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EXISTENCE OF SOLUTIONS FOR SOME THIRD-ORDER BOUNDARY-VALUE PROBLEMS

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Abstract. In this paper concerns the third-order boundary-value problem

$$u'''(t) + f(t, u(t), u'(t), u''(t)) = 0, \quad 0 < t < 1,$$

$$r_1 u(0) - r_2 u'(0) = r_3 u(1) + r_4 u'(1) = u''(0) = 0.$$

By placing certain restrictions on the nonlinear term f, we prove the existence of at least one solution to the boundary-value problem with the use of lower and upper solution method and of Schauder fixed-point theorem. The construction of lower or upper solutions is also presented.

1. Introduction

Recently, third-order boundary-value problems have been considered in many papers. Some problems of regulation and control of some actions by a control level or by a signal reduce to solving the third-order equations. Other applications of third-order differential equations are encountered in the control of a flying apparatus in cosmic space, the deflection of sandwich beam, and the study of draining and coating flows. For details, see the references in this article and the references therein.

As it is pointed out by Anderson *et al.* [1], a large part of the literature on solution to higher-order boundary-value problems seems to be traced to Krasnosel'skii's work on nonlinear operator equations as well as other fixed-point theorem such as Leggett-Williams' fixed-point theorem.

The method of upper and lower solution is extensively developed for lower order equations with linear and nonlinear boundary conditions. But there are only a few applications to higher-order ordinary differential equations. For applications to higher-order ODEs, we refer the reader to Ehme [6], Klaasen [7] and the references therein. Specially, in Cabada [3] and Yao [11], the lower and upper solution method is employed to acquire existence results about some third-order boundary-value problems with some monotonic or quasi-monotonic nonlinear term f which is no dependence on any lower-order derivatives. On the other hand, to my best knowledge, there are few papers referred to lower and upper solutions of third-order

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equation consider the relationship between the property of nonlinear term and the construction of lower and upper solutions.

The purpose of this paper is to study the existence of solution for two class nonlinear third-order boundary-value problems

$$u'''(t) + f(t, u(t), u''(t)) = 0, \quad 0 \le t \le 1, \tag{1.1}$$

$$r_1 u(0) - r_2 u'(0) = r_3 u(1) + r_4 u'(1) = u''(0) = 0, \tag{1.2}$$

and

$$u'''(t) + f(t, u(t), u'(t), u''(t)) = 0, \quad 0 \le t \le 1, \tag{1.3}$$

$$u(0) = u'(1) = u''(0) = 0. \tag{1.4}$$

The method used here is not based on the Krasnosel'skii's fixed-point theorem or monotonic operator theory; rather, it is based on Schauder fixed-point theorem, the appropriate integral transvestites and lower and upper solution method. The construction of lower or upper solution is also presented.

2. Preliminaries

In this section, we consider (1.1)–(1.2), under the assumption that $f : [0,1] \times R^2 \to R$ is continuous, $r_1, r_2, r_3, r_4 \ge 0$ and $\rho := r_2 r_3 + r_1 r_3 + r_1 r_4 > 0$. We give some lemmas which indicate some restrictions on the nonlinear term and let us construct lower or upper solutions.

Definition 2.1. We call α, $\beta \in C^2[0,1] \bigcap C^3(0,1)$ lower and upper solutions of Problem (1.1)–(1.2), respectively, if

$$\alpha'''(t) + f(t, \alpha(t), \alpha''(t)) \ge 0, \quad 0 < t < 1,$$
$$r_1\alpha(0) - r_2\alpha'(0) = r_3\alpha(1) + r_4\alpha'(1) = 0, \quad \alpha''(0) \ge 0;$$

$$\beta'''(t) + f(t, \beta(t), \beta''(t)) \le 0, \quad 0 < t < 1,$$
$$r_1\beta(0) - r_2\beta'(0) = r_3\beta(1) + r_4\beta'(1) = 0, \quad \beta''(0) \le 0.$$

Denote by $G(t,s)$ the Green's function of

$$\begin{gathered} -u''(t) = 0, \quad 0 < t < 1, \\ r_1 u(0) - r_2 u'(0) = r_3 u(1) + r_4 u'(1) = 0, \end{gathered} \tag{2.1}$$

then

$$G(t,s) = \begin{cases} \frac{1}{\rho} x(t) y(s), & 0 \le s < t \le 1, \\ \frac{1}{\rho} x(s) y(t), & 0 \le t < s \le 1, \end{cases} \tag{2.2}$$

where $x(t) := r_3 + r_4 - r_3 t$, $y(t) := r_2 + r_1 t$, for $t \in [0,1]$, $\rho = r_2 r_3 + r_1 r_3 + r_1 r_4$. Clearly, $G(t,s) \ge 0$ for $0 \le t, s \le 1$.

The following Lemma comes from Lian [8] with small modification.

Lemma 2.2 ([8]). *For $G(t,s)$ defined by* (2.2)*, the following holds:*

(R1) $\frac{G(t,s)}{G(s,s)} \le 1$ *for* $t, s \in (0,1)$,

(R2) $\frac{G(t,s)}{G(s,s)} \ge C =: \min\{\frac{r_4}{r_3+r_4}, \frac{r_2}{r_1+r_2}\} \ge 0$, *for* $t, s \in (0,1)$.

Let $\eta = \int_0^1 G(s,s) s \, ds > 0$. Then we have the following results.

Lemma 2.3. *If there exists a constant $M \ge 0$ such that*

$$f(t,s,r) \le M, \quad \text{for } 0 \le t \le 1,\ \eta MC \le s \le \eta M,\ -M \le r \le 0,$$

then Problem (1.1)-(1.2) *has an upper solution.*

Proof. Setting $v(t) = -u''(t)$, Problem (1.1)–(1.2) is equivalent to

$$v'(t) = f(t,(Av)(t),-v(t)), \quad 0 < t < 1, \tag{2.3}$$

$$v(0) = 0, \tag{2.4}$$

where $(Av)(t) = \int_0^1 G(t,s)v(s)ds$ and $G(t,s)$ is defined by (2.2). It is clear that the restriction on f guarantee that $\psi(t) = Mt$ satisfies

$$\psi'(t) - f(t,(A\psi)(t),-\psi(t)) \ge 0, \quad 0 < t < 1,$$
$$\psi(0) \ge 0.$$

This shows that $\beta(t) = (A\psi)(t)$ is an upper solution of Problem (1.1)–(1.2). □

Lemma 2.4. *If there exists a constant $N \le 0$ such that*

$$f(t,s,r) \ge N, \quad \text{for } 0 \le t \le 1,\ \eta N \le s \le \eta NC,\ 0 \le r \le -N,$$

then Problem (1.1)–(1.2) *has a lower solution.*

Proof. Setting $\varphi(t) = Nt, \alpha(t) = (A\varphi)(t)$, we can complete the proof as in Lemma 2.3. □

Remark 2.5. *In fact, we can write the upper solution $\beta(t)$ and lower solution $\alpha(t)$ explicitly. Particularly, if $r_1, r_3 \ne 0$ and $r_2 = r_4 = 0$, then $\beta(t) = -\frac{M}{6}t^3 + \frac{M}{6}t$, $\alpha(t) = -\frac{N}{6}t^3 + \frac{N}{6}t$ are upper and lower solutions of Problem* (1.1), (1.2).

3. Main results

In this section, we give some existence results for Problems (1.1)–(1.2) and (1.3)–(1.4).

Theorem 3.1. *Suppose there are two constants $M \ge 0 \ge N$, $M \ge |N|$ such that*

$$f(t,s,r) \le M, \quad \text{for } 0 \le t \le 1,\ \eta MC \le s \le \eta M,\ -M \le r \le 0, \tag{3.1}$$

$$f(t,s,r) \ge N, \quad \text{for } 0 \le t \le 1,\ \eta N \le s \le \eta NC,\ 0 \le r \le -N. \tag{3.2}$$

If $f(t,s,r)$ is increasing in s, then Problem (1.1)–(1.2) *has a solution $u(t)$ such that $\alpha(t) \le u(t) \le \beta(t)$, $\beta''(t) \le u''(t) \le \alpha''(t)$, $t \in [0,1]$, where*

$$\beta(t) = M\int_0^1 G(t,s)s\,ds, \quad \alpha(t) = N\int_0^1 G(t,s)s\,ds\,.$$

Proof. From Lemmas 2.3 and 2.4 combined, conditions (3.1) and (3.2) yield that (1.1)–(1.2) has a lower solution $\alpha(t)$ and an upper solution $\beta(t)$. Set $\varphi = -\alpha''$, $\psi = -\beta''$, then $\varphi(t) = Nt \le Mt = \psi(t)$. On the other hand, $M \ge 0 \ge N$ implies $\alpha''(t) \ge \beta''(t)$, for $0 < t < 1$. Taking into account boundary condition (1.2) and the fact that the Green's function $G(t,s) \ge 0$ yield $\alpha(t) \le \beta(t)$, for $0 \le t \le 1$.

Now, consider the truncated problem

$$Lv = Fv, \quad v \in \mathrm{dom}(L), \tag{3.3}$$

where $L : \operatorname{dom} L = \{v \in C^1(0,1) \bigcap C[0,1] : v(0) = 0\} \to C[0,1]$ is a derivative operator such that $(Lv)(t) = v'(t), t \in (0,1)$, and $F : C[0,1] \to C[0,1]$ is a continuous operator defined as

$$(Fv)(t) = f\big(t, A[v(t)]_{\varphi(t)}^{\psi(t)}, -[v(t)]_{\varphi(t)}^{\psi(t)}\big), \tag{3.4}$$

where $[v]_{\varphi}^{\psi} = \min\{\psi, \max\{v, \varphi\}\}$.

Firstly, we prove that if v^* is a solution of (3.3), then $\varphi(t) \le v^*(t) \le \psi(t)$, $t \in [0,1]$. Consequently, $v^*(t)$ is a solution of (2.3)–(2.4). Furthermore, $u^*(t) = (Av^*)(t)$ is a solution of (1.1)–(1.2) satisfying $\alpha(t) \le u^*(t) \le \beta(t)$.

In fact, if $\varphi(t) \not\le v^*(t)$, there exists $\bar{t} \in (0,1)$ such that $v^*(\bar{t}) < \varphi(\bar{t})$. As $v^*(0) = 0 = -\alpha''(0) = \varphi(0)$, by the continuity of v^* and φ, there exist $t_1 \in [0,\bar{t}), t_2 \in (\bar{t},1]$ such that $v^*(t_1) = \varphi(t_1)$ and $v^*(t) < \varphi(t)$, for $t \in (t_1,t_2)$. Therefore, for $t \in [t_1,t_2]$,

$$\begin{aligned}(Fv^*)(t) &= f\big(t, A[v^*(t)]_{\varphi(t)}^{\psi(t)}, -[v^*(t)]_{\varphi(t)}^{\psi(t)}\big)\\ &= f\big(t, A[v^*(t)]_{\varphi(t)}^{\psi(t)}, -\varphi(t)\big).\end{aligned}$$

Let $p(t) = v^*(t) - \varphi(t)$, $t \in [t_1,t_2]$, taking into account the monotonicity of f and A, one has

$$\begin{gathered}\varphi'(t) = -\alpha'''(t) \le f(t,\alpha(t),\alpha''(t)) = f(t, A\varphi(t), -\varphi(t)), \quad t \in [0,1],\\ {v^*}'(t) = (Fv^*)(t) = f\big(t, A[v^*(t)]_{\varphi(t)}^{\psi(t)}, -\varphi(t)\big), \quad t \in [t_1,t_2],\end{gathered}$$

so $p'(t) \ge 0$, $t \in [t_1,t_2]$. Thus, $p(t_1) = 0$ implies $p(t) \ge 0$, $t \in [t_1,t_2]$. Namely, $v^*(t) \ge \varphi(t)$, $t \in [t_1,t_2]$. It is a contradiction. Thus, $\varphi(t) \le v^*(t)$ for $t \in [0,1]$. Analogously, we can prove $v^*(t) \le \psi(t)$, $t \in [0,1]$.

Secondly, we prove operator equation (3.3) has a solution. Let $T : C[0,1] \to C[0,1]$ by

$$(Tv)(t) = \int_0^t (Fv)(s)ds, \quad t \in [0,1].$$

It is clear that T is a continuous operator and the fixed point of T is a solution of Problem (3.3). Set $B_M = \{\omega \in C[0,1] : \|\omega\| \le M\}$, taking into account that $|N| = \|\varphi\| \le \|\psi\| = M$, so we deduce that $\varphi, \psi \in B_M$. Now from condition (3.1), one has

$$\begin{gathered}\|Tv\| \le M, \quad v \in B_M,\\ |(Tv)(t) - (Tv)(s)| = \Big|\int_s^t (Fv)(r)dr\Big| \le M|t-s|.\end{gathered}$$

That is, $\{T(B_M)\}$ is equi-continuous and bounded uniformly. From Arzela-Ascoli theorem, we assert that $T : B_M \to B_M$ is a completely continuous operator, furthermore, Schauder fixed-point theorem guarantee that T has a fixed point $v^* \in B_M$. Therefore,

$$u^*(t) = \int_0^1 G(t,s)v^*(s)ds$$

is a solution of Problem (1.1)–(1.2) in $C^2[0,1] \bigcap C^3(0,1)$ such that $\alpha(t) \le u^*(t) \le \beta(t)$, $\beta''(t) \le u^*(t) \le \alpha''(t)$, $\quad t \in [0,1]$. $\square$

From the above proof, we should have noticed that the existence of M is very important. It make us not only acquire an upper solution of Problem (1.1)–(1.2), but guarantee that $T : B_M \to B_M$ is a completely continuous operator. On the other hand, the truncated problem used here is different from other earlier literature. As the second order derivatives of upper and lower solutions are employed to make truncating, we take the operator F instead of the place of the traditional truncated function.

Remark 3.2. If the nonlinear term f satisfies

$$\bar{f}_\infty = \limsup_{u\to+\infty} \max_{0\le t\le 1} \frac{f(t,\eta u,-u)}{u} \le 1,$$

then there exists $M > 0$ such that $\beta(t) = \int_0^1 G(t,s)Ms\,ds$ is an upper solution of (1.1)–(1.2). If the nonlinear term f satisfies

$$\underline{f}_\infty = \liminf_{u\to-\infty} \min_{0\le t\le 1} \frac{f(t,\eta u,-u)}{u} \le 1,$$

then there exists $N < 0$ such that $\alpha(t) = \int_0^1 G(t,s)Ns\,ds$ is a lower solution of (1.1)–(1.2).

Example 3.3. Consider the problem

$$u'''(t) + \frac{1}{3}[t + \ln(1 + u(t)) - u''(t)] = 0, \quad 0 < t < 1 \tag{3.5}$$

$$r_1u(0) - r_2u'(0) = r_3u(1) + r_4u'(1) = u''(0) = 0. \tag{3.6}$$

Let $f(t,s,r) = \frac{1}{3}[t + \ln(1+s) - r]$, it is easy to check that $M = 1, N = 0$ satisfy condition (3.1), (3.2), respectively, therefore,

$$\alpha(t) = 0, \quad \beta(t) = \int_0^1 G(t,s)s\,ds$$

are lower and upper solutions of (3.5)–(3.6). By Theorem 3.1, Problem (3.5)–(3.6) has a positive solution u^* such that $0 \le u^*(t) \le \int_0^1 G(t,s)s\,ds$.

We now give some sufficient conditions with which there is at least one solution to (1.3)–(1.4). In the following we assume $f : [0,1] \times R^3 \to R$ is continuous. It is not difficult to see that (1.3)–(1.4) is equivalent to

$$v'(t) = f(t,(Av)(t),(Bv)(t),-v(t)), \quad 0 < t < 1 \tag{3.7}$$

$$v(0) = 0, \tag{3.8}$$

where $(Av)(t) = \int_0^1 G(t,s)v(s)ds$, $(Bv)(t) = \int_t^1 v(s)ds$. Lemmas similar to Lemmas 2.3 and 2.4 can be obtained analogously and so are omitted. An argument similar to the one in Theorem 3.1 provides the following result about Problem (1.3)–(1.4).

Theorem 3.4. *Suppose there are two constants $M \ge 0 \ge N$, $M \ge |N|$ such that*

$$f(t,s,l,r) \le M, \quad \text{for } t \in [0,1],\ \frac{s}{\eta}, 2l \in [0,M],\ r \in [-M,0], \tag{3.9}$$

$$f(t,s,l,r) \ge N, \quad \text{for } t \in [0,1],\ s \in [\eta N,0],\ 2l, r \in [0,-N]. \tag{3.10}$$

If $f(t,s,l,r)$ is increasing in each of s and l, then (1.3)–(1.4) has a solution $u(t)$ such that $\alpha(t) \le u(t) \le \beta(t)$, $\beta''(t) \le u''(t) \le \alpha''(t)$, $t \in [0,1]$, where $\beta(t) = M\int_0^1 G(t,s)s\,ds$, $\alpha(t) = N\int_0^1 G(t,s)s\,ds$, $\eta = 1/3$.

Example 3.5. Consider the problem

$$u'''(t) + \frac{1}{4}[t + e^{u(t)} + (u'(t))^2 + u''(t)] = 0, \quad 0 < t < 1, \tag{3.11}$$

$$u(0) = u'(1) = u''(0) = 0. \tag{3.12}$$

We can check that $M = 2$, $N = 0$ satisfy (3.9), (3.10), respectively, therefore,

$$\alpha(t) = 0, \quad \beta(t) = 2\int_0^1 G(t,s)s\,ds$$

are lower and upper solutions of (3.11)–(3.12). By Theorem 3.4, Problem (3.11)–(3.12) has a positive solution u^* such that $0 \le u^*(t) \le 2\int_0^1 G(t,s)s\,ds$.

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