

DEGENERATE STATIONARY PROBLEMS WITH HOMOGENEOUS BOUNDARY CONDITIONS

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ABSTRACT. We are interested in the degenerate problem

$$b(v) - \operatorname{div} a(v, \nabla g(v)) = f$$

with the homogeneous boundary condition $g(v) = 0$ on some part of the boundary. The vector field a is supposed to satisfy the Leray-Lions conditions and the functions b, g to be continuous, nondecreasing and to verify the normalization condition $b(0) = g(0) = 0$ and the range condition $R(b + g) = \mathbb{R}$. Using monotonicity methods, we prove existence and comparison results for renormalized entropy solutions in the L^1 setting.

1. INTRODUCTION

Let Ω be a $C^{1,1}$ bounded open subset of \mathbb{R}^N with regular boundary if $N > 1$. We consider the problem, $(P_{b,g}(f))$,

$$\begin{aligned} b(v) - \operatorname{div} a(v, \nabla g(v)) &= f \quad \text{in } \Omega \\ g(v) &= 0 \quad \text{on } \Gamma := \partial\Omega, \end{aligned} \tag{1.1}$$

where $b, g : \mathbb{R} \rightarrow \mathbb{R}$ are nondecreasing, continuous such that $b(0) = g(0) = 0$, $R(b + g) = \mathbb{R}$ and that $f \in L^1(\Omega)$.

The vector field $a : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is supposed to be continuous, to satisfy the growth condition

$$|a(r, \xi) - a(r, 0)| \leq C(|r|)|\xi|^{p-1} \quad \text{for all } (r, \xi) \in \mathbb{R} \times \mathbb{R}^N \tag{1.2}$$

with $C : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ nondecreasing and the weak coerciveness condition

$$(a(r, \xi) - a(r, 0)) \cdot \xi + M(|r|) \geq \lambda(|r|)|\xi|^p \quad \text{for all } r \in \mathbb{R}, \xi \in \mathbb{R}^N \tag{1.3}$$

where $M : \mathbb{R}^+ \rightarrow \mathbb{R}$, $\lambda : \mathbb{R}^+ \rightarrow]0, \infty[$ are continuous functions satisfying, for all $k > 0$, $\lambda_k := \inf_{\{r; |b(r)| \leq k\}} \lambda(r) > 0$ and $M_k := \sup_{\{r; |b(r)| \leq k\}} M(r) < \infty$.

To prove the uniqueness result, we assume that a verifies the additional condition

$$(a(r, \xi) - a(s, \eta)) \cdot (\xi - \eta) + B(r, s)(1 + |\xi|^p + |\eta|^p)|r - s| \geq \Gamma_1(r, s) \cdot \xi + \Gamma_2(r, s) \cdot \eta, \tag{1.4}$$

2000 *Mathematics Subject Classification.* 35K65, 35F30, 35K35, 65M12.

Key words and phrases. Degenerate; homogenous boundary conditions; diffusion; continuous flux.

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Submitted January 8, 2008. Published February 28, 2008.

for all $r, s \in \mathbb{R}$, $\xi, \eta \in \mathbb{R}^N$, for some continuous function $B : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and continuous fields $\Gamma_1, \Gamma_2 : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^N$.

It is well known that the above problem is ill-posed in the variational setting. In the sense that there is no existence and uniqueness of a weak solution in the distributional sense. In order to overcome this difficulty, we use the notion of entropy solution introduced by Krushkov in [19] (see also [20]) and which coincides with the “physical” solution. An other difficulty is related to the irregularity of the data which is only supposed to be in $L^1(\Omega)$. The suitable notion of solution which guarantees existence and uniqueness results in this general framework is the so called renormalized entropy solution (see [6, 10, 5]).

The outline of the paper is as follows: In Section 2, we define the renormalized entropy solution and present our main results. Then, in section 3, we prove the comparison principle for bounded solution. Finally, in Section 4, we prove the existence of a renormalized entropy solution, the comparison result in the L^1 -setting and give some possible extensions of our results.

2. DEFINITIONS, NOTATION AND MAIN RESULTS

Definition 2.1. Let $f \in L^1(\Omega)$. A measurable function $v : \Omega \rightarrow \mathbb{R}$ is said to be a weak solution of (1.1) if $b(v) \in L^1(\Omega)$, $g(v) \in W^{1,p}(\Omega)$ and

$$\int_{\Omega} b(v)\xi + \int_{\Omega} a(v, \nabla g(v)) \cdot \nabla \xi = \int_{\Omega} f\xi$$

for all $\xi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Definition 2.2. Let $f \in L^1(\Omega)$. A measurable function $v : \Omega \rightarrow \mathbb{R}$ is said to be a renormalized entropy solution of (1.1) if $b(v) \in L^1(\Omega)$,

$$g(T_k v) \in W_0^{1,p}(\Omega), \quad \forall k > 0$$

and there exists some families of non-negative bounded measures $\mu_l := \mu_l(v)$ and $\nu_l = \nu_l(v)$ on $\bar{\Omega}$ such that

$$\|\mu_l\|, \|\nu_{-l}\| \rightarrow 0, \quad l \rightarrow \infty,$$

and the following entropy inequalities are satisfied:

For all $k \in \mathbb{R}$, for all $\xi \in C_0^\infty(\mathbb{R}^N)$ such that $\xi \geq 0$ and $\text{sign}^+(-g(k))\xi = 0$ a.e. on Γ , for all $l \geq k$,

$$\begin{aligned} - \int_{\Omega} b(v \wedge l) \chi_{\{v \wedge l > k\}} \xi - \int_{\Omega} \chi_{\{v \wedge l > k\}} (a(v \wedge l, \nabla g(v \wedge l)) - a(k, 0)) \cdot \nabla \xi \\ + \int_{\Omega} \chi_{\{k > v \wedge l\}} f \xi \geq -\langle \mu_l, \xi \rangle \end{aligned} \quad (2.1)$$

and for all $k \in \mathbb{R}$, for all $\xi \in C_0^\infty(\mathbb{R}^N)$ such that $\xi \geq 0$ and $\text{sign}^+(g(k))\xi = 0$ a.e. on Γ , for $l \leq k$,

$$\begin{aligned} \int_{\Omega} b(v \vee l) \chi_{\{k > v \vee l\}} \xi + \int_{\Omega} \chi_{\{k > v \vee l\}} (a(v \vee l, \nabla g(v \vee l)) - a(k, 0)) \cdot \nabla \xi \\ - \int_{\Omega} \chi_{\{k > v \vee l\}} f \xi \geq -\langle \nu_l, \xi \rangle. \end{aligned} \quad (2.2)$$

Remark 2.3. (i) In the case where the data $f \in L^\infty(\Omega)$, it is easily verified that a renormalized entropy solution v of (1.1) is such that $b(v) \in L^1(\Omega)$, $g(v) \in W_0^{1,p}(\Omega)$ and v satisfies the following entropy inequalities: For all $k \in \mathbb{R}$, for all $\xi \in C_0^\infty(\mathbb{R}^N)$ such that $\xi \geq 0$ and $\text{sign}^+(-g(k))\xi = 0$ a.e. on Γ ,

$$-\int_{\Omega} \chi_{\{v>k\}}(a(v, \nabla g(v)) - a(k, 0)) \cdot \nabla \xi + \int_{\Omega} \chi_{\{v>k\}} f \xi \geq \int_{\Omega} b(v) \chi_{\{v>k\}} \xi \quad (2.3)$$

and for all $k \in \mathbb{R}$, for all $\xi \in C_0^\infty(\mathbb{R}^N)$ such that $\xi \geq 0$ and $\text{sign}^+(g(k))\xi = 0$ a.e. on Γ ,

$$\int_{\Omega} \chi_{\{k>v\}}(a(v, \nabla g(v)) - a(k, 0)) \cdot \nabla \xi - \int_{\Omega} \chi_{\{k>v\}} f \xi \geq \int_{\Omega} -b(v) \chi_{\{k>v\}} \xi. \quad (2.4)$$

In this case, v is called weak entropy solution of (1.1).

If moreover, the function b is strictly increasing on \mathbb{R} with $R(b) = \mathbb{R}$, then the weak entropy solution v is also in $L^\infty(\Omega)$.

(ii) If v is a renormalized entropy solution of (1.1), then $-v$ is an entropy solution of

$$\begin{aligned} \bar{b}(v) - \text{div } \bar{a}(v, \nabla \bar{g}(v)) &= \tilde{f} \quad \text{in } \Omega \\ \bar{g}(v) &= 0 \quad \text{on } \Gamma := \partial\Omega, \end{aligned} \quad (2.5)$$

where $\bar{b}(r) = -b(-r)$, $\bar{g}(r) = -g(-r)$ and $\bar{a}(r, \xi) = -a(-r, \xi)$.

The main result of this paper is the following.

Theorem 2.4. *For any $f \in L^1(\Omega)$, there exists a unique pair $(b(v), g(v))$ such that v is a renormalized entropy solution of (1.1).*

The uniqueness result follows as a consequence of an L^1 -comparison principle.

Some notation. Throughout this paper we use the operators

$$H_\delta(s) := \min\left(\frac{s^+}{\delta}, 1\right), \quad H_0(s) = \begin{cases} 1 & \text{if } s > 0 \\ 0 & \text{if } s \leq 0 \end{cases}$$

and we denote

$$E := \{r \in R(g)/(g^{-1})_0 \text{ is discontinuous at } r\}. \quad (2.6)$$

For $k > 0$, T_k is the truncation function defined on \mathbb{R} by

$$T_k(r) = \text{sign}^0(r)(|r| \wedge k)$$

and for $r \in \mathbb{R}$, we define $r^+ = r \vee 0$, $r^- = r \wedge 0$.

3. PROOFS OF COMPARISON AND UNIQUENESS RESULTS

We first prove the comparison result in the L^∞ -setting.

Theorem 3.1. *For $i = 1, 2$, let $f_i \in L^\infty(\Omega)$ and $v_i \in L^\infty(\Omega)$ be a weak entropy solution of $P_{b,g}(f_i)$. Then there exist $\kappa \in L^\infty(\Omega)$ with $\kappa \in \text{sign}^+(v_1 - v_2)$ a.e. in Ω such that, for any $\xi \in \mathcal{D}(\mathbb{R}^N)$, $\xi \geq 0$,*

$$\int_{\Omega} (b(v_1) - b(v_2))^+ \xi \leq \int_{\Omega} \kappa (f_1 - f_2) \xi - \int_{\Omega} \chi_{\{v_1 > v_2\}} (a(v_1, \nabla g(v_1)) - a(v_2, \nabla g(v_2))) \cdot \nabla \xi. \quad (3.1)$$

Lemma 3.2. *Let $f \in L^\infty(\Omega)$ and v be a weak solution of (1.1). Then*

$$\begin{aligned} & \int_{\Omega} \chi_{\{v>k\}}((a(v, \nabla g(v))) - a(k, 0)) \cdot \nabla \xi + \int_{\Omega} \chi_{\{v>k\}}\{b(v)\xi - f\xi\} dx \\ &= - \lim_{\delta \rightarrow 0} \int_{\Omega} (a(v, \nabla g(v)) - a(v, 0)) \cdot \nabla g(v) H'_\delta(g(v) - g(k)) \xi dx, \end{aligned} \quad (3.2)$$

for any $(k, \xi) \in \mathbb{R} \times \mathcal{D}^+(\Omega)$ such that $g(k) \notin E$ and $(g(v) - g(k))^+\xi = 0$ a.e. on Γ . Moreover,

$$\begin{aligned} & \int_{\Omega} \chi_{\{k>v\}}(a(v, \nabla g(v)) - a(k, 0)) \cdot \nabla \xi + \int_{\Omega} \chi_{\{k>v\}}\{b(v)\xi - f\xi\} dx \\ &= \lim_{\delta \rightarrow 0} \int_{\Omega} (a(v, \nabla g(v)) - a(v, 0)) \cdot \nabla g(v) H'_\delta(g(k) - g(v)) \xi dx, \end{aligned} \quad (3.3)$$

for any $(k, \xi) \in \mathbb{R} \times \mathcal{D}^+(\Omega)$ such that $g(k) \notin E$ and $(g(k))^+\xi = 0$ a.e. on Γ .

From now on, we denote $\tilde{a}(r, \xi) = a(r, \xi) - a(r, 0)$, $r \in \mathbb{R}$, $\xi \in \mathbb{R}^N$. The proof of the above lemma follows the same lines as the proof in [8, Lemma 2.5].

Proof of Theorem 3.1. Let $(B_i)_{i=0\dots m}$ be a covering of $\bar{\Omega}$ satisfying $B_0 \cap \partial\Omega = \emptyset$ and such that, for each $i \geq 1$, B_i is a ball contained in some larger ball B_i with $B_i \cap \partial\Omega$ is part of the graph of a Lipschitz function. Let $(\varphi_i)_{i=0\dots m}$ denote a partition of unity subordinate to the covering $(B_i)_i$ and denote by ξ an arbitrary function in $\mathcal{D}(\mathbb{R}^N)$, $\xi \geq 0$.

We use Kruzhkov's technique of doubling variables in order to prove the comparison result (see [19, 20, 10], etc). We choose two variables x and y and consider v_1 as function of y and v_2 as function of $x \in \Omega$. Define the test function $\xi_n^i : (x, y) \mapsto \varphi_i(x)\xi(x)\varrho_n(x-y)$, where $(\varrho_n)_n$ is a sequence of mollifiers in \mathbb{R}^N such that $x \mapsto \varrho_n(x-y) \in \mathcal{D}(\Omega)$, for all $y \in B_i$, $\sigma_n(x) = \int_{\Omega} \varrho_n(x-y) dy$ is an increasing sequence for all $x \in B_i$, and $\sigma_n(x) = 1$ for all $x \in B_i$ with $d(x, \mathbb{R}^N \setminus \Omega) > \frac{c}{N}$ for some $c = c(i)$ depending on B_i . Then, for n sufficiently large,

$$\begin{aligned} y &\mapsto \xi_n^i(x, y) \in \mathcal{D}(\mathbb{R}^N), \quad \text{for any } x \in \Omega, \\ x &\mapsto \xi_n^i(x, y) \in \mathcal{D}(\Omega), \quad \text{for any } y \in \Omega \\ \text{supp}_y(\xi_n^i(x, \cdot)) &\subset B_i, \quad \text{for any } x \in \text{supp}(\varphi_i). \end{aligned}$$

For convenience, we sometimes omit the index i and simply set $\varphi = \varphi_i$, $B = B_i$ and $\xi_n^i = \xi_n$. Then $\hat{\zeta}_n(x) := \xi(x)\varphi(x)\sigma_n(x)$ satisfies $\hat{\zeta}_n \in \mathcal{D}(\Omega)$, $0 \leq \hat{\zeta}_n \leq \xi$, for all $n \in \mathbb{N}$. Let

$$\Omega_1 := \{y \in \Omega / v_1(y) \in E\}, \quad \Omega_2 := \{x \in \Omega / v_2(x) \in E\}.$$

Then, $\nabla_y g(v_1) = 0$ a.e in Ω_1 and $\nabla_x g(v_2) = 0$ a.e in Ω_2 . Moreover, $H_0(v_1 - v_2) = H_0(g(v_1) - g(v_2))$ a.e in $(\Omega \setminus \Omega_1) \times \Omega \cup \Omega \times (\Omega \setminus \Omega_2)$.

First inequality. We first prove the following inequality:

$$\begin{aligned} & \int_{\Omega} (b(v_1^+) - b(v_2^+)) \xi \varphi \\ & \leq \int_{\Omega} \kappa_1 \chi_{\{v_1 > 0\}} (f_1 - \chi_{\{v_2 \geq 0\}} f_2) \xi \varphi \\ & \quad - \int_{\Omega} \chi_{\{v_1^+ > v_2^+\}} (a(v_1^+, \nabla g(v_1^+)) - a(v_2^+, \nabla g(v_2^+)) \cdot \nabla_x(\xi \varphi) + \lim_{n \rightarrow \infty} \mathcal{L}(\xi \varphi \sigma_n) \end{aligned} \quad (3.4)$$

where $\kappa_1 \in L^\infty(\Omega)$, $\kappa_1 \in \text{sign}^+(v_1 - v_2^+)$ and \mathcal{L} is a linear operator which will be defined later.

As v_1 satisfies (3.2) (with $v = v_1$, $f = f_1$), choosing $k = v_2^+(x)$ and $\xi(y) = \zeta_n(x, y)$ in (2.3), integrating in x over Ω , we find

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \int_{\{\Omega \setminus \Omega_1\} \times \{\Omega \setminus \Omega_2\}} \tilde{a}(v_1, \nabla_y g(v_1)) \cdot \nabla_y g(v_1) H'_\delta(g(v_1) - g(v_2^+)) \zeta_n \\ & = \lim_{\delta \rightarrow 0} \int_{\Omega \times \{\Omega \setminus \Omega_2\}} \tilde{a}(v_1, \nabla_y g(v_1)) \cdot \nabla_y g(v_1) H'_\delta(g(v_1) - g(v_2^+)) \zeta_n \\ & \leq - \int_{\Omega \times \Omega} \chi_{\{v_1 > v_2^+\}} \{b(v_1) \zeta_n - f_1 \zeta_n + \tilde{a}(v_1^+, \nabla_y g(v_1^+)) \cdot \nabla_y \zeta_n \\ & \quad + (a(v_1^+, 0) - a(v_2^+, 0)) \cdot \nabla_y \zeta_n\}. \end{aligned} \quad (3.5)$$

Now, since $x \mapsto \zeta_n(x, y) H_\delta(g(v_1^+) - g(v_2^+)) \in \mathcal{D}(\Omega)$ for a.e. $y \in \Omega$, we have

$$\int_{\Omega \times \Omega} \tilde{a}(v_1^+, \nabla_y g(v_1^+)) \cdot \nabla_x (H_\delta(g(v_1^+) - g(v_2^+)) \zeta_n) = 0. \quad (3.6)$$

Therefore, going to the limit in δ , we get

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \int_{\{\Omega \setminus \Omega_1\} \times \{\Omega \setminus \Omega_2\}} \tilde{a}(v_1^+, \nabla_y g(v_1^+)) \cdot \nabla_x g(v_2^+) H'_\delta(g(v_1^+) - g(v_2^+)) \zeta_n \\ & = \int_{\Omega \times \Omega} H_0(g(v_1^+) - g(v_2^+)) \tilde{a}(v_1^+, \nabla_y g(v_1^+)) \cdot \nabla_x \zeta_n \\ & = \int_{\Omega \times \Omega} H_0(v_1^+ - v_2^+) \tilde{a}(v_1, \nabla_y g(v_1^+)) \cdot \nabla_x \zeta_n. \end{aligned} \quad (3.7)$$

Arguing as in [8], inequality (3.5) can be written as

$$\begin{aligned} & \int_{\Omega \times \Omega} \{-\tilde{a}(v_1^+, \nabla_y g(v_1^+)) \cdot \nabla_{x+y} \zeta_n - b(v_1^+) \zeta_n + f_1 \zeta_n \\ & \quad - (a(v_1^+, 0) - a(v_2^+, 0)) \cdot \nabla_y \zeta_n\} H_0(v_1^+ - v_2^+) \\ & \geq \lim_{\delta \rightarrow 0} \int_{\{\Omega \setminus \Omega_1\} \times \{\Omega \setminus \Omega_2\}} \tilde{a}(v_1^+, \nabla_y g(v_1^+)) \cdot \nabla_{x+y} (g(v_1^+) - g(v_2^+)) \\ & \quad \times H'_\delta(g(v_1^+) - g(v_2^+)) \zeta_n \end{aligned} \quad (3.8)$$

with $\nabla_{x+y}(\cdot) := \nabla_x(\cdot) + \nabla_y(\cdot)$. Now, as v_2 is a weak entropy solution of (1.1) with f_2 instead of f , choosing $k = v_1^+(y)$, $\xi(x) = \zeta_n(x, y)$ in (3.3) (with $v = v_2$, $f = f_2$),

integrating in y over Ω , we find

$$\begin{aligned}
& - \lim_{\delta \rightarrow 0} \int_{\{\Omega \setminus \Omega_1\} \times \Omega} \tilde{a}(v_2, \nabla_x g(v_2)) \cdot \nabla_x g(v_2) H'_\delta(g(v_1^+) - g(v_2)) \zeta_n \\
& = - \lim_{\delta \rightarrow 0} \int_{\{\Omega \setminus \Omega_1\} \times \{\Omega \setminus \Omega_2\}} \tilde{a}(v_2, \nabla_x g(v_2)) \cdot \nabla_x g(v_2) H'_\delta(g(v_1^+) - g(v_2)) \zeta_n \quad (3.9) \\
& \leq \int_{\Omega \times \Omega} \chi_{\{v_1^+ > v_2\}} \{b(v_2) \zeta_n - f_2 \zeta_n + \tilde{a}(v_2, \nabla_x g(v_2)) \cdot \nabla_x \zeta_n + (a(v_1^+, 0) \\
& \quad - a(v_2, 0)) \cdot \nabla_x \zeta_n\}
\end{aligned}$$

It is easily verified that

$$\begin{aligned}
& \int_{\{\Omega \setminus \Omega_1\} \times \{\Omega \setminus \Omega_2\}} \tilde{a}(v_2, \nabla_x g(v_2)) \cdot \nabla_x g(v_2) H'_\delta(g(v_1^+) - g(v_2)) \zeta_n \\
& = \int_{\{\Omega \setminus \Omega_1\} \times \{\Omega \setminus \Omega_2\}} \tilde{a}(v_2^+, \nabla_x g(v_2^+)) \cdot \nabla_x g(v_2^+) H'_\delta(g(v_1^+) - g(v_2^+)) \zeta_n \quad (3.10) \\
& \quad + \int_{\{\Omega \setminus \Omega_1\} \times \{\Omega \setminus \Omega_2\}} \tilde{a}(v_2^-, \nabla_x g(v_2^-)) \cdot \nabla_x g(v_2^-) H'_\delta(g(v_1^+) - g(v_2^-)) \zeta_n
\end{aligned}$$

and that the second term in the right hand side of (3.10) converges to 0 with $\delta \rightarrow 0$. Moreover, the right hand side of (3.9) is equal to

$$\begin{aligned}
& \int_{\Omega \times \Omega} \chi_{\{v_1^+ > v_2^+\}} \{b(v_2^+) \zeta_n - \chi_{\{v_2 \geq 0\}} f_2 \zeta_n + (\tilde{a}(v_2^+, \nabla_x g(v_2^+)) - a(v_1^+, 0) \\
& \quad + a(v_2^+, 0)) \cdot \nabla_x \zeta_n\} + \int_{\Omega \times \Omega} \chi_{\{v_2 < 0\}} \{b(v_2) \zeta_n - f_2 \zeta_n - a(v_2, \nabla_x g(v_2)) \cdot \nabla_x \zeta_n\}. \quad (3.11)
\end{aligned}$$

Since $y \mapsto \zeta_n(x, y) H'_\delta(g(v_1^+) - g(v_2^+)) \in \mathcal{D}(\Omega)$ for a.e. $(x) \in \Omega$, we have

$$\int_{\Omega \times \Omega} \tilde{a}(v_2^+, \nabla_x g(v_2^+)) \cdot \nabla_y (H'_\delta(g(v_1^+) - g(v_2^+)) \zeta_n) = 0. \quad (3.12)$$

Therefore,

$$\begin{aligned}
& - \lim_{\delta \rightarrow 0} \int_{\{\Omega \setminus \Omega_1\} \times \{\Omega \setminus \Omega_2\}} \tilde{a}(v_2^+, \nabla_x g(v_2^+)) \cdot \nabla_y g(v_1^+) H'_\delta(g(v_1^+) - g(v_2^+)) \zeta_n \\
& = \int_{\Omega \times \Omega} H_0(g(v_1^+) - g(v_2^+)) \tilde{a}(v_2^+, \nabla_x g(v_2^+)) \nabla_y \zeta_n. \quad (3.13)
\end{aligned}$$

Then, the inequality (3.9) can be equivalently written as

$$\begin{aligned}
& \int_{\Omega \times \Omega} b(v_2^+) H_0(v_1^+ - v_2^+) \zeta_n - \int_{\Omega \times \Omega} \chi_{\{v_1^+ > v_2^+\}} \chi_{\{v_2 \geq 0\}} f_2 \zeta_n \\
& + \int_{\Omega \times \Omega} H_0(v_1^+ - v_2^+) \tilde{a}(v_2^+, \nabla_x g(v_2^+)) \cdot (\nabla_y \zeta_n + \nabla_x \zeta_n) \\
& - \int_{\Omega \times \Omega} H_0(v_1^+ - v_2^+) (a(v_1^+, 0) - a(v_2^+, 0)) \cdot \nabla_x \zeta_n \\
& + \int_{\Omega \times \Omega} \chi_{\{v_2 < 0\}} \{b(v_2) \zeta_n - f_2 \zeta_n - a(v_2, \nabla_x g(v_2)) \cdot \nabla_x \zeta_n\} \\
& \geq \lim_{\delta \rightarrow 0} \int_{\{\Omega \setminus \Omega_1\} \times \{\Omega \setminus \Omega_2\}} \tilde{a}(v_2^+, \nabla_x g(v_2^+)) \cdot (\nabla_x g(v_2^+) - \nabla_y g(v_1^+)) \\
& \quad \times H'_\delta(g(v_1^+) - g(v_2^+)) \zeta_n.
\end{aligned} \tag{3.14}$$

Summing up inequalities (3.8) and (3.14), we get

$$\begin{aligned}
& \lim_{\delta \rightarrow 0} \int_{(\Omega \setminus \Omega_1) \times (\Omega \setminus \Omega_2)} (\tilde{a}(v_1^+, \nabla_y g(v_1^+)) - \tilde{a}(v_2^+, \nabla_x g(v_2^+))) \cdot (\nabla_y g(v_1^+) - \nabla_x g(v_2^+)) \\
& \quad \times H'_\delta(g(v_1^+) - g(v_2^+)) \zeta_n \\
& \leq - \int_{\Omega \times \Omega} (b(v_1^+) - b(v_2^+))^+ \xi \varrho_n - \int_{\Omega \times \Omega} b(v_2) \chi_{\{v_2 < 0\}} \zeta_n \\
& \quad + \int_{\Omega \times \Omega} \chi_{\{v_1^+ > v_2^+\}} \chi_{\{v_1 > 0\}} (f_1 - \chi_{\{v_2 \geq 0\}} f_2) \zeta_n + \int_{\Omega \times \Omega} \chi_{\{v_2 < 0\}} f_2 \zeta_n \\
& \quad - \int_{\Omega \times \Omega} (a(v_1^+, \nabla_y g(v_1^+)) - a(v_2^+, \nabla_x g(v_2^+))) \cdot (\nabla_{x+y} \zeta_n) H_0(v_1^+ - v_2^+) \\
& \quad - \int_{\Omega \times \Omega} \chi_{\{0 > v_2\}} a(v_2, \nabla_x g(v_2)) \cdot \nabla_x \zeta_n.
\end{aligned} \tag{3.15}$$

Denote the integrals on the right hand side of (3.15) by I_1, \dots, I_6 successively. Going to the limit with n , one get

$$\begin{aligned}
\lim_{n \rightarrow \infty} I_1 &= - \int_{\Omega} (b(v_1^+) - b(v_2^+))^+ \xi \varrho, \\
\limsup_{n \rightarrow \infty} I_3 &\leq \int_{\Omega} \kappa_1 \chi_{\{v_1 > 0\}} (f_1 - \chi_{\{v_2 \geq 0\}} f_2) \xi \varrho
\end{aligned}$$

for some

$$\kappa_1 \in L^\infty(\Omega) \text{ with } \kappa_1 \in \text{sign}^+(v_1 - v_2^+) \text{ a.e. in } \Omega, \tag{3.16}$$

$$\limsup_{m, n \rightarrow +\infty} I_5 = - \int_{\Omega} H_0(v_1^+ - v_2^+) (a(v_1, \nabla g(v_1^+)) - a(v_2, \nabla g(v_2^+))) \cdot \nabla(\xi \varrho),$$

It remains to estimate

$$I_2 + I_4 + I_6 = \int_{\Omega} \chi_{\{v_2 < 0\}} \{b(v_2) \hat{\zeta}_n - f_2 \hat{\zeta}_n + a(v_2, \nabla_x g(v_2)) \cdot \nabla_x \hat{\zeta}_n\}.$$

Define the functional \mathcal{L} on $\mathcal{D}(\Omega)$ by

$$\mathcal{L}(\zeta) = \int_{\Omega} b(v_2) \chi_{\{v_2 < 0\}} \zeta - \chi_{\{0 > v_2\}} f_2 \zeta + \int_{\Omega} \chi_{\{0 > v_2\}} a(v_2, \nabla_x g(v_2)) \cdot \nabla_x \zeta. \tag{3.17}$$

As v_2 is an entropy solution, we have $\mathcal{L}(\zeta) \geq 0$ for all $\zeta \in \mathcal{D}(\Omega)$, $\zeta \geq 0$, a.e. \mathcal{L} is a positive linear functional on $\mathcal{D}(\Omega)$. Since $(\hat{\zeta})_n = (\xi\sigma_n)_n \subset \mathcal{D}(\Omega)$ is an increasing sequence satisfying $0 \leq \xi\sigma_n \leq \xi\varphi$, $\mathcal{L}(\hat{\zeta}_n)$ is a bounded and increasing sequence and thus converges. As a consequence, $I_2 + I_4 + I_6 = \mathcal{L}(\xi\varphi\sigma_n)$ converges as $n \rightarrow \infty$.

To estimate the first term in the left hand side of (3.15), we use the additional hypothesis (1.4) on the vector field a :

$$\begin{aligned} & \int_{(\Omega \setminus \Omega_1) \times (\Omega \setminus \Omega_2)} (a(v_1^+, \nabla_y g(v_1^+) - a(v_2^+, \nabla_x g(v_2^+))) \cdot (\nabla_y g(v_1^+) - \nabla_x g(v_2^+)) \\ & \times H'_\delta(g(v_1^+) - g(v_2^+)) \zeta_n \\ & \geq -\frac{1}{\delta} \int_{(\Omega \setminus \Omega_1) \times (\Omega \setminus \Omega_2)} \zeta_n B(v_1^+, v_2^+) \times (1 + |\nabla_y g(v_1^+)|^p + |\nabla_x g(v_2^+)|^p) |v_1^+ - v_2^+| \\ & \quad \times \chi_{\{0 \leq g(v_1^+) - g(v_2^+) \leq \delta\}} \\ & \quad + \frac{1}{\delta} \int_{(\Omega \setminus \Omega_1) \times (\Omega \setminus \Omega_2)} \zeta_n \Gamma_1(v_1^+, v_2^+) \cdot \nabla_y g(v_1^+) \chi_{\{0 \leq g(v_1^+) - g(v_2^+) \leq \delta\}} \\ & \quad + \frac{1}{\delta} \int_{(\Omega \setminus \Omega_1) \times (\Omega \setminus \Omega_2)} \zeta_n \Gamma_2(v_1^+, v_2^+) \cdot \nabla_x g(v_2^+) \chi_{\{0 \leq g(v_1^+) - g(v_2^+) \leq \delta\}}. \end{aligned} \tag{3.18}$$

The two last terms in the right hand side of (3.18) can be estimated as follows

$$\begin{aligned} & \frac{1}{\delta} \int_{(\Omega \setminus \Omega_1) \times (\Omega \setminus \Omega_2)} \zeta_n \Gamma_1(v_1^+, v_2^+) \cdot \nabla_y g(v_1^+) \chi_{\{0 \leq g(v_1^+) - g(v_2^+) \leq \delta\}} \\ & = \int_{(\Omega \setminus \Omega_1) \times (\Omega \setminus \Omega_2)} \left(\int_0^{\gamma(v_1, v_2)} \Gamma_1((g^{-1})_0(g(v_2^+) + \delta r), (g^{-1})_0(g(v_2^+))) dr \right) \nabla_y \zeta_n \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{\delta} \int_{(\Omega \setminus \Omega_1) \times (\Omega \setminus \Omega_2)} \zeta_n \Gamma_2(v_1^+, v_2^+) \cdot \nabla_x g(v_2^+) \chi_{\{0 \leq g(v_1^+) - g(v_2^+) \leq \delta\}} \\ & = \int_{(\Omega \setminus \Omega_1) \times (\Omega \setminus \Omega_2)} \left(\int_0^{\gamma(v_1, v_2)} \Gamma_2((g^{-1})_0(g(v_1^+)), (g^{-1})_0(g(v_1^+) - \delta r)) dr \right) \nabla_x \zeta_n, \end{aligned}$$

where

$$\gamma(v_1, v_2) := \inf(g(v_1^+) - g(v_2^+))^+ / \delta, 1).$$

Due to the continuity of $\Gamma((g^{-1})_0(r), \xi)$ in $r, g(r) \notin E$, it follows that the two terms converge to 0 with δ . In order to estimate the remaining term, we use the estimation

$$|r - s| = |(g^{-1})_0(g(r)) - (g^{-1})_0(g(s))| \leq C|g(r) - g(s)|, \quad g(r) \notin E, g(s) \notin E$$

where C is the Lipschitz constant of $(g^{-1})_0$ on $\{r \in \mathbb{R}, b(r) \notin E, |r| \leq |g(v_1) + g(v_2)|\}$. Then, we have

$$\begin{aligned} & - \lim_{\delta \rightarrow 0} \frac{1}{\delta} \int_{(\Omega \setminus \Omega_1) \times (\Omega \setminus \Omega_2)} \zeta_n B(v_1^+, v_2^+) \times (1 + |\nabla_y g(v_1^+)|^p + |\nabla_x g(v_2^+)|^p) |v_1^+ - v_2^+| \\ & \quad \times \chi_{\{0 \leq g(v_1^+) - g(v_2^+)\}} \\ & \geq -C \lim_{\delta \rightarrow 0} \int_{(\Omega \setminus \Omega_1) \times (\Omega \setminus \Omega_2)} \zeta_n B(v_1^+, v_2^+) \times (1 + |\nabla_y g(v_1^+)|^p + |\nabla_x g(v_2^+)|^p) \\ & \quad \times \chi_{\{0 \leq g(v_1^+) - g(v_2^+)\}} = 0. \end{aligned}$$

Using similar arguments, we prove that

$$\begin{aligned} & - \lim_{\delta \rightarrow 0} \int_{(\Omega \setminus \Omega_1) \times (\Omega \setminus \Omega_2)} (a(v_1^+, 0) - a(v_2^+, 0)) \cdot (\nabla_y g(v_1^+) - \nabla_x g(v_2^+)) \\ & \times H'_\delta(g(v_1^+) - g(v_2^+)) \zeta_n = 0. \end{aligned}$$

Combining the estimates of I_1, \dots, I_6 , we get

$$\begin{aligned} & \int_{\Omega} (b(v_1^+) - b(v_2^+))^+ \xi \wp \\ & \leq \int_{\Omega} \kappa_1 \chi_{\{v_1 > 0\}} (f_1 - \chi_{\{v_2 \geq 0\}} f_2) \xi \wp + \lim_{n \rightarrow \infty} \mathcal{L}(\xi \wp \sigma_n) \\ & \quad - \int_{\Omega} (a(v_1^+, \nabla_x g(v_1^+)) - a(v_2^+, \nabla g(v_2^+))) \cdot \nabla_x (\xi \wp) \chi_{\{v_1^+ > v_2^+\}}. \end{aligned} \quad (3.19)$$

This is “half” of the inequality to be proved.

Second inequality: In view of Remark 2.3, inequality (3.19) is still true when v_1 is replaced by $-v_2$, v_2 is replaced by $-v_1$, f_1 by $-f_2$, f_2 by $-f_1$, b by \bar{b} , g by \bar{g} and a by \bar{a} . Then we have

$$\begin{aligned} & \int_{\Omega} (b(v_1^-) - b(v_2^-))^+ \xi \wp_i \\ & \leq \int_{\Omega} \kappa_2 \chi_{\{v_2 < 0\}} (\chi_{\{v_1 \leq 0\}} f_1 - f_2) \xi \wp_i \\ & \quad - \int_{\Omega} \chi_{\{v_1^- \geq v_2^-\}} (a(v_1^-, \nabla g(v_1^-)) - a(v_2^-, \nabla g(v_2^-))) \cdot \nabla_x (\xi \wp_i) + \lim_{n \rightarrow \infty} \mathcal{L}(\xi \sigma_n \wp_i), \end{aligned} \quad (3.20)$$

where

$$\mathcal{L}(\xi) := \int_{\Omega} (b(v_1))^+ \zeta + \int_{\Omega} \chi_{\{v_1 > 0\}} \{a(v_1, \nabla g(v_1)) \cdot \nabla_y \zeta + f_1 \zeta\}.$$

Using the same arguments as above, we can prove that $(\mathcal{L}(\xi \sigma_n \wp_i))$ converges (as $\mathcal{L}(\xi \sigma_n \wp_i)$) with n .

Therefore, summation of (3.19) and (3.20) yields

$$\begin{aligned} & \int_{\Omega} (b(v_1) - b(v_2))^+ \xi \wp_i \\ & \leq \int_{\Omega} \kappa (f_1 - f_2) \xi \wp_i - \int_{\Omega} \chi_{\{v_1 \geq v_2\}} (a(v_1, \nabla g(v_1)) - a(v_2, \nabla g(v_2))) \cdot \nabla_x (\xi \wp_i) \\ & \quad + \lim_{n \rightarrow \infty} \mathcal{L}(\xi \wp_i \sigma_n) + \lim_{n \rightarrow \infty} \mathcal{L}(\xi \wp_i \sigma_n), \end{aligned} \quad (3.21)$$

for any $\xi \in \mathcal{D}(\mathbb{R}^N)$, $\xi \geq 0$, for all $i \in \{1, \dots, m\}$.

Remark 3.3. The method of doubling variables allows to prove the following local comparison result: for all $\xi \in \mathcal{D}(\Omega)$, here exists $\kappa \in L^\infty(\Omega)$ with $\kappa \in \text{sign}^+(v_1 - v_2)$ a.e. in Ω such that, for any $\zeta \in \mathcal{D}(\Omega)$, $\zeta \geq 0$,

$$\begin{aligned} & \int_{\Omega} (b(v_1) - b(v_2))^+ \zeta + \int_{\Omega} \chi_{\{v_1 > v_2\}} (a(v_1, \nabla g(v_1)) - a(v_2, \nabla g(v_2))) \cdot \nabla \zeta \\ & \leq \int_{\Omega} \kappa (f_1 - f_2) \zeta. \end{aligned} \quad (3.22)$$

The proof in this case is easier as the global comparison result. Indeed, as $\xi = 0$ on Γ , we can choose $k = v_2(x)$ (resp $k = v_1(s, x)$) in (2.3) (resp in 2.4) and we have only to add the obtained inequalities, then to go to the limit on n in order to get (3.22).

As $\xi = \xi(1 - \sigma_m) + \xi\sigma_m$ and $\xi\sigma_m \in \mathcal{D}(\Omega)$ for m sufficiently large, applying the local comparison principle (3.22) with $\zeta = \xi\sigma_m$, the global estimate (3.21) with $\xi(1 - \sigma_m)$, we obtain

$$\begin{aligned} & - \int_{\Omega} (b(v_1) - b(v_2))^+ \xi \wp_i - \chi_{\{v_1 \geq v_2\}} (a(v_1, \nabla g(v_1)) - a(v_2, \nabla g(v_2))) \cdot \nabla_x (\xi \wp_i) \\ & \geq \int_{\Omega} (b(v_1) - b(v_2))^+ (\xi(1 - \sigma_m)) \wp_i + \int_{\Omega} \kappa(f_1 - f_2) \xi(1 - \sigma_m) \wp_i \\ & \quad - \int_{\Omega} \chi_{\{v_1 \geq v_2\}} (a(v_1, \nabla g(v_1)) - a(v_2, \nabla g(v_2))) \cdot \nabla_x (\xi(1 - \sigma_m) \wp_i) \\ & \geq - \lim_{n \rightarrow \infty} \mathcal{L}(\xi \wp_i (1 - \sigma_m) \sigma_n) - \lim_{n \rightarrow \infty} \mathcal{L}(\xi \wp_i (1 - \sigma_m) \sigma_n) \\ & = - \lim_{n \rightarrow \infty} \mathcal{L}(\xi \wp_i (\sigma_n - \sigma_m \sigma_n)) - \lim_{n \rightarrow \infty} \mathcal{L}(\xi \wp_i (\sigma_n - \sigma_m \sigma_n)). \end{aligned}$$

Note that $\wp_i \sigma_n \sigma_m = \wp_i \sigma_m$ for n sufficiently large. Therefore,

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \mathcal{L}(\xi \wp_i (\sigma_n - \sigma_m \sigma_n)) = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \mathcal{L}(\xi \wp_i (\sigma_n - \sigma_m \sigma_n)) = 0,$$

and thus, passing to the limit with $m \rightarrow \infty$ in the preceding inequality yields

$$\begin{aligned} & \int_{\Omega} (b(v_1) - b(v_2))^+ \xi \wp_i + \chi_{\{v_1 \geq v_2\}} (a(v_1, \nabla g(v_1)) - a(v_2, \nabla g(v_2))) \cdot \nabla_x (\xi \wp_i) \\ & \leq \int_{\Omega} \kappa(f_1 - f_2) \xi \wp_i \end{aligned}$$

After summation over i , we deduce (3.1). \square

4. EXISTENCE OF ENTROPY SOLUTION

The proof of the existence result consists of two steps. In a first step, we prove existence of a bounded entropy solution of the problem

$$\begin{aligned} b_{\alpha}(v) - \operatorname{div} a(v, \nabla g(v)) &= f \quad \text{in } \Omega \\ g(v) &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{4.1}$$

where $f \in L^1(\Omega)$ and b_{α} is an increasing Lipschitz continuous function on \mathbb{R} such that $b_{\alpha}(0) = 0$ and $\lim_{\alpha \rightarrow 0} b_{\alpha}(r) = b(r)$, for all $r \in \mathbb{R}$.

This is done via approximation with the elliptic-parabolic problems with homogeneous boundary conditions:

$$\begin{aligned} b_{\alpha}(v) - \operatorname{div} a(v, \nabla g_{\varepsilon}(v)) &= f \quad \text{in } \Omega \\ v &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{4.2}$$

where $g_{\varepsilon}(r) = g(r) + \varepsilon r$. In the second step, we pass to the limit with α to 0 and prove the existence result for L^1 -data.

4.1. First step.

Proposition 4.1. *For all $\varepsilon > 0$ and $f \in L^\infty(\Omega)$, there exists a unique $v \in L^\infty(\Omega)$ entropy solution of (4.2) i.e. $v \in W_0^{1,p}(\Omega)$ and v satisfies the following entropy inequalities: For all $k \in \mathbb{R}$, for all $\xi \in C_0^\infty(\mathbb{R}^N)$ such that $\xi \geq 0$ and $\text{sign}^+(-k)\xi = 0$ a.e. on Γ ,*

$$\int_{\Omega} b_\alpha(v)\chi_{\{v>k\}}\xi \leq \int_{\Omega} \chi_{\{v>k\}}(f\xi - (a(v, \nabla g_\varepsilon(v)) - a(k, 0)) \cdot \nabla \xi) \quad (4.3)$$

and for all $k \in \mathbb{R}$, for all $\xi \in C_0^\infty(\mathbb{R}^N)$ such that $\xi \geq 0$ and $\text{sign}^+(k)\xi = 0$ a.e. on Γ ,

$$\int_{\Omega} -b_\alpha(v)\chi_{\{k>v\}}\xi \leq - \int_{\Omega} \chi_{\{k>v\}}(f\xi - (a(v, \nabla g_\varepsilon(v)) - a(k, 0)) \cdot \nabla \xi). \quad (4.4)$$

Proof. The existence of a unique weak solution v of (4.2) is already proved in [4]. Indeed the Problem can be equivalently formulated as follows:

$$\begin{aligned} (b_\alpha \circ g_\varepsilon^{-1})(v) - \text{div } a(g_\varepsilon^{-1}(v), \nabla v) &= f \quad \text{in } \Omega \\ v &= 0 \quad \text{on } \Gamma. \end{aligned} \quad (4.5)$$

As $(r, \xi) \mapsto a(g_\varepsilon^{-1}(v), \xi)$, $r \in \mathbb{R}$, $\xi \in \mathbb{R}^N$ satisfies the same hypothesis as the vector field a thanks to the strict monotonicity of g_ε , it is sufficient to apply the results of [18]. In order to prove that the week solution satisfies the entropy inequalities, we proceed as in [8]. \square

Proposition 4.2. *For all $f \in L^\infty(\Omega)$, there exists a unique $v \in L^\infty(\Omega)$ weak (and entropy) solution of (4.1) i.e. $g(v) \in W_0^{1,p}(\Omega)$ and v satisfies the following entropy inequalities: For all $k \in \mathbb{R}$, for all $\xi \in C_0^\infty(\mathbb{R}^N)$ such that $\xi \geq 0$ and $\text{sign}^+(-g(k))\xi = 0$ a.e. on Γ ,*

$$\int_{\Omega} b_\alpha(v)\chi_{\{v>k\}}\xi \leq \int_{\Omega} \chi_{\{v>k\}}(f\xi - (a(v, \nabla g(v)) - a(k, 0)) \cdot \nabla \xi) \quad (4.6)$$

and for all $k \in \mathbb{R}$, for all $\xi \in C_0^\infty(\mathbb{R}^N)$ such that $\xi \geq 0$ and $\text{sign}^+(g(k))\xi = 0$ a.e. on Γ ,

$$\int_{\Omega} -b_\alpha(v)\chi_{\{k>v\}}\xi \leq - \int_{\Omega} \chi_{\{k>v\}}(f\xi - (a(v, \nabla g(v)) - a(k, 0)) \cdot \nabla \xi). \quad (4.7)$$

Proof. According to Proposition 4.1, for $f \in L^\infty(\Omega)$, there exists a unique $v_\varepsilon \in L^\infty(\Omega)$ entropy solution of (4.2). i.e. $v_\varepsilon \in L^\infty(\Omega)$, $g_\varepsilon(v_\varepsilon) \in W_0^{1,p}(\Omega)$ and v_ε satisfies the entropy inequalities (4.3) and (4.4):

With a particular choice of test functions and thanks to the strict monotonicity of b_α , one can prove that $(v_\varepsilon)_\varepsilon$ and $(|\nabla g_\varepsilon(v_\varepsilon)|)_\varepsilon$ are uniformly bounded in $L^\infty(\Omega)$ and $L^p(\Omega)$ respectively. Thanks to the growth condition (1.2) on a , it follows that $(a(v_\varepsilon, \nabla g_\varepsilon(v_\varepsilon)))_\varepsilon$ is bounded in $L^{p'}(\Omega)^N$ as well. Following classical arguments, extracting a subsequence if necessary, we can prove that as $\varepsilon \rightarrow 0$,

$$g(v_\varepsilon) \text{ converges to some } w \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$$

weakly in $W_0^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$. Moreover,

$$a(v_\varepsilon, \nabla g_\varepsilon(v_\varepsilon)) \text{ converges weakly in } L^{p'}(\Omega)^N \text{ to some } \chi \in L^{p'}(\Omega)^N.$$

In order to prove the strong convergence of v_ε (in L_{Loc}^1 for example) to some v , we can use the method of compensated compactness (see [15] and [16]) but this

requires some additional conditions on the flux function Φ . An other approach consists in using the L^∞ uniform bound on (v_ε) in order to deduce the weak-* convergence of (v_ε) to a function v . Then, going to the limit in the approximate entropy inequalities, we prove that v is an entropy process solution of (4.1) (see Definition 4.5 below). Finally using a “stronger” principle of uniqueness, we show that v is the entropy solution of (4.1) and that the convergence holds strongly in $L^1(\Omega)$. \square

Definition 4.3. Let Ω be an open subset of \mathbb{R}^N ($N \geq 1$), (u_n) be a bounded sequence of $L^\infty(\Omega)$ and $u \in L^\infty(\Omega \times (0, 1))$. The sequence (u_n) converges towards u in the “nonlinear weak-* sense” if

$$\int_{\Omega} g(u_n(x))\psi(x) dx \rightarrow \int_0^1 \int_{\Omega} g(u(x, \mu))\psi(x) dx d\mu, \quad \text{as } n \rightarrow \infty, \quad (4.8)$$

for all $\psi \in L^1(\Omega)$, for all $g \in \mathcal{C}(\mathbb{R}, \mathbb{R})$.

Lemma 4.4. Let Ω be an open subset of \mathbb{R}^N ($N \geq 1$) and (u_n) be a bounded sequence of $L^\infty(\Omega)$. Then (u_n) admits a subsequence converging in the nonlinear weak-* sense.

For the proof of the above lemma see [17, 11]. According to Lemma 4.4, the sequence (v_ε) is convergent in the nonlinear weak-* sense to some $v \in L^\infty(\Omega \times (0, 1))$. We will prove that v is a weak entropy process solution of (4.1) in the following sense.

Definition 4.5. Let $u \in L^\infty((0, 1) \times \Omega)$ with $g(u) \in W_0^{1,p}(\Omega)$. The function u is a weak entropy process solution of (4.1) if for all $k \in \mathbb{R}$, for all $\xi \in C_0^\infty(\mathbb{R}^N)$ such that $\xi \geq 0$ and $\text{sign}^+(-g(k))\xi = 0$ a.e. on $\partial\Omega$,

$$\int_0^1 \int_{\Omega} b_\alpha(u)\chi_{\{u>k\}}\xi d\mu \leq \int_0^1 \int_{\Omega} \chi_{\{u>k\}}(f\xi - (a(u, \nabla g(u)) - a(k, 0)) \cdot \nabla \xi) d\mu \quad (4.9)$$

and for all $k \in \mathbb{R}$, for all $\xi \in C_0^\infty(\mathbb{R}^N)$ such that $\xi \geq 0$ and $\text{sign}^+(g(k))\xi = 0$ a.e. on $\tilde{\Sigma}$,

$$-\int_0^1 \int_{\Omega} b_\alpha(u)\chi_{\{k>u\}}\xi d\mu \leq -\int_0^1 \int_{\Omega} \chi_{\{k>u\}}(f\xi - (a(u, \nabla g(u)) - a(k, 0)) \cdot \nabla \xi) d\mu. \quad (4.10)$$

Taking into account the above estimates, it follows that

$$g(v_\varepsilon) \text{ converges to } g(v) \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega) \quad (4.11)$$

strongly in $L^p(\Omega)$ and weakly in $W^{1,p}(\Omega)$. In particular, it follows that $g(v)$ is independent of μ .

To pass to the limit in (4.3) and (4.4), it remains to prove that

$$\int_{\Omega} a(v_\varepsilon, \nabla g_\varepsilon(v_\varepsilon)) \cdot \nabla \xi \rightarrow \int_0^1 \left(\int_{\Omega} a(v, \nabla g(v)) \cdot \nabla \xi \right) d\mu \quad (4.12)$$

By the Minty Browder argument, we have only to prove that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} a(v_\varepsilon, \nabla g(v_\varepsilon)) \cdot \nabla (g(v_\varepsilon) - g(v)) = 0.$$

As v_ε is also a weak solution of (4.2), we have

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\Omega} a(v_\varepsilon, \nabla g(v_\varepsilon)) \cdot \nabla (g(v_\varepsilon) - g(v)) \\ &= - \lim_{\varepsilon \rightarrow 0} \left[\int_{\Omega} b(v_\varepsilon)(g(v_\varepsilon) - g(v)) + \int_{\Omega} f(g(v_\varepsilon) - g(v)) \right] = 0 \end{aligned}$$

where the last equality follows by the strong convergence in $L^p(\Omega)$ of $g(v_\varepsilon)$ to $g(v)$ and the weak*-convergence of v_ε to v . By the standard pseudo-monotonicity argument it follows that

$$\int_{\Omega} \chi \cdot \nabla \xi = \int_{\Omega} a(v, \nabla g(v)) \cdot \nabla \xi \quad \text{for all } \xi \in \mathcal{D}(\Omega). \quad (4.13)$$

Indeed, for $\xi \in \mathcal{D}(\Omega)$, $\xi \geq 0$, $\alpha \in \mathbb{R}$, we have

$$\begin{aligned} \alpha \int_{\Omega} \chi \nabla \xi &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \alpha a(v_\varepsilon, \nabla g_\varepsilon(v_\varepsilon)) \cdot \nabla \xi \\ &\geq \limsup_{\varepsilon \rightarrow 0} \int_{\Omega} a(v_\varepsilon, \nabla g_\varepsilon(v_\varepsilon)) \cdot \nabla (g_\varepsilon(v_\varepsilon) - g(v) + \alpha \xi) \\ &\geq \limsup_{\varepsilon \rightarrow 0} \int_{\Omega} a(v_\varepsilon, \nabla (g(v) - \alpha \xi)) \cdot \nabla (g_\varepsilon(v_\varepsilon) - g(v) + \alpha \xi) \\ &\geq \int_{\Omega} \alpha a(v, \nabla (g(v) - \alpha \xi)) \cdot \nabla \xi. \end{aligned}$$

Dividing by $\alpha > 0$ (resp. $\alpha < 0$), passing to the limit with $\alpha \rightarrow 0$, we obtain (4.13). We can now pass to the limit in (4.3) and (4.4) to get for all $k \in \mathbb{R}$, for all $\xi \in C_0^\infty(\mathbb{R}^N)$ such that $\xi \geq 0$ and $\text{sign}^+(-g(k))\xi = 0$ a.e. on Γ ,

$$\int_0^1 \int_{\Omega} b_\alpha(v) \chi_{\{v > k\}} \xi \leq \int_0^1 \int_{\Omega} \chi_{\{v > k\}} (f \xi - (a(v, \nabla g(v)) - a(k, 0)) \cdot \nabla \xi) \quad (4.14)$$

and for all $k \in \mathbb{R}$, for all $\xi \in C_0^\infty(\mathbb{R}^N)$ such that $\xi \geq 0$ and $\text{sign}^+(g(k))\xi = 0$ a.e. on Γ ,

$$\int_0^1 \int_{\Omega} -b_\alpha(v) \chi_{\{k > v\}} \xi \leq - \int_0^1 \int_{\Omega} \chi_{\{k > v\}} (f \xi + (a(v, \nabla g(v)) - a(k, 0)) \cdot \nabla \xi). \quad (4.15)$$

Hence we have shown that v is a weak entropy process solution of (4.1). Now, to prove that v is the weak entropy solution of (4.1), we use the following ‘‘reinforced’’ comparison principle.

Proposition 4.6. *Let $f_i \in L^\infty(\Omega)$ and $v_i \in L^\infty(\Omega \times (0, 1))$ be a weak entropy process of $P_{b_\alpha, g}(f_i)$ $i = 1, 2$. Then there exists $\kappa \in L^\infty(\Omega \times (0, 1))$ with $\kappa \in \text{sign}^+(v_1 - v_2)$ a.e. in $\Omega \times (0, 1)$ such that*

$$\int_0^1 \int_{\Omega} (b_\alpha(v_1(x, \alpha)) - b_\alpha(v_2(x, \mu)))^+ \xi \, dx \, d\alpha \, d\mu \leq \int_0^1 \int_{\Omega} \kappa (f_1 - f_2) \xi \, dx.$$

In particular, when $f_1 = f_2$, we have

$$v_1(x, \alpha) = v_2(x, \mu) \quad \text{for a.e. } (x, \alpha, \mu) \in \Omega \times (0, 1) \times (0, 1).$$

Defining the function $w(x) = \int_0^1 v_1(x, \alpha) \, d\alpha$, we deduce that $w(x) = v_1(x, \alpha) = v_2(x, \beta)$ for a.e. $(x, \alpha, \beta) \in \Omega \times (0, 1) \times (0, 1)$.

The proof of Proposition 4.6 follows the same lines as those of Theorem 3.1 and is omitted. The reader is referred among others to [21] and [11] in order to verify

the technical tools which are necessary to deal with measure-valued functions. The result of Proposition 4.6 implies that v is the unique weak entropy solution of (4.1) and the first step of the proof is complete.

4.2. Second step. The comparison principle is again the main tool in this last step: Let $f \in L^1(\Omega)$. For $m, n \in \mathbb{N}$, let $f_{m,n} = f \wedge m \vee (-n)$ and define $b_{m,n} : r \mapsto b(r) + \frac{1}{m}r^+ - \frac{1}{n}r^-$. Denote by $v_{m,n}$ the unique weak entropy solution of $P_{b_{m,n},g}(f_{m,n})$ (which exists by the result of the first step). Then,

$$0 \leq \int_{\Omega} -\chi_{\{v_{m,n} > k\}} \{ (a(v_{m,n}, \nabla g(v_{m,n})) - a(k, 0)) \cdot \nabla \xi + f_{m,n} \xi - b_{m,n}(v_{m,n}) \xi \} \quad (4.16)$$

for any $\xi \in \mathcal{D}(\mathbb{R}^N)$, $\xi \geq 0$, for all $k \in \mathbb{R}$ such that $\text{sign}^+(-g(k))\xi = 0$ on Γ ,

$$0 \leq \int_{\Omega} \chi_{\{k > v_{m,n}\}} \{ (a(v_{m,n}, \nabla g(v_{m,n})) - a(k, 0)) \cdot \nabla \xi - f_{m,n} \xi + b_{m,n}(v_{m,n}) \xi \} \quad (4.17)$$

for any $\xi \in \mathcal{D}(\mathbb{R}^N)$, $\xi \geq 0$, for all $k \in \mathbb{R}$ such that $\text{sign}^+(g(k))\xi = 0$ on Γ .

By Theorem 3.1, there exists $\kappa_{m_1, m_2} \in L^\infty(\Omega)$ and $\kappa_{n_1, n_2} \in L^\infty(\Omega)$ with $\kappa_{m_1, m_2} \in \text{sign}^+(v_{m_1, n} - v_{m_2, n})$, $\kappa_{n_1, n_2} \in \text{sign}^+(v_{m, n_1} - v_{m, n_2})$ such that, for all $\xi \in \mathcal{D}^+(\mathbb{R}^N)$, $\xi \geq 0$,

$$\begin{aligned} & \int_{\Omega} \left(\frac{1}{m_2} (v_{m_1, n}^+) - \frac{1}{m_2} (v_{m_2, n}^+) \right)^+ \xi + \frac{1}{n} (-v_{m_1, n}^- + v_{m_2, n}^-)^+ \xi \\ & \leq - \int_{\Omega} (b(v_{m_1, n}) - b(v_{m_2, n}))^+ \xi + \int_{\Omega} \kappa_{m_1, m_2} \left(\frac{1}{m_2} - \frac{1}{m_1} \right) v_{m_1, n}^+ \xi \\ & \quad - \int_{\Omega} \chi_{\{v_{m_1, n} > v_{m_2, n}\}} (a(v_{m_1, n}, \nabla g(v_{m_1, n})) - a(v_{m_2, n}, \nabla g(v_{m_2, n}))) \cdot \nabla \xi. \end{aligned} \quad (4.18)$$

and

$$\begin{aligned} & \int_{\Omega} \left(\frac{1}{n_2} v_{m, n_1}^- - \frac{1}{n_2} v_{m, n_2}^- \right)^+ \xi + \frac{1}{m} (v_{m, n_1}^+ - v_{m, n_2}^+)^+ \xi \\ & \leq \int_{\Omega} -(b(v_{m, n_1}) - b(v_{m, n_2}))^+ \xi - \int_{\Omega} \kappa_{n_1, n_2} \left(\frac{1}{n_2} - \frac{1}{n_1} \right) v_{m, n_1}^- \xi \\ & \quad + \int_{\Omega} \chi_{\{v_{m, n_1} > v_{m, n_2}\}} (a(v_{m, n_1}, \nabla g(v_{m, n_1})) - a(v_{m, n_2}, \nabla g(v_{m, n_2}))) \cdot \nabla \xi. \end{aligned} \quad (4.19)$$

This yields that $v_{m_1, n} \leq v_{m_2, n}$ for $m_1 \leq m_2$ and $v_{m, n_1} \leq v_{m, n_2}$ for $n_1 \geq n_2$. Therefore, $v_{m,n} \uparrow_m v_n$ a.e. on Ω where $v_n : \Omega \rightarrow \mathbb{R}$ is a measurable function. Here, we use the notation \uparrow_n resp. \downarrow_n to denote convergence of a sequence which is monotone increasing, resp. decreasing in n . Moreover, from (4.18) and (4.19), it follows that

$$b(v_{m,n})_m \rightarrow b(v_n) \quad \text{in } L^1(\Omega). \quad (4.20)$$

Applying a diagonal argument, we may assume that for some subsequence $(m(n))_n$ we have

$$v_{m(n),n} \rightarrow v_n \quad \text{a.e. in } \Omega, \quad b(v_{m(n),n}) \rightarrow b(v_n) \quad \text{in } L^1(\Omega). \quad (4.21)$$

where v_n is the weak entropy solution of $P_{b_n, g}(f_n)$ with $b_n := b_{m(n),n}$, $f_n := f_{m(n),n}$. Next, we prove that v_n is finite a.e. in Ω : Suppose first that $b(+\infty) := \lim_{r \rightarrow +\infty} b(r) < \infty$. Then, by the Range condition, it follows that $\lim_{r \rightarrow +\infty} g(r) = \infty$.

As $v_{m,n}$ is a weak solution of $P_{b_{m,n},g}(f_{m,n})$, choosing $g(T_k(v_{m,n}^+))$ as test function, taking into account the growth condition on a , we find

$$\lambda_{b(+\infty)} \int_{\Omega} |\nabla g(T_k v_{m,n}^+)|^p \leq M_{b(+\infty)} + g(k) \int_{\Omega} |f_{m,n}|$$

(see condition (1.2) on a). Hence, by Poincaré’s inequality,

$$|\{v_{m,n}^+ \geq k\}| \leq \frac{C(1 + g(k))}{g(k)^p}$$

for some constant C independent of m, n and k . Passing the limit with $m \rightarrow \infty$ and then with $k \rightarrow \infty$ in the above inequality, we find that v_n is finite a.e. on Ω . In the case where $b(+\infty) = +\infty$, the last assertion follows from (4.20). Using $T_k g(v_{m,n})$ as test function in the weak formulation, by the coerciveness assumption on a , we obtain

$$\int_{\Omega} |\nabla T_k g(v_{m,n})|^p \leq C(k),$$

for a constant $C(k)$ depending only on k . Therefore, we can assume that the sequence $(T_k g(v_{m(n),n}))_n$ converges weakly in $W_0^{1,p}(\Omega)$ to $T_k g(v_n)$. Going to the limit with $n \rightarrow \infty$, proceeding as above, we can extract a subsequence still denoted $(v_n)_n$ such that

$$v_n \rightarrow v \quad \text{a.e. in } \Omega, \quad b(v_n) \rightarrow b(v) \quad \text{in } L^1(\Omega)$$

where v is finite a.e. in Ω . Moreover, $T_k g(v) \in W_0^{1,p}(\Omega)$ and $(T_k g(v_n))_n$ converges weakly in $W_0^{1,p}(\Omega)$ to $T_k g(v)$. Applying again the argument of Minty Browder, we can prove for our diagonal sequence that $a(T_k v_n, \nabla g(T_k v_n)) \rightarrow a(T_k v, \nabla g(T_k v))$ weakly in $(L^{p'}(\Omega))^N$. It remains only to prove the inequalities (2.1) and (2.2). To this end, let us first verify that v_n satisfies (2.1) and (2.2) for all $n \in \mathbb{N}$: For all $k \in \mathbb{R}$, for all $l \geq k$, for any $\xi \in \mathcal{D}(\mathbb{R}^N)$, $\xi \geq 0$, we have

$$\begin{aligned} & \int_{\Omega} -b_n(v_n \wedge l) \chi_{\{v_n \wedge l > k\}} \xi + \chi_{\{v_n \wedge l > k\}} f_n \xi \\ & - \chi_{\{v_n \wedge l > k\}} (a(v_n \wedge l, \nabla g(v_n \wedge l)) - a(k, 0)) \cdot \nabla \xi \\ & = \int_{\Omega} \chi_{\{v_n > k\}} \{-(b_n(v_n) - f_n) \xi - (a(v_n, \nabla g(v_n)) - a(k, 0)) \cdot \nabla \xi\} \\ & \quad + \int_{\Omega} \chi_{\{v_n > l\}} (b_n(v_n) - b_n(l) - f_n) \xi + (a(v_n, \nabla g(v_n)) - a(l, 0)) \cdot \nabla \xi + f_n^- \\ & \geq \int_{\Omega} \chi_{\{v_n > l\}} \{(b_n(v_n) - b_n(l) + f_n) \xi + (a(v_n, \nabla g(v_n)) - a(l, 0)) \cdot \nabla \xi - f_n^- \xi\}, \end{aligned}$$

Let

$$\langle \mu_l^n, \xi \rangle := - \int_{\Omega} \chi_{\{v_n > l\}} \{(b_n(v_n) - b_n(l)) \xi + f_n \xi + (a(v_n, \nabla g(v_n)) - a(l, 0)) \cdot \nabla \xi - f_n^- \xi\}.$$

Then, μ_l^n is a non-negative measure on $\bar{\Omega}$ and $\mu_l^n \equiv 0$ for $l \geq \|v_n\|_{L^\infty(\Omega)}$. Moreover,

$$\|\mu_l^n\| \leq \int_{\Omega} |f_n| \chi_{\{v_n > l\}}.$$

Working on the second entropy inequality, we construct a family of bounded non-negative measures $(\nu_l^n)_l$ on $\bar{\Omega}$

$$\langle \nu_l^n, \xi \rangle := - \int_{\Omega} \chi_{\{l > v_n\}} \{ (b_n(l) - (b_n(v_n))) \xi - f_n^+ \xi - f_n \xi + (a(l, 0) - a(v_n, \nabla g(v_n))) \cdot \nabla \xi \}$$

such that

$$\int_{\Omega} \chi_{\{k > v_n \vee l\}} \{ b_n(v_n \vee l) \xi - f_n \xi + a(v_n \vee l, \nabla g(v_n \vee l)) \cdot \nabla \xi \} \geq - \langle \nu_l^n, \xi \rangle$$

for all $\xi \in \mathcal{D}^+(\Omega)$ and $k \in \mathbb{R}$ with $(g(k))^+ \xi = 0$ on Γ and $\|\nu_l^n\| \leq \int_{\Omega} |f_n| \chi_{\{v_n < l\}}$. It follows that $(\mu_l^n)_n$ and $(\nu_l^n)_n$ are uniformly bounded with respect to n . Therefore, we can extract two subsequences still denoted by $(\mu_l^n)_n$ and $(\nu_l^n)_n$ which are convergent with respect to the weak- $*$ topology on $C(\bar{\Omega})$ to μ_l and ν_l respectively. Now, combining all the estimates on the sequence $(v_n)_n$, we can pass the limit in the above inequalities to (2.1) and (2.2). The measures μ_l and ν_l are defined as follows:

$$\begin{aligned} \langle \mu_l, \xi \rangle &:= - \int_{\Omega} \chi_{\{v > l\}} \{ (b(v) - b(l)) \xi - f \xi + (a(v, Dg(v)) - a(l, 0)) \cdot \nabla \xi - f^- \xi \}, \\ \langle \nu_l, \xi \rangle &:= - \int_{\Omega} \chi_{\{l > v\}} \{ (b(l) - b(v)) \xi + (a(l, 0) - a(v, Dg(v))) \cdot \nabla \xi + f \xi - f^+ \xi \}. \end{aligned}$$

Here, $Dg(v)$ is defined by $\chi_{\{-k < v < k\}} Dg(v) = \nabla g(T_k v)$ for all $k > 0$.

The uniqueness result in the L^1 setting follows from the following proposition.

Proposition 4.7. *Let $f_1 \in L^\infty(\Omega)$, $f_2 \in L^1(\Omega)$ and v_1, v_2 be an entropy solution and a renormalized entropy solution of (1.1) with f_1 instead of f , and (1.1) with f_2 in stead of f , respectively. Then, there exists $\kappa \in L^\infty(\Omega)$ with $\kappa \in \text{sign}^+(v_1 - v_2 \vee l)$ a.e. in Ω such that, for any $\zeta \in \mathcal{D}^+(\mathbb{R}^N)$,*

$$\begin{aligned} - \langle \nu_l, \zeta \rangle &\leq - \int_{\Omega} \chi_{\{v_1 > v_2 \vee l\}} (a(v_1, \nabla g(v_1)) - a(v_2 \vee l, \nabla g(v_2 \vee l))) \cdot \nabla \zeta \\ &\quad - \int_{\Omega} (b(v_1) - b(v_2 \vee l))^+ \zeta + \int_{\Omega} \kappa (f_1 - f_2) \zeta. \end{aligned} \tag{4.22}$$

For the proof of the above proposition can be found in [3]. Let us show how to deduce uniqueness of the renormalized entropy solution: Let v be a renormalized entropy solution of (1.1) and v_n be the entropy solution of $P_{b,g}(f_n)$ constructed above. Then by Proposition 4.7, there exists $\kappa_n \in L^\infty(\Omega)$ with $\kappa_n \in \text{sign}^+(v_n - v \vee l_n)$ a.e. in Ω such that, for any $\zeta \in \mathcal{D}(\mathbb{R}^N)$, $\zeta \geq 0$, for any $l_n \geq n$,

$$\begin{aligned} - \langle \nu_{-l_n}, \zeta \rangle &\leq - \int_{\Omega} \chi_{\{v_n > v \vee (-l_n)\}} (a(v_n, \nabla g(v_n)) - a(v \vee (-l_n), \nabla g(v \vee (-l_n)))) \cdot \nabla \zeta \\ &\quad - \int_{\Omega} (b_n(v_n) - b_n(v \vee (-l_n)))^+ \zeta + \int_{\Omega} \kappa_n (f_n - f) \zeta. \end{aligned} \tag{4.23}$$

Similarly, we prove that there exists $\tilde{\kappa}_n \in L^\infty(\Omega)$ with $\tilde{\kappa}_n \in \text{sign}^+(v \wedge l_n - v_1)$ a.e. in Ω such that, for any $\zeta \in \mathcal{D}(\mathbb{R}^N)$, $\zeta \geq 0$, for any $l_n \geq n$,

$$\begin{aligned} - \langle \mu_{l_n}, \zeta \rangle &\leq \int_{\Omega} - \chi_{\{v_n < v \wedge l_n\}} (a(v \wedge l_n, \nabla g(v \wedge l_n)) - a(v_n, \nabla g(v_n))) \cdot \nabla \zeta \\ &\quad - \int_{\Omega} (b_n(v \wedge l_n) - b_n(v_n))^+ \zeta + \int_{\Omega} \tilde{\kappa}_n (f - f_n) \zeta. \end{aligned} \tag{4.24}$$

Summing up (4.23) and (4.24), letting $n \rightarrow +\infty$, we get $b(v) = \lim_{n \rightarrow +\infty} b(v_n)$.

Let us define the operator $A_{b,g}$ in $L^\infty(\Omega) \times L^\infty(\Omega) \subset L^1(\Omega) \times L^1(\Omega)$ by $(u, f) \in A_{b,g}$ if and only if there exists v measurable such that $b(v) = u$ and v is an entropy solution of $P_{b,g}(f + u)$.

Proposition 4.8. *Let b be strictly increasing with $b(0) = 0$. Then*

(i) *The operator $A_{b,g}$ is T -accretive in $L^1(\Omega)$; i.e., for all $(u_i, f_i) \in A_{b,g}$,*

$$\int_{\Omega} \kappa(f_1 - f_2) \geq 0 \quad \text{for some } \kappa \in \text{sign}(v_1 - v_2).$$

(ii) *For any $\alpha > 0$, $R(I + \alpha A_{b,g}) = L^\infty(\Omega)$,*

(iii) *$\overline{D(A_{b,g})}^{L^1(\Omega)} = \{u \in L^1(\Omega), u(x) \in \overline{R(b)} \text{ a.e. } x \in \Omega\}$*

Proof. (i) and (ii) are direct consequences of Theorem 3.1 and the existence result. To prove (iii), let $f \in L^\infty(\Omega)$ be such that $f \pm \epsilon \in R(b)$ and let v_h be an entropy solution of

$$\begin{aligned} b(v) - h \operatorname{div} a(v, \nabla g(v)) &= f \quad \text{in } \Omega \\ g(v) &= 0 \quad \text{on } \Gamma \end{aligned}$$

with $h > 0$. Then

$$\|b(v_h)\|_{L^q(\Omega)} \leq \|f\|_{L^q(\Omega)} \quad (4.25)$$

for every $1 \leq q \leq +\infty$. In particular, $\|v_h\|_{L^\infty(\Omega)} \leq C(f)$ and by the growth condition, it follows that $h \operatorname{div} a(v_h, \nabla g(v_h)) \rightarrow 0$ in $\mathcal{D}'(\Omega)$. Therefore, $b(v_h) \rightarrow f$ in $\mathcal{D}'(\Omega)$ and weakly in $L^p(\Omega)$. Whence, $\liminf_{h \rightarrow 0} \|b(v_h)\|_{L^p(\Omega)} \geq \|f\|_{L^p(\Omega)}$. Taking into account (4.25), we deduce that $b(v_h) \rightarrow f$ strongly in $L^1(\Omega)$. The proof is complete. \square

Remark 4.9. Proposition 4.8 allows to study the Cauchy problem associated to (1.1) from the point of view of semi-groups theory. The elliptic parabolic problem

$$b(v)_t - \operatorname{div} a(v, \nabla g(v)) = f \in (0, T) \times \Omega$$

with initial condition and general boundary condition will be treated by the first author in a forthcoming paper.

Corollary 4.10. *For every $f \in L^1((0, T) \times \Omega)$ and every $v_0 \in \overline{D(A_{b,g})}$, there exists a unique integral solution of*

$$\begin{aligned} u_t + A_{b,g}(u) &\ni f \\ v(0) &= v_0, \end{aligned} \quad (4.26)$$

with u in $C([0, T], L^1(\Omega))$. Moreover, a comparison principle holds.

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