is inside a strip of width $1/|H|$.

When Ω is a strip on the plane, the existence of constant mean curvature graphs with prescribed boundary was considered by Lopez in [11]. The approach used in [11] is a modified version of the classical Perron's method of super- sub- solutions. The subsolution used in [11] is a solution to the minimal surface equation (i.e. a solution to (1.1)-(1.2) with $\Lambda = 0$), while the role played by a supersolution is replaced by a family of turned to side nodoids (the use of turned to side nodiods was adopted from an idea used by Finn [5]) that were used to prove that liftings from subfunctions will be bounded uniformly on any compact subset of Ω.

When Ω is an unbounded domain inside a cone or cylinder, we proved in [9] the existence of solutions to (1.1)-(1.2) for certain class of functions Λ. The approach used in [9] is also a modified version of the classical Perron's method. There are new difficulties in carrying out the Perron's method when Λ is not a constant and Ω is not a slab. The main difficulty is that the family of turned to side nodiods cannot be used anymore. The difficulty was overcome in [9] by constructing a family of auxiliary functions that were used to prove that liftings from subfunctions will be bounded uniformly on any compact subset of Ω. However when Ω is outside a cone (in the compliment of a cone) or inside a parabola-shaped region, the family of auxiliary functions used in [9] can no longer be used.In this paper, we construct a new family of auxiliary functions so that we can use the Perron's method to prove the existence of solutions to (1.1)-(1.2). As a by product, we can also derive the growth estimates for solutions u to (1.1)-(1.2).

For more historical notes and references on prescribed mean curvature equations, we refer readers to [2], [4], [5], [6], [10], [11], [14].

We will consider only those domains that are inside some special regions. The first kind of regions is the compliment of a cone in $\mathbf{R}^n$ ($n \geq 2$) defined by (we use the notation $\mathbf{x}^* = (x_1, x_2, \cdots, x_{n-1})$)

$$P(n) = \{\mathbf{x} \in \mathbf{R}^n : |x_n| < \frac{1}{240n}|\mathbf{x}^*|\}.$$

The second kind of regions is a parabola-shaped region defined by

$$P(n, \alpha, b) = \{\mathbf{x} \in \mathbf{R}^n : |x_n| < b|\mathbf{x}^*|^\alpha\}.$$

for some fixed positive constants α, b, $0 < \alpha < 1$.

For a general domain Ω inside $P(n)$, we can estimate the growth rate of a solution.

Theorem 1.1. *Let Ω be a domain inside $P(n)$, $|\Lambda(\mathbf{x})|$ satisfy*

$$|\Lambda(\mathbf{x})| \leq \frac{15(n-1)}{14(n+1)} \frac{1}{|\mathbf{x}^*|} \quad \text{on } \Omega, \tag{1.3}$$

then any $C^2(\Omega) \cap C^0(\overline{\Omega})$ solution u to (1.1)-(1.2) satisfies that on Ω,

$$|u(\mathbf{x})| \leq \frac{1}{240n}|\mathbf{x}^*| + \sup\{|\phi(\mathbf{p}, q)| : (\mathbf{p}, q) \in \partial\Omega, \frac{1}{2}|\mathbf{x}^*| \leq |\mathbf{p}| \leq 2|\mathbf{x}^*|\}. \tag{1.4}$$

When Ω satisfies more geometric conditions, the existence of solutions to (1.1)-(1.2) can be proved. First we list a set of conditions that will guarantee a solution to the minimal surface equation with the same boundary data on the same domain:

(A1) There is a sequence of subdomains Ω_j such that $\Omega_j \subset \Omega_{j+1} \subset \Omega$ for all $j \geq 1$, $\cup \Omega_j = \Omega$;
(A2) Each Ω_j is a $C^{2,\gamma}$ bounded domain and has positive mean curvature on $\partial\Omega_j$ with respect to the inner normal on $\partial\Omega_j$;
(A3) $dist(\mathbf{0}, \Omega \setminus \Omega_j) \to \infty$ as $j \to \infty$.

The next condition on Ω will be used to prove the solution obtained by Perron's method takes boundary data ϕ continuously.
Serrin's condition: The mean curvature function H' on $\partial\Omega$ with respect to the inner normal satisfies

$$H' > \frac{n}{n-1} |\Lambda(\mathbf{x})| \quad on \quad \partial\Omega. \tag{1.5}$$

Remark 1.2. Conditions (A1)-(A3) and Serrin's condition (1.5) are the same as those used in [9].

Here is the first existence result.

Theorem 1.3. *Assume* (A1)–(A3)*, Serrin's condition* (1.5) *and* Ω *is inside* $P(n)$*. If* $\Lambda(\mathbf{x})$ *satisfies* (1.3)*, then* (1.1)–(1.2) *has a solution* $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$.

When the domains are inside $P(n, \alpha, b)$, we assume Ω is not very close to the origin:

$$|\mathbf{x}^*| \geq (120nb(\frac{3}{2})^\alpha)^{\frac{1}{1-\alpha}} \quad \text{for any} \quad \mathbf{x} \in \Omega. \tag{1.6}$$

Remark 1.4. Condition (1.6) is not absolutely necessary, we use it here so that we can state results more clearly. Without (1.6), the following results are still true as long as $\Lambda(\mathbf{x})$ is bounded appropriately where (1.6) does not hold.

The growth estimate now is as follows.

Theorem 1.5. *Let* Ω *be a domain inside* $P(n, \alpha, b)$*. If* Ω *satisfies* (1.6) *and*

$$|\Lambda(\mathbf{x})| \leq \frac{(n-1)}{56n(n+1)} (\frac{1}{3})^\alpha \frac{1}{b|\mathbf{x}^*|^\alpha} \quad on\ \Omega, \tag{1.7}$$

then any $C^2(\Omega) \cap C^0(\overline{\Omega})$ *solution* u *to* (1.1)-(1.2) *satisfies that on* Ω

$$|u(\mathbf{x})| \leq \frac{1}{2} (\frac{3}{2})^\alpha |\mathbf{x}^*|^\alpha + \sup\{|\phi(\mathbf{p}, q)| : (\mathbf{p}, q) \in \partial\Omega, \frac{1}{2}|\mathbf{x}^*| \leq |\mathbf{p}| \leq 2|\mathbf{x}^*|\}. \tag{1.8}$$

Here is the existence results for domains in $P(n, \alpha, b)$.

Theorem 1.6. *Assume* (A1)–(A3)*, Serrin's condition* (1.5) *and* Ω *is inside* $P(n, \alpha, b)$ *satisfying* (1.6)*. Then if* $|\Lambda(\mathbf{x})|$ *satisfies* (1.7)*,* (1.1)-(1.2) *has a solution* $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$.

2. A family of auxiliary functions and growth estimates

In this section, we construct a family of auxiliary functions and derive growth estimates for solutions of (1.1)-(1.2). The construction is adapted from that in [7] and [8] to fit our needs here (in turn, the constructions in [7] and [8] were inspired by [5] and [13]). Set

$$Qz \equiv \frac{((1 + |Dz|^2)\delta_{ij} - D_i z D_j z)}{n + (n-1)|Dz|^2} D_{ij} z \tag{2.1}$$

We first prove the existence of a family auxiliary functions that will suit our needs later.

Lemma 2.1. *For any numbers $M > 0$, $H \geq 2$, and any point $\mathbf{x}_0^* \in R^{n-1}$, there are positive decreasing functions $\chi(t)$ (depending on n only), $h_a(t)$ (with the inverse h_a^{-1}) and a positive increasing function $A(t)$ (depending on n, H and M only) such that for any constant γ, the function*

$$z = z(\mathbf{x}) = \gamma + A(H)e^{\chi(H)} - \{(h_a^{-1}(x_n + M))^2 - |\mathbf{x}^* - \mathbf{x}_0^*|^2\}^{1/2} \tag{2.2}$$

satisfies

$$Qz \leq -\frac{n-1}{28(n+1)MH} \cdot \frac{(1+|Dz|^2)^{3/2}}{n+(n-1)|Dz|^2} \quad in \quad \Omega_{\mathbf{x}_0^*,H,M} \tag{2.3}$$

where

$$\Omega_{\mathbf{x}_0^*,H,M} = \{\mathbf{x} : |x_n| < M, |\mathbf{x}^* - \mathbf{x}_0^*| < h_a^{-1}(x_n + M)\}. \tag{2.4}$$

Furthermore

$$z(\mathbf{x}_0^*, x_n) \leq \gamma + \frac{M}{H} \quad for \quad -M \leq x_n \leq M. \tag{2.5}$$

Proof. Set $E = \frac{1}{n-1}$, $G = \frac{1}{2n-1}$, $c_2 = \frac{2+E}{G} = 4n + \frac{1}{n-1}$, and $\Phi_1(\rho) = \rho^{-2}$ if $0 < \rho < 1$, $\Phi_1(\rho) = c_2$ if $\rho \geq 1$. We define a function χ by

$$\chi(\alpha) = \int_\alpha^\infty \frac{d\rho}{\rho^3\Phi_1(\rho)} \quad \text{for } \alpha > 0.$$

It is clear that $\chi(\alpha)$ is a decreasing function with range $(0, \infty)$. Let η be the inverse of χ. Then η is a positive, decreasing function with range $(0, \infty)$.

For $\alpha > 1$, we have

$$\chi(\alpha) = \int_\alpha^\infty \frac{d\rho}{\rho^3\Phi_1(\rho)} = \int_\alpha^\infty \frac{d\rho}{c_2\rho^3} = \frac{1}{2c_2}\alpha^{-2} < 1. \tag{2.6}$$

Thus

$$\eta(\beta) = (2c_2\beta)^{-1/2} \quad \text{for } 0 < \beta < (2c_2)^{-1}. \tag{2.7}$$

For $H \geq 2$, since $\eta(\chi(H)) = H$ and η is decreasing, we have $\eta(\beta) > H$ and $\eta(\beta) = (2c_2\beta)^{-1/2}$ for $0 < \beta < \chi(H)$. We define a function $A(H) = A(H, M)$ by

$$A(H) = 2M(\int_1^{e^{\chi(H)}} \eta(\ln t)dt)^{-1}. \tag{2.8}$$

For the rest of this article, we set $a = A(H)$ and define

$$h_a(r) = \int_r^{ae^{\chi(H)}} \eta(\ln\frac{t}{a})dt \quad \text{for } a \leq r \leq ae^{\chi(H)}. \tag{2.9}$$

Then

$$h_a(ae^{\chi(H)}) = 0, \quad h_a(a) = h_{A(H)}(A(H)) = 2M. \tag{2.10}$$

For $a < r \leq ae^{\chi(H)}$,

$$h_a'(r) = -\eta(\ln\frac{r}{a}) < 0, \quad |h_a'(r)| > H, \quad h_a''(r) = \frac{1}{r}(\eta(\ln\frac{r}{a}))^3\Phi_1(\eta(\ln\frac{r}{a})). \tag{2.11}$$

Thus for $a < r \leq ae^{\chi(H)}$,

$$\frac{h_a''(r)}{(h_a'(r))^2} = -\frac{h_a'(r)}{r}\Phi_1(-h_a'(r)). \tag{2.12}$$

Let h_a^{-1} be the inverse of h_a. Then h_a^{-1} is decreasing and

$$h_a^{-1}(0) = A(H)e^{\chi(H)}, \quad h_a^{-1}(2M) = A(H). \tag{2.13}$$

Further for $-M \le x_n \le M$,

$$(h_a^{-1})'(x_n+M) = \frac{1}{h_a'(h_a^{-1}(x_n+M))},$$

$$\begin{aligned}(h_a^{-1})''(x_n+M) &= (\frac{1}{h_a'(h_a^{-1}(x_n+M))})' \\ &= -\frac{h_a''(h_a^{-1}(x_n+M))(h_a^{-1})'(y+M)}{(h_a'(h_a^{-1}(x_n+M)))^2} \\ &= -\frac{h_a''(h_a^{-1}(x_n+M))}{(h_a'(h_a^{-1}(x_n+M)))^3} \\ &= \frac{1}{h_a^{-1}(x_n+M)}\Phi_1(-h_a'(h_a^{-1}(x_n+M))).\end{aligned}$$

Thus for $-M < x_n < M$,

$$(h_a^{-1})''(y+M)h_a^{-1}(x_n+M) = \Phi_1(-h_a'(h_a^{-1}(x_n+M))). \tag{2.14}$$

For $H \ge 2$, by (2.6), (2.7), we have

$$\begin{aligned}A(H)^{-1} &= (2M)^{-1}\int_1^{e^{\chi(H)}} \eta(\ln t)\,dt \\ &= (2M)^{-1}\int_0^{\chi(H)} \eta(m)e^m\,dm \\ &= (2M)^{-1}\int_0^{\chi(H)} \frac{e^m}{\sqrt{2c_2m}}\,dm.\end{aligned}$$

From

$$\int_0^{\chi(H)} \frac{1}{\sqrt{2c_2m}}dm \le \int_0^{\chi(H)} \frac{e^m}{\sqrt{2c_2m}}dm \le \frac{e^{\chi(H)}}{\sqrt{2c_2}}\int_0^{\chi(H)} m^{-1/2}dm,$$

we have

$$\frac{1}{c_2H} = \frac{2\sqrt{\chi(H)}}{\sqrt{2c_2}} \le \int_0^{\chi(H)} \frac{e^m}{\sqrt{2c_2m}}dm \le \frac{2e^{\chi(H)}\sqrt{\chi(H)}}{\sqrt{2c_2}} = \frac{e^{\frac{1}{2c_2H^2}}}{c_2H}.$$

Thus

$$2Mc_2H \ge A(H) \ge 2Mc_2He^{-\chi(H)} = 2Mc_2He^{-\frac{1}{2c_2H^2}}. \tag{2.15}$$

For $\mathbf{x}_0 \in \mathbf{R}^{n-1}$, $H \ge 2$ and $M > 0$, we define a domain $\Omega_{\mathbf{x}_0^*,H,M}$ in $\mathbf{x}$ space by (2.4) and define a function $z = z_{\mathbf{x}_0^*,H,M}(\mathbf{x})$ by (2.2). It is clear that the function z is well defined on $\Omega_{\mathbf{x}_0^*,H,M}$. Let

$$S = ((h_a^{-1}(x_n+M))^2 - |\mathbf{x}^* - \mathbf{x}_0^*|^2)^{1/2}. \tag{2.16}$$

then for $1 \le i \le n-1$, we have

$$\frac{\partial z}{\partial x_i} = \frac{1}{S}(x_i - x_{0i}), \quad \frac{\partial z}{\partial x_n} = -\frac{1}{S}h_a^{-1}(h_a^{-1})'. \tag{2.17}$$

Since $h_a^{-1}(r)$ and η are decreasing functions, for $H \ge 2$, $|y| \le M$, we have

$$\begin{aligned}0 < -(h_a^{-1})' &= \frac{-1}{h_a'(h_a^{-1}(x_n+M))} = \frac{1}{\eta(\ln(\frac{1}{a}h_a^{-1}(x_n+M)))} \\ &\le \frac{1}{\eta(\ln e^{\chi(H)})} = \frac{1}{\eta(\chi(H))} = \frac{1}{H} \quad \text{for} |x_n| \le -M.\end{aligned} \tag{2.18}$$

Then (2.5) follows from the facts that $z(\mathbf{x}_0^*, -M) = \gamma + A(H)e^{\chi(H)} - h_a^{-1}(0) = \gamma$ and

$$\frac{\partial z}{\partial x_n}(\mathbf{x}_0^*, x_n) = -\frac{1}{S}h_a^{-1}(h_a^{-1})' = -(h_a^{-1})' \le \frac{1}{H}.$$

Now if $|\mathbf{x}^* - \mathbf{x}_0^*| \ge \frac{1}{2}h_a^{-1}(x_n + M))$ and $H \ge 2$,

$$(\frac{\partial z}{\partial x_n})^2 = \frac{1}{S^2}(h_a^{-1})^2((h_a^{-1})')^2 \le \frac{1}{S^2}(\frac{1}{2}h_a^{-1})^2\frac{4}{H^2} \le \sum_{i=1}^{n-1}(\frac{\partial z}{\partial x_i})^2. \tag{2.19}$$

If $|\mathbf{x}^* - \mathbf{x}_0^*| \le \frac{1}{2}h_a^{-1}(x_n + M))$ and $H \ge 2$, then

$$S^2 = (h_a^{-1}(x_n + M)))^2 - |\mathbf{x}^* - \mathbf{x}_0^*|^2 \ge \frac{3}{4}(h_a^{-1}(x_n + M)))^2,$$

and

$$(\frac{\partial z}{\partial x_n})^2 = \frac{1}{S^2}(h_a^{-1})^2((h_a^{-1})')^2 \le \frac{1}{S^2}(h_a^{-1})^2\frac{1}{H^2} \le \frac{4}{3H^2} \le 1. \tag{2.20}$$

Therefore,

$$(\frac{\partial z}{\partial x_n})^2 \le \sum_{i=1}^{n-1}(\frac{\partial z}{\partial x_i})^2 + 1.$$

We set the notation

$$a_{ij} = \frac{(1 + |p|^2)\delta_{ij} - p_i p_j}{n + (n-1)|p|^2}, p_i = \frac{\partial z}{\partial x_i} \quad 1 \le i, j \le n.$$

Then $|p_n|^2 \le \sum_{i=1}^{n-1} p_i^2 + 1$ and

$$a_{nn} = \frac{1 + \sum_{i=1}^{n-1} p_i^2}{n + (n-1)|p|^2} \ge \frac{1 + \sum_{i=1}^{i=n-1} p_i^2}{2n - 1 + 2(n-1)\sum_{i=1}^{i=n-1} p_i^2} \ge \frac{1}{2n-1} = G \tag{2.21}$$

and

$$\sum_{i,j=1}^{n} a_{ij}\frac{\partial z}{\partial x_i}\frac{\partial z}{\partial x_j} = \frac{|p|^2}{n + (n-1)|p|^2} \le \frac{1}{n-1} = E. \tag{2.22}$$

Thus on $\Omega_{\mathbf{x}_0^*, H, M}$, we have

$$\begin{aligned}
Qz &= \sum_{i,j=1}^{n} a_{ij}D_{ij}z \\
&= \frac{1}{S}\sum_{i=1}^{n-1} a_{ii} + \frac{1}{S^3}\sum_{i,j=1}^{n-1} a_{ij}(x_i - x_i^0)(x_j - x_j^0) - \frac{1}{S^3}\sum_{i=1}^{n-1} a_{in}(x_i - x_i^0)h_a^{-1}(h_a^{-1})' \\
&\quad - \frac{1}{S}a_{nn}((h_a^{-1})^2 + h_a^{-1}(h_a^{-1})'') + \frac{1}{S^3}a_{nn}(h_a^{-1})^2((h_a^{-1})')^2 \\
&= \frac{1}{S}\{1 - a_{nn} + \sum_{i,j=1}^{n} a_{ij}\frac{\partial z}{\partial x_i}\frac{\partial z}{\partial x_j} - a_{nn}((h_a^{-1})^2 + h_a^{-1}(h_a^{-1})'')\} \\
&\le \frac{1}{S}\{1 + \sum_{i,j=1}^{n} a_{ij}\frac{\partial z}{\partial x_i}\frac{\partial z}{\partial x_j} - a_{nn}h_a^{-1}(h_a^{-1})''\} \\
&\le \frac{1}{S}\{1 + E - Gh_a^{-1}(h_a^{-1})''\} = \frac{-1}{S}
\end{aligned}$$

(by (2.14)), (2.21), (2.22) and the definition of Φ). Then (2.3) follows from the following inequality

$$\frac{n-1}{28(n+1)MH}\cdot\frac{(1+|Dz|^2)^{3/2}}{n+(n-1)|Dz|^2}\le\frac{1}{S}. \tag{2.23}$$

To prove (2.23), since $\chi(H)<1$ for $H\ge 2$, we have

$$\begin{aligned}
&\frac{(1+|Dz|^2)^{3/2}}{n+(n-1)|Dz|^2}\\
&\le\frac{1}{n-1}(1+|Dz|^2)^{1/2}\\
&=\frac{1}{n-1}(1+\frac{1}{S^2}(|\mathbf{x}^*-\mathbf{x}_0^*|^2+(h_a^{-1})^2((h_a^{-1})')^2))^{1/2}\quad\text{by (2.17)}\\
&=\frac{1}{(n-1)S}(S^2+|\mathbf{x}^*-\mathbf{x}_0^*|^2+(h_a^{-1})^2((h_a^{-1})')^2)^{1/2}\\
&=\frac{1}{(n-1)S}((h_a^{-1})^2+(h_a^{-1})^2((h_a^{-1})')^2)^{1/2}\quad\text{by (2.16)}\\
&=\frac{1}{(n-1)S}(h_a^{-1})(1+((h_a^{-1})')^2)^{1/2}\\
&\le\frac{1}{(n-1)S}(h_a^{-1})(1+\frac{1}{H^2})^{1/2}\le\frac{1}{(n-1)S}A(H)e^{\chi(H)}(1+\frac{1}{4})^{1/2}\quad\text{by (2.13), (2.18)}\\
&\le\frac{1}{(n-1)S}(\frac{5}{4})^{1/2}2c_2e^{\chi(H)}MH\le\frac{1}{(n-1)S}c_2 5^{1/2}eMH\quad\text{by (2.15)}\\
&=\frac{1}{(n-1)S}(4n+\frac{1}{n-1})5^{1/2}eMH\le\frac{28(n+1)}{n-1}MH\frac{1}{S}\quad\text{by the definition of } c_2.
\end{aligned}$$

□

Lemma 2.2. *Let ϕ be a continuous function defined on $\partial\Omega$. For any $\mathbf{x}_0^*\in R^{n-1}$, we set*

$$\gamma=\gamma(\mathbf{x}_0^*)=\sup\{|\phi(\mathbf{x})|:\mathbf{x}\in\partial\Omega,\quad\frac{1}{2}|\mathbf{x}_0^*|\le|\mathbf{x}^*|\le\frac{3}{2}|\mathbf{x}_0^*|\}. \tag{2.24}$$

For any $\mathbf{x}_0^*\in R^{n-1}$ such that $(\mathbf{x}_0^*,x_n)\in\Omega$ for some x_n, in the function $z=z_{\mathbf{x}_0^*}$ defined in (2.2), we set

$$\gamma=\gamma(\mathbf{x}_0^*),\quad H=2,\quad M=\frac{1}{120n}|\mathbf{x}_0^*|. \tag{2.25}$$

Then $z=z_{\mathbf{x}_0^*}$ satisfies

$$Qz\le-n\Lambda_0(\mathbf{x})\frac{(1+|Dz|^2)^{3/2}}{n+(n-1)|Dz|^2}\quad\text{in}\quad\Omega_{\mathbf{x}_0,H,M}\cap P(n) \tag{2.26}$$

where

$$\Lambda_0(\mathbf{x})=\frac{15(n-1)}{14(n+1)|\mathbf{x}^*|}. \tag{2.27}$$

Furthermore

(i) $z(\mathbf{x}_0^*,x_n)\le\frac{1}{240n}|\mathbf{x}_0^*|+\gamma(\mathbf{x}_0^*)$ *for* $|x_n|<M$*;*
(ii) $\partial\Omega_{\mathbf{x}_0^*,M,H}\cap P(n)\subset\{\mathbf{x}:|x_n|<M\}$*;*

(iii) *For the unit outer normal* $\mathbf{n}$ *on* $\partial\Omega_{\mathbf{x}_0^*,M,H}\cap P(n)$,

$$\frac{\partial z}{\partial \mathbf{n}} = +\infty \quad on\ \partial\Omega_{\mathbf{x}_0^*,M,H}\cap P(n).$$

Proof. From the choices of H and M and the definition of $\Omega_{\mathbf{x}_0^*,H,M}$, we have that for any $\mathbf{x}\in\Omega_{\mathbf{x}_0^*,H,M}$, by (2.15),

$$|\mathbf{x}^*-\mathbf{x}_0^*|\le h_a^{-1}(x_n+M)\le A(H)e^{\chi(H)}\le 2c_2eHM\le 60nM=\frac{1}{2}|\mathbf{x}_0^*|.$$

Thus for any $\mathbf{x}\in\Omega_{\mathbf{x}_0^*,H,M}$, $\frac{1}{2}|\mathbf{x}_0^*|\le|\mathbf{x}^*|\le\frac{3}{2}|\mathbf{x}_0^*|$. Again from the choices of M and H, we have

$$\frac{n-1}{28(n+1)MH}=\frac{n-1}{56(n+1)M}=\frac{15n(n-1)}{7(n+1)|\mathbf{x}_0^*|}.$$

Then if we set

$$\Lambda_0(\mathbf{x})=\frac{15(n-1)}{14(n+1)|\mathbf{x}^*|},$$

we have that on $\Omega_{\mathbf{x}_0^*,H,M}$

$$n\Lambda_0(\mathbf{x})=\frac{15n(n-1)}{14(n+1)|\mathbf{x}^*|}\le\frac{15n(n-1)}{7(n+1)|\mathbf{x}_0^*|}=\frac{n-1}{28(n+1)MH}.$$

Now (2.26) follows from Lemma 2.1.

(i) is clear from (2.5) and the definitions of M and H.

(ii) follows from $|\mathbf{x}^*|\le\frac{3}{2}|\mathbf{x}_0^*|$ and $|x_n|\le\frac{1}{240n}|\mathbf{x}^*|$ for all $\mathbf{x}\in\Omega_{\mathbf{x}_0^*,M,H}\cap P(n)$:

$$|x_n|\le\frac{1}{240n}|\mathbf{x}^*|\le\frac{3}{480n}|\mathbf{x}_0^*|<\frac{1}{120n}|\mathbf{x}_0^*|=M\quad for\quad \mathbf{x}\in\Omega_{\mathbf{x}_0^*,M,H}\cap P(n).$$

(iii) is obvious since $|\mathbf{x}^*-\mathbf{x}_0^*|=h_a^{-1}(x_n+M)$ on $\partial\Omega_{\mathbf{x}_0^*,M,H}\cap P(n)$.

□

Lemma 2.3. *Let* ϕ *be a continuous function defined on* $\partial\Omega$. *For any* $\mathbf{x}_0^*\in R^{n-1}$ *such that* $|\mathbf{x}_0^*|\ge(120bn(\frac{3}{2})^\alpha)^{\frac{1}{1-\alpha}}$ *and* $(\mathbf{x}_0^*,x_n)\in\Omega$ *for some* x_n, *in the function* $z=z_{\mathbf{x}_0^*}$ *defined in (2.2), we set* ($\gamma(\mathbf{x}_0^*)$ *is defined in (2.24))*

$$\gamma=\gamma(\mathbf{x}_0^*),\quad H=2,\quad M=(\frac{3}{2})^\alpha b|\mathbf{x}_0^*|^\alpha. \tag{2.28}$$

Then $z=z_{\mathbf{x}_0^*}$ *satisfies*

$$Qz\le -n\Lambda_1(\mathbf{x})\frac{(1+|Dz|^2)^{3/2}}{n+(n-1)|Dz|^2}\quad in\quad \Omega_{\mathbf{x}_0,H,M}\cap P(n,\alpha,b) \tag{2.29}$$

where

$$\Lambda_1(\mathbf{x})=\frac{(n-1)}{56n(n+1)b}(\frac{2}{3})^\alpha\frac{1}{|\mathbf{x}_0^*|^\alpha}. \tag{2.30}$$

Furthermore

(i) $z(\mathbf{x}_0^*,x_n)\le\frac{1}{2}(\frac{3}{2})^\alpha b|\mathbf{x}_0^*|^\alpha+\gamma(\mathbf{x}_0^*)$ *for* $|x_n|<M$;

(ii) $\partial\Omega_{\mathbf{x}_0^*,M,H}\cap P(n,\alpha,b)\subset\{\mathbf{x}:|x_n|<M\}$;

(iii) *For the unit outer normal* $\mathbf{n}$ *on* $\partial\Omega_{\mathbf{x}_0^*,M,H}\cap P(n,\alpha,b)$,

$$\frac{\partial z}{\partial \mathbf{n}} = +\infty \quad on\ \partial\Omega_{\mathbf{x}_0^*,M,H}\cap P(n,\alpha,b).$$

Proof. From the choices of H and M and the definition of $\Omega_{\mathbf{x}_0^*,H,M}$, we have that for any $\mathbf{x} \in \Omega_{\mathbf{x}_0^*,H,M}$, by (2.15),

$$|\mathbf{x}^* - \mathbf{x}_0^*| \le h_a^{-1}(x_n + M) \le A(H)e^{\chi(H)} \le 2c_2 eHM \le 60nM < 60nb(\frac{3}{2})^\alpha |\mathbf{x}_0^*|^\alpha.$$

Thus when $|\mathbf{x}_0^*| \ge (120nb(\frac{3}{2})^\alpha)^{\frac{1}{1-\alpha}}$, for any $\mathbf{x} \in \Omega_{\mathbf{x}_0^*,H,M}$, $\frac{1}{2}|\mathbf{x}_0^*| \le |\mathbf{x}^*| \le \frac{3}{2}|\mathbf{x}_0^*|$. Again from the choices of M and H, we have

$$\frac{n-1}{28(n+1)MH} = \frac{n-1}{56(n+1)M} = \frac{(n-1)}{56(n+1)}(\frac{2}{3})^\alpha \frac{1}{b|\mathbf{x}_0^*|^\alpha}.$$

Then if we set

$$\Lambda_1(\mathbf{x}) = \frac{(n-1)}{56n(n+1)}(\frac{1}{3})^\alpha \frac{1}{b|\mathbf{x}^*|^\alpha},$$

we have that on $\Omega_{\mathbf{x}_0^*,H,M}$

$$n\Lambda_1(\mathbf{x}) \le \frac{n-1}{28(n+1)MH}.$$

Now (2.29) follows from Lemma 2.1. (i) and (iii) are proved in the same way as in that of Lemma 2.2. (ii) follows from $|\mathbf{x}^*| < \frac{3}{2}|\mathbf{x}_0^*|$ and $|x_n| \le b|\mathbf{x}^*|^\alpha$ for all $\mathbf{x} \in \Omega_{\mathbf{x}_0^*,M,H} \cap P(n,\alpha,b)$:

$$|x_n| \le b|\mathbf{x}^*|^\alpha \le (\frac{3}{2})^\alpha b|\mathbf{x}_0^*|^\alpha = M for \mathbf{x} \in \Omega_{\mathbf{x}_0^*,M,H} \cap P(n,\alpha,b).$$

□

Now we are ready to prove growth estimates for solutions.

Lemma 2.4. *Let Ω be a domain inside $P(n)$ and $\Lambda(\mathbf{x})$ satisfy* (1.3)*, then any $C^2(\Omega) \cap C^0(\overline{\Omega})$ solution v to* (1.1)-(1.2) *satisfies*

$$|v| \le z_{\mathbf{x}_0^*} \quad on\ \Omega_{\mathbf{x}_0^*,M,H} \cap \Omega \tag{2.31}$$

where $z_{\mathbf{x}_0^*}$ is defined in Lemma 2.2.

Proof. Since Λ_0 defined in (1.3) is positive, (2.26) implies that $z_{\mathbf{x}_0^*}$ is also a super-solution to (1.1) on $\Omega_{\mathbf{x}_0^*,M,H} \cap \Omega$. Furthermore, the definition of $z_{\mathbf{x}_0^*}$ implies that $z_{\mathbf{x}_0^*} \ge |\phi|$ on $\Omega_{\mathbf{x}_0^*,M,H} \cap \partial\Omega$. Now $v - z_{\mathbf{x}_0^*}$ cannot achieve its maximum value in $\overline{\Omega_{\mathbf{x}_0^*,M,H} \cap \Omega}$ on $\partial\Omega_{\mathbf{x}_0^*,M,H} \cap \Omega$ since the directional derivative of $z_{\mathbf{x}_0^*}$ with respect to outer normal is $+\infty$ on $\partial\Omega_{\mathbf{x}_0^*,M,H} \cap \Omega$ by (iii) in Lemma 2.2. A comparison argument also concludes that $v - z_{\mathbf{x}_0^*}$ cannot achieve a local maximum inside $\Omega_{\mathbf{x}_0^*,M,H} \cap \Omega$. Thus $v - z_{\mathbf{x}_0^*}$ achieves its maximum value in $\overline{\Omega_{\mathbf{x}_0^*,M,H} \cap \Omega}$ on $\Omega_{\mathbf{x}_0^*,M,H} \cap \partial\Omega$. Then $z_{\mathbf{x}_0^*} \ge |\phi|$ on $\Omega_{\mathbf{x}_0^*,M,H} \cap \partial\Omega$. implies $v - z_{\mathbf{x}_0^*} \le 0$ on $\Omega_{\mathbf{x}_0^*,M,H} \cap \Omega$. Now we apply the same argument with $z_{\mathbf{x}_0^*}$ and $-v$ ($-v$ satisfies (1.1) with Λ replaced by $-\Lambda$ and boundary data $-\phi$), we can conclude that $-v \le z_{\mathbf{x}_0^*}$. □

Lemma 2.5. *Let Ω be a domain inside $P(n,\alpha,b)$ satisfying* (1.6) *and $\Lambda(\mathbf{x})$ satisfy* (1.7)*, then any $C^2(\Omega) \cap C^0(\overline{\Omega})$ solution v to* (1.1)-(1.2) *satisfies*

$$|v| \le z_{\mathbf{x}_0^*} \quad on\ \Omega_{\mathbf{x}_0^*,M,H} \cap \Omega \tag{2.32}$$

where $z_{\mathbf{x}_0^*}$ is defined in Lemma 2.3.

The proof of the above lemma is completely parallel to that of Lemma 2.4 with Lemma 2.2 replaced by 2.3.

Proof of Theorem 1.1. From Lemma 2.4, for any $(\mathbf{x}_0^*, x_n) \in \Omega$,

$$|v(\mathbf{x})| \le z_{\mathbf{x}_0^*}(\mathbf{x}) \quad \text{on } \Omega_{\mathbf{x}_0^*,M,H} \cap \Omega.$$

In particular,

$$|v(\mathbf{x}_0^*, x_n)| \le z_{\mathbf{x}_0^*}(\mathbf{x}_0^*, x_n) \quad \text{on } \Omega_{\mathbf{x}_0^*,M,H} \cap \Omega.$$

Then Theorem 1.1 follows from (i) in Lemma 2.2, the definition of $\gamma(\mathbf{x}_0^*)$ and the fact that $(\mathbf{x}_0^*, x_n)$ can be an arbitrary point in Ω. □

Proof of Theorem 1.5. The proof is completely parallel to that of Theorem 1.1 with Lemma 2.4 replaced by Lemma 2.5. □

3. Proofs of Theorems 1.3 and 1.6

We will give only the proof for Theorem 1.3. The proof of Theorem 1.6 is completely parallel to that of Theorem 1.3 with Lemma 2.2 replaced by Lemma 2.3 and Lemma 2.4 replaced by Lemma 2.5.

Once we have proved the lemmas in the previous section, the proof of Theorem 1.3 is very similar to the [9, Theorems 1.2 and 1.3] that in turn is similar that of [11, Theorem 1]. For reader's convenience, we still carry out the details here.

We define Π, a family of open subsets of Ω, as follows: If $\mathbf{x}_1 \in \Omega$, we choose a small ball O centered $\mathbf{x}_1$ such that $O \subset \Omega$ and the mean curvature function H' on ∂O with respect to inner normal satisfies

$$H' \ge \frac{n-1}{n}|\Lambda(\mathbf{x})|. \tag{3.1}$$

If $\mathbf{x}_1 \in \partial\Omega$, we choose a domain O such that $O \subset \Omega$, O has $C^{2,\mu}$ boundary and $\partial O \cap \partial\Omega$ is a neighborhood of $\mathbf{x}_1$ in $\partial\Omega$. Furthermore, the mean curvature function H' on ∂O with respect to inner normal satisfies (3.1). The existence of such domains can be proved under the assumption that Ω satisfies Serrin's condition (1.5), for details of a proof, one may see [9, Lemma A.3].

Let $v > 0$ be a continuous function on $\overline{\Omega}$, for each open set $O \in \Pi$, we define a new function $M_O(v)$, called the lifting of v over O as follows:

$$M_O(v)(\mathbf{x}) = v(\mathbf{x}) \quad \text{if } \mathbf{x} \in \Omega \setminus O, \quad M_O(v)(\mathbf{x}, y) = w(\mathbf{x}) \quad \text{if } \mathbf{x} \in O$$

where $w(\mathbf{x})$ is the solution of the boundary-value problem

$$((1+|Dw|^2)\delta_{ij} - D_iwD_jw)D_{ij}w = n\Lambda(\mathbf{x})(1+|Dw|^2)^{3/2} \quad \text{in } O, \tag{3.2}$$

$$w = v \quad \text{on } \partial O\,. \tag{3.3}$$

Remark 3.1. By (3.1), Lemma 2.4 and [13] or [6, Theorem 16.9], there is a unique solution $w \in C^2(O) \cap C^0(\overline{\Omega})$ to (3.2)-(3.3). Thus $M_O(v)$ is well defined.

We define a class Ξ of functions v, called subfunctions, such that:

(1) $v \in C^0(\overline{\Omega})$ and $v \le \phi$ on $\partial\Omega$;
(2) For any $O \in \Pi$, $v \le M_O(v)$;
(3) $v \le z_{\mathbf{x}_0^*}$ on $\Omega \cap \Omega_{\mathbf{x}_0^*,M,H}$ for any $\mathbf{x}_0^* \in R^{n-1}$ such that $(\mathbf{x}_0^*, x_n) \in \Omega$ for some x_n, where $z_{\mathbf{x}_0^*}$ are those functions defined in Lemma 2.2.

Now we prove some properties for subfunctions in the class Ξ.

Lemma 3.2. *If $v_1 \le v_2$, then $M_O(v_1) \le M_O(v_2)$ for any $O \in \Pi$.*

Proof. Let w_1, w_2 be the solutions of the following two problems, respectively:

$$((1+|Dw_k|^2)\delta_{ij} - D_i w_k D_j w_k) D_{ij} w_k = n\Lambda(\mathbf{x})(1+|Dw_k|^2)^{3/2} \text{in} O,$$
$$w_k = v_k \quad \text{on } \partial O, \quad k = 1, 2.$$

Since $w_1 = v_1 \le v_2 = w_2$ on ∂O, by a comparison principle for quasilinear elliptic equations (e.g. see [6, Theorem 10.1]), we have $w_1 \le w_2$ on O. On $\Omega\setminus O$, $M_O(v_1) = v_1$, $M_O(v_2) = v_2$. Thus $M_O(v_1) \le M_O(v_2)$. $\square$

Lemma 3.3. *If* $v_1 \in \Xi$, $v_2 \in \Xi$, *then* $\max\{v_1, v_2\} \in \Xi$.

Proof. If $v_1 \in \Xi$, $v_2 \in \Xi$, then $\max\{v_1, v_2\} \in C^0(\overline{\Omega})$, and $\max\{v_1, v_2\} \le \phi$ on $\partial\Omega$. It is also clear that $\max\{v_1, v_2\} \le z_{\mathbf{x}_0^*}$ on $\Omega_{\mathbf{x}_0^*,M,H} \cap \Omega$.

Since $v_1 \le \max\{v_1, v_2\}$, $v_2 \le \max\{v_1, v_2\}$, we have (by Lemma 3.2) that for any $O \in \Pi$,

$$M_O(v_1) \le M_O(\max\{v_1, v_2\}), \quad M_O(v_2) \le M_O(\max\{v_1, v_2\}).$$

Since $v_1 \in \Xi$ and $v_2 \in \Xi$ imply $v_1 \le M_O(v_1)$, $v_2 \le M_O(v_2)$, we have $\max\{v_1, v_2\} \le M_O(\max\{v_1, v_2\})$. Thus $\max\{v_1, v_2\} \in \Xi$. $\square$

Lemma 3.4. *If* $v \in \Xi$, *then* $M_O(v) \in \Xi$ *for any* $O \in \Pi$.

Proof. By the definition of $M_O(v)$, it is clear that $M_O(v) \in C^0(\overline{\Omega})$ and $M_O(v) \le \phi$ on $\partial\Omega$. First we show that for any $O_1 \in \Pi$,

$$M_O(v)(\mathbf{x}) \le M_{O_1}(M_O(v))(\mathbf{x}). \tag{3.4}$$

We need to prove only that (3.4) is true for $\mathbf{x} \in O_1$. Since $v \le M_O(v)$ on Ω, we have (by Lemma 3.2) $M_{O_1}(v) \le M_{O_1}(M_O(v))$. Combining this with $v \le M_{O_1}(v)$, we have $v \le M_{O_1}(M_O(v))$. Thus for $\mathbf{x} \in O_1 \setminus O$,

$$M_O(v)(\mathbf{x}) = v(\mathbf{x}) \le M_{O_1}(M_O(v))(\mathbf{x}). \tag{3.5}$$

That is, (3.4) is true on $O_1 \setminus O$, Now for $\Omega_1 = O_1 \cap O$, if we set

$$M_O(v) = w_1, \quad M_{O_1}(M_O(v)) = w_2,$$

we have that on Ω_1, $k = 1, 2$,

$$((1+|Dw_k|^2)\delta_{ij} - D_i w_k D_j w_k) D_{ij} w_k = n\Lambda(\mathbf{x})(1+|Dw_k|^2)^{3/2}.$$

On $\partial\Omega_1$, $w_1 \le w_2$ on $O_1 \cap \partial O$ by (3.5) and $w_1 \le w_2$ on $\partial O_1 \cap O$ since (3.4) is true on $\Omega \setminus O_1$. Then a comparison argument implies $w_1 \le w_2$ on Ω_1. Thus (3.4) is true on $O_1 \cap O$ and on O_1. $\square$

Now we prove that $M_O(v) \le z_{\mathbf{x}_0^*}$ on $\Omega_{\mathbf{x}_0^*,M,H} \cap \Omega$. Since $v \in \Xi$, $v \le z_{\mathbf{x}_0^*}$ on $\Omega_{\mathbf{x}_0^*,M,H} \cap \Omega$. Thus by the definition of $M_O(v)$, we only need to show $M_O(v) \le z_{\mathbf{x}_0^*}$ on $\Omega_{\mathbf{x}_0^*,M,H} \cap O$. If O does not intersect with $\Omega_{\mathbf{x}_0^*,M,H}$, the conclusion is trivial. In the case that O is at least partly covered by $\Omega_{\mathbf{x}_0^*,M,H}$. $M_O(v) - z_{\mathbf{x}_0^*}$ cannot achieve its maximum value in $\overline{\Omega_{\mathbf{x}_0^*,M,H} \cap O}$ on $\partial\Omega_{\mathbf{x}_0^*,M,H} \cap O$ since the directional derivative of $z_{\mathbf{x}_0^*}$ with respect to outer normal is $+\infty$ on $\partial\Omega_{\mathbf{x}_0^*,M,H} \cap O$ by (iii) in Lemma 2.2. Furthermore since $z_{\mathbf{x}_0^*}$ satisfies (2.26) and $|\Lambda(\mathbf{x})| \le \Lambda_0(\mathbf{x})$ by (1.3) and (2.27), a comparison argument concludes that $M_O(v) - z_{\mathbf{x}_0^*}$ cannot achieve a local maximum inside $\Omega_{\mathbf{x}_0^*,M,H} \cap O$. Thus $M_O(v) - z_{\mathbf{x}_0^*}$ achieves its maximum value in $\overline{\Omega_{\mathbf{x}_0^*,M,H} \cap O}$ on $\Omega_{\mathbf{x}_0^*,M,H} \cap \partial O$. Then $M_O(v) - z_{\mathbf{x}_0^*} \le 0$ on $\Omega_{\mathbf{x}_0^*,M,H} \cap O$ follows from $M_O(v) - z_{\mathbf{x}_0^*} = v - z_{\mathbf{x}_0^*} \le 0$ on $\Omega_{\mathbf{x}_0^*,M,H} \cap \partial O$.

Now we will show that Ξ is not empty by proving the existence of a solution to the minimal surface equation with the same boundary-value and on the same domain.

Lemma 3.5. *If $v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a solution of the problem*

$$((1+|Dv|^2)\delta_{ij} - D_i v D_j v) D_{ij} v = 0 \text{in} \Omega, v = \phi on \partial\Omega. \tag{3.6}$$

Then for any $(\mathbf{x}_0^*, x_n) \in \Omega$,

$$|v| \le z_{\mathbf{x}_0^*} \quad \text{on } \Omega_{\mathbf{x}_0^*,M,H} \cap \Omega \tag{3.7}$$

The proof of the above lemma is just a special case of Lemma 2.4 with $\Lambda(\mathbf{x}) = 0$.

Lemma 3.6. *Assume* (A1)–(A3). *Then the boundary-value problem*

$$((1+|Dv|^2)\delta_{ij} - D_i v D_j v) D_{ij} v = 0 \quad \text{in } \Omega, \quad v = \phi \quad \text{on } \partial\Omega. \tag{3.8}$$

has a solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$.

Proof. This is [9, Lemma 4.5] (Though a slight difference should be noted there. That is, the bound for solutions of the minimal surface equation is given by (3.7) which will play the same role as the Lemma 4.4 in [9] (By the way, the Lemma 3.1 quoted in the proof of Lemma 4.5 in [9] should be Lemma 4.4 in [9]). □

Now we prove the Theorem 1.3. We set

$$u(\mathbf{x}) = \sup\{v(\mathbf{x}) : v \in \Xi\}, \mathbf{x} \in \overline{\Omega}.$$

We first consider the case that $\Lambda(\mathbf{x}) \le 0$ on Ω. For such a choice of $\Lambda(\mathbf{x})$, we will show that u is in $C^0(\overline{\Omega}) \cap C^2(\Omega)$ satisfying (1.1)-(1.2). It is well known and standard (for example, see [3]) that by Perron's method, we can prove that u is in $C^2(\Omega)$ and satisfies (1.1). Indeed, let $\mathbf{x}_1 \in \Omega$. By the definition of $u(\mathbf{x}_1)$, there is a sequence of functions v_i in Ξ such that

$$u(\mathbf{x}_1) = \lim_{i\to\infty} v_i(\mathbf{x}_1).$$

Let v_0 be a solution of (3.8). Since $\Lambda(\mathbf{x}) \le 0$ on Ω, by Lemma 3.5, it is easy to check that $v_0 \in \Xi$. By Lemma 3.3 and replacing v_i by $\max\{v_i, v_0\}$, we may assume that $v_i \ge v_0$ on Ω. Let O be an open set in Π such that $\mathbf{x}_1 \in O$. We replace v_i by $M_O(v_i)$. Then we have a sequence of functions z_i defined on O satisfying

$$u(\mathbf{x}_1) = \lim_{i\to\infty} z_i(\mathbf{x}_1),$$
$$((1+|Dz_i|^2)\delta_{pq} - D_p z_i D_q z_i) D_{pq} z_i = n\Lambda(\mathbf{x})(1+|Dz_i|^2)^{3/2} \quad \text{on } O,$$
$$z_i = v_i \quad \text{on } \partial O.$$

Since for all i, if $O \cap \Omega_{\mathbf{x}_0^*,M,H}$ is not empty,

$$v_0 \le v_i \le z_i \le z_{\mathbf{x}_0^*} \quad \text{on } O \cap \Omega_{\mathbf{x}_0^*,M,H},$$

and we can cover O by finitely many such domains $\Omega_{\mathbf{x}_0^*,M,H}$, thus there is a number K_3 independent of i, such that for all i,

$$v_0 \le z_i \le K_3 \quad \text{in } O.$$

By [6, Corollarys 16.6, 16.7], there is a subsequence of z_i, for convenience still denoted by z_i, converges to a $C^2(O)$ function $z(x)$ in $C^2(O)$. Thus $z(x)$ satisfies

$$((1+|Dz|^2)\delta_{pq} - D_p z D_q z) D_{pq} z = n\Lambda(\mathbf{x})(1+|Dz|^2)^{3/2} \quad \text{on } O.$$

Note that $u(\mathbf{x}_1) = z(\mathbf{x}_1)$ and $u(\mathbf{x}) \geq z(\mathbf{x})$ on O. We claim that $u = z$ on O. Indeed, if there is another point $\mathbf{x}_2 \in O$ such that $u(\mathbf{x}_2)$ is not equal to $z(\mathbf{x}_2)$, we must have $u(\mathbf{x}_2) > z(\mathbf{x}_2)$. Then there is a function $u_0 \in \Xi$, such that

$$z(\mathbf{x}_2) < u_0(\mathbf{x}_2) \leq u(\mathbf{x}_2).$$

Now the sequence $\max\{u_0, M_O(v_i)\}$ satisfying

$$v_i \leq \max\{u_0, M_O(v_i)\} \leq u.$$

Then similar to the way we obtain z, $M_O(\max\{u_0, M_O(v_i)\})$ will produce a C^2 function z_1 satisfying

$$\begin{gathered}((1+|Dz_1|^2)\delta_{pq} - D_p z_1 D_q z_1) D_{pq} z_1 = n\Lambda(\mathbf{x})(1+|Dz_1|^2)^{3/2} \quad \text{on } O,\\ z \leq z_1 \quad \text{on } O, \quad z(\mathbf{x}_2) < u_0(\mathbf{x}_2) \leq z_1(\mathbf{x}_2),\\ z(\mathbf{x}_1) = z_1(\mathbf{x}_1) = u(\mathbf{x}_1).\end{gathered}$$

That is, $z_1(\mathbf{x}) - z(\mathbf{x})$ is non-negative, not identically zero on O and achieves its minimum value zero inside O. However, from the equations satisfied by z and z_1, we have that on O,

$$\begin{aligned}&((1+|Dz_1|^2)\delta_{pq} - D_p z_1 D_q z_1) D_{pq}(z_1 - z)\\ &= E(\mathbf{x}, z_1, z, Dz, Dz_1, D^2 z, D^2 z_1)(Dz_1 - Dz) \quad \text{on } O\end{aligned}$$

for some continuous function E. Then by the standard maximum principle (for example, see [6, Theorem 3.5]), we get a contradiction. Thus $u = z$ on O. Therefore $u \in C^2(\Omega)$ and

$$((1+|Du|^2)\delta_{ij} - D_i u D_j u) D_{ij} u = n\Lambda(\mathbf{x})(1+|Du|^2)^{3/2} \quad \text{on } \Omega.$$

Since $v_0 = \phi$ on $\partial\Omega$, from the definition of u we see that $u = \phi$ on $\partial\Omega$. We still need to prove that $u \in C^0(\overline{\Omega})$.

Since Ω satisfies Serrin's condition (1.5), we can find a $C^{2,\gamma}$ domain $\Omega_1 \subset \Omega$, $\Omega_1 \in \Pi$, (for the existence of Ω_1, see Lemma A.3 in Appendix in [9]), such that $\partial\Omega_1 \cap \partial\Omega$ is an open neighborhood of $\mathbf{x}_1$ in $\partial\Omega$ and on $\partial\Omega_1$, the mean curvature H' with respect to inner normal of $\partial\Omega_1$ satisfies

$$H' > \frac{n}{n-1}|\Lambda(\mathbf{x})| \quad \text{on } \partial\Omega_1. \tag{3.9}$$

Since Ω_1 can be covered by finitely many $\Omega_{\mathbf{x}_0^*,M,H}$, there is a number $K_4 > 0$, such that for all $v \in \Xi$,

$$v \leq K_4 \quad \text{on } \overline{\Omega}_1. \tag{3.10}$$

Now on $\partial\Omega_1$, we choose a smooth function ϕ^* as follows. $\phi^* = K_4$ on $\partial\Omega_1 \cap \Omega$. $\phi^* = \phi$ in a neighborhood of $\mathbf{x}_1$ in $\partial\Omega_1$ and $\phi^* \geq \phi$ on the rest of $\partial\Omega_1$ (since (3.10) implies $\phi \leq K_4$ on $\partial\Omega_1 \cap \partial\Omega$, this is possible). Now we consider the boundary-value problem

$$((1+|Du|^2)\delta_{ij} - D_i u D_j u) D_{ij} u = n\Lambda(\mathbf{x})(1+|Du|^2)^{3/2} \quad \text{on } \Omega_1, \tag{3.11}$$

$$u = \phi^* \quad \text{on } \partial\Omega_1. \tag{3.12}$$

From (3.9), Lemma 2.4 and [13] or [6, Theorem 16.9], (3.11)-(3.12) has a solution $u_1 \in C^2(\Omega_1) \cap C^0(\overline{\Omega}_1)$. From the definition of u_1, (3.10) and the fact that $v = \phi$ on $\partial\Omega$ for any $v \in \Xi$, a comparison argument shows that for any $v \in \Xi$,

$$M_{\Omega_1}(v) \leq u_1 \quad \text{on } \Omega_1 \quad \text{for any} \quad v \in \Xi.$$

Therefore,

$$u \le u_1 \quad \text{on } \Omega_1. \tag{3.13}$$

Since we always have

$$u \ge v_0 \quad \text{on } \Omega$$

for the solution v_0 of (3.8), we have

$$v_0 \le u \le u_1 \quad \text{on } \Omega_1. \tag{3.14}$$

Then the continuity of u at $\mathbf{x}_1$ follows from the fact that $v_0 = u_1 = \phi$ on a neighborhood of $\mathbf{x}_1$ in $\partial\Omega$ and both v_0 and u_1 are continuous in a neighborhood of $\mathbf{x}_1$ in $\overline{\Omega}$. Since $\mathbf{x}_1 \in \partial\Omega$ can be arbitrary, we have $u \in C^0(\overline{\Omega})$. Thus under the additional assumption that $\Lambda(\mathbf{x}) \le 0$ on Ω, we have proved Theorem 1.3.

In the case that $\Lambda(\mathbf{x}) \ge 0$ on Ω, repeating above proof, we can find a function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfying

$$((1+|Du|^2)\delta_{ij} - D_iuD_ju)D_{ij}u = -n\Lambda(\mathbf{x})(1+|Du|^2)^{3/2} \quad \text{on } \Omega,$$
$$u = -\phi \quad \text{on } \partial\Omega.$$

Then $-u$ will satisfy (1.1)-(1.2).

In the general case of $\Lambda(\mathbf{x})$, we first find a function $u_0 \in C^1(\Omega)\cap C^0(\overline{\Omega})$ satisfying

$$((1+|Du|^2)\delta_{ij} - D_iuD_ju)D_{ij}u = n|\Lambda(\mathbf{x})|(1+|Du|^2)^{3/2} \quad \text{on } \Omega,$$
$$u = \phi \quad \text{on } \partial\Omega.$$

In the proof for the case that $\Lambda \le 0$, we replace v_0 (the solution of (3.8)) by u_0, without changing the rest of the proof, now we will obtain a function $u \in C^1(\Omega) \cap C^0(\overline{\Omega})$ satisfies (1.1)-(1.2). This completes the proof for Theorems 1.3.

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