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LINEAR UNCERTAIN NON-AUTONOMOUS TIME-DELAY SYSTEMS: STABILITY AND STABILIZABILITY VIA RICCATI EQUATIONS

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Abstract. This paper addresses the problem of exponential stability for a class of uncertain linear non-autonomous time-delay systems. Here, the parameter uncertainties are time-varying and unknown but norm-bounded and the delays are time-varying. Based on combination of the Riccati equation approach and the use of suitable Lyapunov-Krasovskii functional, new sufficient conditions for the robust stability are obtained in terms of the solution of Riccati-type equations. The approach allows to compute simultaneously the two bounds that characterize the exponential stability rate of the solution. As an application, sufficient conditions for the robust stabilization are derived. Numerical examples illustrated the results are given.

1. Introduction

Since time-delay systems are frequently encountered in various areas, including physical and chemical processes, biology, economics, engineering etc., the stability problem of linear time-delay systems has attracted a lot of attention in the past decades, e.g. see [3, 7, 12, 15] and the references therein. The main technique using in stability investigation relies on the use of the Lyapunov functional method [18]. The results concerning Lyapunov's direct method for time-invariant systems provide stability sufficient conditions in terms of linear matrix inequalities (LMIs) [2, 9]. More recently, simple and systematic procedure for finding exponential stability conditions using the Lyapunov-Krasovskii functionals has been proposed in [8, 11] for autonomous systems and in [4, 16] for non-autonomous systems. Among the usual approaches to studying stabilization problem of autonomous systems, the effective approach is to design linear feedback control via solving algebraic Riccati equations [1, 6]. However, for the non-autonomous systems, the solution of Riccati differential equation (RDE) is in general not bounded from above and below such that it can not served as the candidate of the Lyapunov function. Moreover, the approach used in the mentioned above papers can not be readily applied to the non-autonomous systems. Some sufficient conditions for global stabilizability for non-autonomous periodical systems via controllability are given in [14]. Among

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the results on stability analysis of uncertain linear autonomous systems, it is worth to mention the papers [13, 19], where the uncertainties must verify some matching/bounded conditions or must have a particular structure and intensive computation is needed to test the robust stability. Applications to linear time-delay control systems without uncertainties are given in [12, 16, 17, 20].

Motivated by the result on exponential stability of linear non-autonomous delay systems in [16], we develop sufficient conditions for the exponential stability of a class of uncertain linear non-autonomous time-delay systems. The parameter uncertainties are time-varying and unknown but norm-bounded and the delays are time-varying. Stability and stabilization conditions are formulated in terms of the solution of Riccati-like equations, which allow to compute the decay rate as well as the constant stability factor. These conditions generalize and improve the LMI conditions obtained earlier for autonomous delay systems.

The paper is organized as follows. Section 2 presents notations, definitions and some auxiliary propositions needed in the proof of main results. In Section 3, based on the Lyapunov-Krasovskii functional method, sufficient conditions for the exponential stability and stabilization are presented. Numerical examples illustrated the obtained results are given in Section 4. The paper ends with conclusions and cited references.

2. Preliminaries

The following notation will be used in this paper: $\mathbb{R}^+$ denotes the set of all real non-negative numbers; $\mathbb{R}^n$ denotes the n-dimensional space with the scalar product $\langle .,. \rangle$ and the vector norm $\|\cdot\|$; $M^{n\times r}$ denotes the space of all matrices of $(n\times r)$-dimensions.

A^T denotes the transpose of the vector/matrix A; A is symmetric if $A = A^T$; I denotes the identity matrix; $\lambda(A)$ denotes the set of all eigenvalues of A; $\lambda_{\max}(A) = \max\{\operatorname{Re}\lambda : \lambda \in \lambda(A)\}$.

$x_t := \{x(t+s) : s \in [-h, 0]\}$, $\|x_t\| = \sup_{s\in[-h,0]} \|x(t+s)\|$. $C([0,t],\mathbb{R}^n)$ denotes the set of all $\mathbb{R}^n$-valued continuous functions on $[0,t]$; $L_2([0,t],\mathbb{R}^m)$ denotes the set of all the $\mathbb{R}^m$-valued square integrable functions on $[0,t]$;

Matrix A is called semi-positive definite ($A \geq 0$) if $\langle Ax, x\rangle \geq 0$, for all $x \in \mathbb{R}^n$; A is positive definite ($A > 0$) if $\langle Ax, x\rangle > 0$ for all $x \neq 0$; $A > B$ means $A - B > 0$. $BM^+(0,\infty)$ denotes the set of all symmetric semi-positive definite matrix functions bounded on $[0,\infty)$;

In the sequel, sometimes for the sake of brevity, we will omit the arguments of matrix-valued functions, if it does not cause any confusion.

Consider the uncertain linear non-autonomous system with time-varying delay

$$
\begin{aligned}
\dot{x}(t) &= [A_0(t) + \Delta A_0(t)]x(t) + [A_1(t) + \Delta A_1(t)]x(t-h(t)) \\
&\quad + [B(t) + \Delta B(t)]u(t), \\
x(t) &= \phi(t), \quad t \in [-h, 0],
\end{aligned} \tag{2.1}
$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m, A_i(t)$, $i = 0,1, B(t)$ are given matrix functions continuous on $[0,\infty), 0 \leq h(t) \leq h, h > 0$. Consider the initial function $\phi(t) \in C([-h,0],\mathbb{R}^n)$ with the norm $\|\phi\| = \sup_{t\in[-h,0]} \|\phi(t)\|$, and the admissible control $u(\cdot) \in L_2([0,t],\mathbb{R}^m)$, for all $t \in \mathbb{R}^+$. The delay $h(t)$ is a continuously differentiable function satisfying

$$
0 \leq h(t) \leq h, \quad \dot{h}(t) \leq \delta < 1.
$$

The uncertainties $\Delta A_0, \Delta A_1, \Delta B$ are time-varying and satisfy the condition:

$$\begin{gathered}\Delta A_i(t) = G_i(t)F(t)H_i(t), \quad i = 0, 1,\\ \Delta B(t) = G_2(t)F(t)H_2(t),\\ \|F(t)\| \le 1, \quad \forall t \in \mathbb{R}^+,\end{gathered}$$

where $G_i(t), H_i(t)$, $i = 0, 1, 2$ are given matrix functions of appropriate dimensions.

Definition The system (2.1), where $u(t) = 0$, is robustly exponentially stable, if there exist numbers $\alpha > 0$, $N > 0$ such that every solution $x(t, \phi)$ of the system satisfies the inequality

$$\|x(t, \phi)\| \le N\|\phi\|e^{-\alpha(t-t_0)}, \quad \forall t \ge t_0 \ge 0,$$

for all uncertainties $\Delta A_0, \Delta A_1$.

The system (2.1) is robustly stabilizable if there is a control $u(t) = K(t)x(t)$ such that the closed-loop system

$$\dot{x}(t) = [A_0(t) + (B(t) + \Delta B(t))K(t) + \Delta A_0(t)]x(t) + [A_1(t) + \Delta A_1(t)]x(t - h(t))$$

is robustly exponentially stable. The function $u(t) = K(t)x(t)$ is called a feedback stabilizing control of the system.

Proposition 2.1 (Completing the square)**.** *Assume that $S \in M^{n\times n}$ is a symmetric positive definite matrix. Then for every $Q \in M^{n\times n}$:*

$$2\langle Qy, x\rangle - \langle Sy, y\rangle \le \langle QS^{-1}Q^T x, x\rangle, \quad \forall x, y \in \mathbb{R}^n.$$

Proposition 2.2 ([20])**.** *Let G, H, F be real matrices of appropriate dimensions with $\|F\| < 1$. Then*

(i) *For any $\epsilon > 0 : GFH + H^T F^T G^T \le \frac{1}{\epsilon}GG^T + \epsilon H^T H$.*

(ii) *For any $\epsilon > 0$ such that $\epsilon I - HH^T > 0$,*

$$(A + GFH)(A + GFH)^T \le AA^T + AH^T(\epsilon I - HH^T)^{-1}HA^T + \epsilon GG^T.$$

Proposition 2.3 (Schur complement lemma [2])**.** *Given constant symmetric matrices X, Y, Z where $Y > 0$. Then $X + Z^T Y^{-1} Z < 0$ if and only if*

$$\begin{pmatrix} X & Z^T \\ Z & -Y \end{pmatrix} < 0 \quad \textit{or} \quad \begin{pmatrix} -Y & Z \\ Z^T & X \end{pmatrix} < 0.$$

Proposition 2.4 ([7])**.** *Consider the time-delay system*

$$\dot{x}(t) = f(t, x_t), x(t) = \phi(t), \quad t \in [-h, 0].$$

If there exist a Lyapunov function $V(t, x_t)$ and $\lambda_1, \lambda_2 > 0$ such that for every solution $x(t)$ of the system, the following conditions hold

(i) $\lambda_1\|x(t)\|^2 \le V(t, x_t) \le \lambda_2\|x_t\|^2$,

(ii) $\dot{V}(t, x_t) \le 0$,

then the solution of the system is bounded; i.e., there exists $N > 0$ such that $\|x(t, \phi)\| \le N\|\phi\|, \forall t \ge 0$.

3. Main results

Given numbers $\epsilon > 0$, $\epsilon_0 > 0$, $\epsilon_1 > 0$, $\alpha > 0$, $h > 0$, we set

$$\begin{aligned} P_\epsilon(t) &= P(t) + \epsilon I, \quad A_{0,\alpha}(t) = A_0(t) + \alpha I, \\ Q(t) &= \epsilon_0 H_0^T(t) H_0(t) + I, \quad S(t) = \epsilon_1 I - H_1(t) H_1^T(t), \\ R(t) &= \frac{e^{2\alpha h}}{1-\delta}[A_1(t)A_1^T(t) + \epsilon_1 G_1(t) G_1^T(t) \\ &\quad + A_1(t) H_1^T(t) S^{-1}(t) H_1(t) A_1^T(t)] + \epsilon_0^{-1} G_0(t) G_0^T(t). \end{aligned}$$

Consider the Riccati differential equation

$$\dot{P}_\epsilon(t) + P_\epsilon(t) A_{0,\alpha}(t) + A_{0,\alpha}^T(t) P_\epsilon(t) + P_\epsilon(t) R(t) P_\epsilon(t) + Q(t) = 0. \tag{3.1}$$

Theorem 3.1. *The uncertain linear non-autonomous system* (2.1)*, where* $u(t) = 0$*, is robustly exponentially stable if there exist positive numbers* $\alpha, \epsilon, \epsilon_0, \epsilon_1$*, and a matrix function* $P(t) \in BM^+(0,\infty)$ *such that* $\epsilon_1 I - H_1(t) H_1^T(t) > 0$ *and the RDE* (3.1) *holds. Moreover, the solution* $x(t,\phi)$ *satisfies the inequality*

$$\|x(t,\phi)\| \le N \|\phi\| e^{-\alpha t}, \quad t \in \mathbb{R}^+,$$

where

$$N = \sqrt{\frac{\lambda_{\max}(P(0))}{\epsilon} + \frac{1}{2\alpha\epsilon}(1 - e^{-2\alpha h}) + 1}.$$

Proof. Let $P_\epsilon(t) \in BM^+(0,\infty), t \in \mathbb{R}^+$, be a solution of the RDE (3.1). We take the change of the state variable

$$y(t) = e^{\alpha t} x(t), \quad t \in \mathbb{R}^+, \tag{3.2}$$

then the linear delay system (2.1), where $u(t) = 0$, is transformed to the delay system

$$\begin{gathered} \dot{y}(t) = [A_{0,\alpha}(t) + \Delta A_0(t)] y(t) + e^{\alpha h(t)} [A_1(t) + \Delta A_1(t)] y(t - h(t)), \\ y(t) = e^{\alpha t} \phi(t), \quad t \in [-h, 0], \end{gathered} \tag{3.3}$$

Consider the following time-varying Lyapunov function for the system (3.3),

$$V(t, y_t) = \langle P(t) y(t), y(t) \rangle + \epsilon \|y(t)\|^2 + \int_{t-h(t)}^t \|y(s)\|^2 ds.$$

It is easy to see that

$$\epsilon \|y(t)\|^2 \le V(t, y_t) \le (p + \epsilon + h) \|y_t\|^2, \tag{3.4}$$

where $p = \max_{t \ge 0} |P(t)|$ which is a finite number because $P(t) \in BM^+(0,\infty)$ and hence $P(t)$ is a bounded function. Taking the derivative of $V(\cdot)$ in t along the solution of $y(t)$ of system (3.3), we have

$$\begin{aligned}
\dot{V}(t,y_t) &= \langle \dot{P}(t)y(t),y(t)\rangle + 2\langle P_\epsilon(t)\dot{y}(t),y(t)\rangle + \|y(t)\|^2 - (1-\dot{h}(t))\|y(t-h(t))\|^2 \\
&= \langle \dot{P}(t)y(t),y(t)\rangle + 2\langle P_\epsilon(t)A_{0,\alpha}(t)y(t),y(t)\rangle + 2\langle P_\epsilon(t)G_0F(t)H_0y(t),y(t)\rangle \\
&\quad + \|y(t)\|^2 - (1-\dot{h}(t))\|y(t-h(t))\|^2 \\
&\quad + 2e^{\alpha h(t)}\langle P_\epsilon(t)[A_1(t)+G_1F(t)H_1]y(t-h),y(t)\rangle \\
&\le \langle \dot{P}(t)y(t),y(t)\rangle + 2\langle P_\epsilon(t)A_{0,\alpha}(t)y(t),y(t)\rangle \\
&\quad + 2\langle P_\epsilon(t)G_0F(t)H_0y(t),y(t)\rangle + \|y(t)\|^2 - (1-\delta)\|y(t-h(t))\|^2 \\
&\quad + 2\langle e^{\alpha h(t)}P_\epsilon(t)[A_1(t)+G_1F(t)H_1]y(t-h(t)),y(t)\rangle.
\end{aligned}$$

From Proposition 2.1 it follows that

$$\begin{aligned}
&2\langle e^{\alpha h(t)}P_\epsilon(t)[A_1(t)+G_1(t)F(t)H_1(t)]y(t-h(t)),y(t)\rangle - (1-\delta)\|y(t-h(t))\|^2 \\
&\le \frac{e^{2\alpha h(t)}}{1-\delta}\langle P_\epsilon(t)[A_1(t)+G_1(t)F(t)H_1(t)][A_1(t)+G_1(t)F(t)H_1(t)]^TP_\epsilon(t)y(t),y(t)\rangle.
\end{aligned}$$

Using Proposition 2.2, for numbers ϵ_0,ϵ_1 such that

$$2\langle P_\epsilon G_0FH_0y,y\rangle \le \frac{1}{\epsilon_0}\langle P_\epsilon G_0G_0^TP_\epsilon y,y\rangle + \epsilon_0\langle H_0^TH_0y,y\rangle,$$

$$[A_1+G_1FH_1][A_1+G_1FH_1]^T \le A_1A_1^T + A_1H_1^T(\epsilon_1I - H_1H_1^T)^{-1}H_1A_1^T + \epsilon_1G_1G_1^T,$$

provided $\epsilon_1I - H_1H_1^T > 0$. Furthermore, note that $e^{2\alpha h(t)} \le e^{2\alpha h}, \forall t \in \mathbb{R}^+$, we obtain

$$\begin{aligned}
&\dot{V}(t,y(t)) \\
&\le \langle \dot{P}(t)y(t),y(t)\rangle + 2\langle P_\epsilon(t)A_{0,\alpha}(t)y(t),y(t)\rangle \\
&\quad + \frac{1}{\epsilon_0}\langle P_\epsilon G_0G_0^TP_\epsilon y,y\rangle + \epsilon_0\langle H_0^TH_0y(t),y(t)\rangle + \|y(t)\|^2 \\
&\quad + \frac{e^{2\alpha h}}{1-\delta}\langle\{P_\epsilon[A_1A_1^T + A_1H_1^T(\epsilon_1I - H_1H_1^T)^{-1}H_1A_1^T + \epsilon_1G_1^TG_1]P_\epsilon\}y(t),y(t)\rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\dot{V}(t,y(t)) \le \langle\{&\dot{P}_\epsilon + P_\epsilon A_{0,\alpha} + A_{0,\alpha}^TP_\epsilon + \epsilon_0^{-1}P_\epsilon G_0G_0^TP_\epsilon + \epsilon_1H_0^TH_0 + I \\
&+ \frac{e^{2\alpha h}}{1-\delta}[P_\epsilon A_1A_1^TP_\epsilon + P_\epsilon A_1H_1^T(\epsilon_1I - H_1H_1^T)^{-1}H_1A_1^TP_\epsilon \\
&+ \epsilon_1P_\epsilon G_1^TG_1P_\epsilon]\}y(t),y(t)\rangle.
\end{aligned}$$

Since $P(t)$ is the solution of (3.1), we obtain

$$\dot{V}(t,y(t)) \le 0, \quad \forall t \in \mathbb{R}^+. \tag{3.5}$$

Thus, from (3.4), (3.5) and Proposition 2.4 it follows the boundedness of the solution $y(t,\phi)$ for the system (3.3); i.e., there exists $N > 0$ such that

$$\|y(t,\phi)\| \le N\|\phi\|, \quad \forall t \ge 0.$$

Returning to the solution $x(t,\phi)$ of the system (2.1) by the transformation (3.2), we obtain

$$\|x(t,\phi)\| \le N\|\phi\|e^{-\alpha t}, \quad \forall t \ge 0,$$

which gives the exponential stability of (2.1). To determine the stability factor N, we integrate both sides of (3.5) from 0 to t we find

$$V(t, y(t)) - V(0, y(0)) \le 0, \quad \forall t \in \mathbb{R}^+,$$

and hence

$$\begin{aligned} &\langle P(t)y(t), y(t)\rangle + \epsilon\|y(t)\|^2 + \int_{t-h(t)}^{t} \|y(s)\|^2 ds \\ &\le \langle P_0 y(0), y(0)\rangle + \epsilon\|y(0)\|^2 + \int_{-h(0)}^{0} \|y(s)\|^2 ds. \end{aligned}$$

Since

$$\langle P(t)y, y\rangle \ge 0, \quad \int_{t-h(t)}^{t} \|y(s)\|^2 ds \ge 0,$$

and

$$\int_{-h(0)}^{0} \|y(s)\|^2 ds \le \|\phi\|^2 \int_{-h(0)}^{0} e^{2\alpha s} ds = \frac{1}{\alpha}(1 - e^{-2\alpha h(0)})\|\phi\|^2 \le \frac{1}{2\alpha}(1 - e^{-2\alpha h})\|\phi\|^2,$$

we have

$$\epsilon\|y(t)\|^2 \le \lambda_{\max}(P(0))\|y(0)\|^2 + \epsilon\|y(0)\|^2 + \frac{1}{2\alpha}(1 - e^{-2\alpha h})\|\phi\|^2.$$

Returning to the solution $x(t, \phi)$ of system (2.1) and noting that

$$\|y(0)\| = \|x(0)\| = \phi(0) \le \|\phi\|,$$

we have

$$\|x(t, \phi)\| \le N\|\phi\|e^{-\alpha t}, \quad \forall t \in \mathbb{R}^+,$$

where

$$N = \sqrt{\frac{\lambda_{\max}(P(0))}{\epsilon} + \frac{1}{2\alpha\epsilon}(1 - e^{-2\alpha h}) + 1}.$$

The proof of the theorem is complete. □

Remark 3.2. When the system is time-invariant, using the Schur complement lemma (Proposition 2.3), the RDE (3.1) can be rewritten in terms of the LMI:

$$\begin{pmatrix} X(P) & P_\epsilon A_{1,\alpha} & P_\epsilon G_1 & P_\epsilon G_0 & P_\epsilon A_{1,\alpha} H_1^T \\ A_{1,\alpha}^T P_\epsilon & -\eta e^{-2\alpha h} I & 0 & 0 & 0 \\ G_1^T P_\epsilon & 0 & -\eta\epsilon_1^{-1} e^{-2\alpha h} I & 0 & 0 \\ G_0^T P_\epsilon & & 0 & -\epsilon_0 I & 0 \\ H_1 A_{1,\alpha}^T P_\epsilon & 0 & 0 & 0 & -\eta e^{-2\alpha h} S \end{pmatrix} < 0,$$

where $\eta = 1 - \delta$ and

$$X(P) := P_\epsilon A_{0,\alpha} + A_{0,\alpha}^T P_\epsilon + \epsilon_0 H_0^T H_0.$$

We now apply the stability result to global stabilization of the uncertain linear time-delay control system (2.1). For this, given numbers $\epsilon > 0$, $\epsilon_0 > 0$, $\epsilon_1 > 0$,

$\alpha > 0$, $h > 0$, we set

$$
\begin{aligned}
P_\epsilon(t) &= P(t) + \epsilon I, \quad A_{0,\alpha}(t) = A_0(t) + \alpha I,\\
Q(t) &= \epsilon_0 H_0^T(t)H_0(t) + I, \quad S(t) = \epsilon_1 I - H_1(t)H_1^T(t),\\
R_1(t) &= \frac{e^{2\alpha h}}{1-\delta}[A_1(t)A_1^T(t) + \epsilon_1 G_1(t)G_1^T(t) + A_1(t)H_1^T(t)S^{-1}(t)H_1(t)A_1^T(t)]\\
&\quad - B(t)B^T(t) + \frac{1}{4}\epsilon_0 B(t)H_2^T(t)H_2(t)B^T(t)\\
&\quad + \epsilon_0^{-1}[G_0(t)G_0^T(t) + G_2(t)G_2^T(t)].
\end{aligned}
$$

Consider the Riccati differential equation

$$
\dot P_\epsilon(t) + P_\epsilon(t)A_{0,\alpha}(t) + A_{0,\alpha}^T P_\epsilon(t) + P_\epsilon(t)R_1(t)P_\epsilon(t) + Q(t) = 0. \tag{3.6}
$$

Theorem 3.3. *Uncertain linear non-autonomous control delay system* (2.1) *is robustly stabilizable if there exist positive numbers* $\alpha, \epsilon, \epsilon_0, \epsilon_1$, *and a matrix function* $P(t) \in BM^+(0,\infty)$ *such that* $\epsilon_1 I - H_1(t)H_1^T(t) > 0$ *and the RDE* (3.6) *holds. Moreover, the feedback stabilizing control is given by*

$$
u(t) = -\frac{1}{2}B^T(t)P(t)x(t). \tag{3.7}
$$

Proof. Let us define

$$
\begin{gathered}
\bar A_0(t) = A(t) + B(t)K(t), \quad \bar G_0(t) = [G_0(t)\ G_2(t)], \quad \bar H_0(t) = \begin{bmatrix} H_0(t) \\ H_2(t)K(t) \end{bmatrix},\\
\bar F(t) = \begin{pmatrix} F(t) & 0 \\ 0 & F(t) \end{pmatrix}, \quad \Delta \bar A_0(t) = \bar G_0(t)\bar F(t)\bar H_0(t),
\end{gathered}
$$

where $K(t) := -\frac{1}{2}B^T(t)P(t)$. With the feedback control (3.7), the closed-loop system of the system (2.1) is

$$
\dot x(t) = [\bar A_0(t) + \Delta \bar A_0(t)]x(t) + [A_1(t) + \Delta A_1(t)]x(t - h(t)). \tag{3.8}
$$

Therefore, the proof of the theorem is completed by using Theorem 3.1 for the uncertain unforced system (3.8) with the following transformations

$$
\begin{gathered}
\bar G_0(t)\bar G_0^T(t) = G_0(t)G_0^T(t) + G_2(t)G_2^T(t),\\
\bar H_0^T(t)\bar H_0(t) = H_0^T(t)H_0(t) + \frac{1}{4}P(t)B(t)H_2^T(t)H_2(t)B^T(t)P(t),\\
\bar A_{0,\alpha}(t) = A_{0,\alpha}(t) - \frac{1}{2}B(t)B^T(t)P(t),\\
P_\epsilon(t)\bar A_{0,\alpha}(t) + \bar A_{0,\alpha}^T(t)P_\epsilon(t) = P_\epsilon(t)A_{0,\alpha}(t) + A_{0,\alpha}^T(t)P_\epsilon(t) - P(t)B(t)B^T(t)P(t).
\end{gathered}
$$

□

Remark 3.4. As in Remark 3.2, for the time-invariant systems the Riccati equation (3.6) can be replaced by the LMI:

$$\begin{pmatrix} X(P) & P_\epsilon A_{1,\alpha} & P_\epsilon G_1 & P_\epsilon G_0 & P_\epsilon A_{1,\alpha}H_1^T & P_\epsilon B & P_\epsilon G_2 & P_\epsilon B H_2^T \\ A_{1,\alpha}^T P_\epsilon & -\eta e^{-2\alpha h} I & 0 & 0 & 0 & 0 & 0 & 0 \\ G_1^T P_\epsilon & 0 & -\eta\epsilon_1^{-1} e^{-2\alpha h} I & 0 & 0 & 0 & 0 & 0 \\ G_0^T P_\epsilon & & 0 & -\epsilon_0 I & 0 & 0 & 0 & 0 \\ H_1 A_{1,\alpha}^T P_\epsilon & 0 & 0 & 0 & -\eta e^{-2\alpha h} S & 0 & 0 & 0 \\ B^T P_\epsilon & 0 & 0 & 0 & & I & 0 & \\ G_2^T P_\epsilon & 0 & 0 & 0 & 0 & 0 & -\epsilon_0 I & 0 \\ H_2 B^T P_\epsilon & 0 & 0 & 0 & 0 & 0 & 0 & -4\epsilon_0^{-1} \end{pmatrix} < 0,$$

where $\eta = 1 - \delta$, and

$$X(P) := P_\epsilon A_{0,\alpha} + A_{0,\alpha}^T P_\epsilon + \epsilon_0 H_0^T H_0.$$

The stability conditions are given in terms of the solution of some RDEs. Although the problem of solving RDEs is in general still not easy, various effective approaches for finding the solutions of RDEs can be found in [5, 10, 21].

4. Examples

Example 4.1. Consider the uncertain linear non-autonomous unforced system with time-varying delay (2.1), where $u(t) = 0$, with any initial function $\phi(t)$ and time-delay function $h(t) = 3\sin^2(2/3)t$ and

$$A_0(t) = \begin{bmatrix} -\frac{1}{2} & -1 \\ 0 & e^{-2t} - \frac{3}{4} \end{bmatrix}, \quad A_1(t) = \begin{bmatrix} -\frac{\sqrt{2-e^{-2t}}}{e^3(e^{-2t}+1)} & 0 \\ 0 & -\frac{\sqrt{2-e^{-2t}}}{2e^3} \end{bmatrix},$$

$$G_0(t) = \begin{bmatrix} \frac{e^t}{e^{-2t}+1} & 0 \\ 0 & \frac{e^{-t}}{\sqrt{2}} \end{bmatrix}, \quad H_0(t) = \begin{bmatrix} e^{-t} & e^{-t} \\ e^t & e^{-t} \end{bmatrix},$$

$$G_1(t) = \begin{bmatrix} \frac{e^t}{e^3(e^{-2t}+1)} & 0 \\ 0 & \frac{e^{-t}}{e^3} \end{bmatrix}, \quad H_1(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{bmatrix}.$$

We see that $h = 3$, and $\dot{h}(t) = 2\sin(4/3t)$ and then $\delta = 2$. Taking $\alpha = \epsilon = \epsilon_0 = 1$ and $\epsilon_1 = 2$, we have

$$\epsilon_1 I - H_1(t)H_1^T(t) = \begin{bmatrix} 2 - e^{-2t} & 0 \\ 0 & 2 - e^{-2t} \end{bmatrix} > 0.$$

We can verify that the matrix $P(t) = \begin{bmatrix} e^{-2t} & 0 \\ 0 & 1 \end{bmatrix}$ is a solution of (3.1). Therefore, by Theorem 3.1 the system is robustly exponentially stable and the solution satisfies

$$\|x(t,\phi)\| \le (3 - e^{-3})\|\phi\| e^{-t}, \quad t \in \mathbb{R}^+.$$

Example 4.2. Consider the uncertain linear non-autonomous control system with time-varying delay (2.1) with any initial function $\phi(t)$ and time-delay function

$h(t) = 2\sin^2 t$ and

$$A_0(t) = \begin{bmatrix} -e^{-2t}+1 & 0 \\ -1 & -e^{-2t}+1 \end{bmatrix}, \quad A_1(t) = \begin{bmatrix} -\frac{\sqrt{2-e^{-2t}}}{e^2(e^{-2t}+1)} & 0 \\ 0 & -\frac{\sqrt{2-e^{-2t}}}{e^2(e^{-2t}+1)} \end{bmatrix},$$

$$B(t) = \begin{bmatrix} -\frac{1}{e^{-2t}+1} & 0 \\ 0 & -\frac{1}{e^{-2t}+1} \end{bmatrix}, \quad G_0(t) = \begin{bmatrix} \frac{e^t}{e^{-2t}+1} & 0 \\ 0 & \frac{e^{-t}}{\sqrt{2}} \end{bmatrix}, \quad H_0(t) = \begin{bmatrix} e^{-t} & e^t \\ e^{-t} & e^{-t} \end{bmatrix},$$

$$G_1(t) = \begin{bmatrix} \frac{e^{-2t}}{e^2(e^{-2t}+1)} & 0 \\ 0 & \frac{e^t}{\sqrt{2}e^2(e^{-2t}+1)} \end{bmatrix}, \quad H_1(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{bmatrix},$$

$$G_2(t) = \begin{bmatrix} 0 & 0 \\ 0 & \frac{e^{-2t}}{(e^{-2t}+1)} \end{bmatrix}, \quad H_2(t) = \begin{bmatrix} 4e^{-2t} & 0 \\ 0 & 2e^{-t} \end{bmatrix}.$$

We see that $h = 2, \delta = 2$. Taking $\alpha = \epsilon = \epsilon_0 = 1$ and $\epsilon_1 = 2$, we have

$$\epsilon_1 I - H_1(t)H_1^T(t) = \begin{bmatrix} 2-e^{-2t} & 0 \\ 0 & 2-e^{-2t} \end{bmatrix} > 0.$$

We can verify that the matrix $P(t) = \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-2t} \end{bmatrix}$ is a solution of the RDE (3.6). Therefore, by Theorem 3.3 the system is robustly stabilizable and the feedback stabilizing control is given by

$$u(t) = \begin{bmatrix} \frac{e^{-2t}}{2(e^{-2t}+1)} & 0 \\ 0 & \frac{e^{-2t}}{2(e^{-2t}+1)} \end{bmatrix} x(t), \quad t \geq 0.$$

Conclusions. Based on combination of the Riccati equation approach and the use of suitable Lyapunov-Krasovskii functional, sufficient conditions for the exponential stability and stabilizability of linear non-autonomous delay systems with time-varying and norm-bounded uncertainties have been established. The conditions are formulated in terms of the solution of curtain Riccati differential equations, which allow to compute the decay rate as well as the constant stability factor.

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