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A NONHOMOGENEOUS BACKWARD HEAT PROBLEM: REGULARIZATION AND ERROR ESTIMATES

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Abstract. We consider the problem of finding the initial temperature, from the final temperature, in the nonhomogeneous heat equation

$$u_t - u_{xx} = f(x,t), \quad (x,t) \in (0,\pi) \times (0,T),$$
$$u(0,t) = u(\pi,t) = 0, \quad (x,t) \in (0,\pi) \times (0,T).$$

This problem is known as the backward heat problem and is severely ill-posed. Our goal is to present a simple and convenient regularization method, and sharp error estimates for its approximate solutions. We illustrate our results with a numerical example.

1. Introduction

For a positive number T, we consider the problem of finding the temperature $u(x,t)$, $(x,t) \in (0,\pi) \times [0,T]$, such that

$$u_t - u_{xx} = f(x,t), \quad (x,t) \in (0,\pi) \times (0,T), \tag{1.1}$$
$$u(0,t) = u(\pi,t), \quad (x,t) \in (0,\pi) \times [0,T], \tag{1.2}$$
$$u(x,T) = g(x), \quad x \in (0,\pi). \tag{1.3}$$

where $g(x), f(x,z)$ are given. The problem is called the backward heat problem, the backward Cauchy problem, or the final value problem. As is known, the nonhomogeneous problem is severely ill-posed; i.e., solutions do not always exist, and in the case of existence, these do not depend continuously on the given data. In fact, from small noise contaminated physical measurements, the corresponding solutions have large errors. It makes difficult to do numerical calculations. Hence, a regularization is in order. Lattes and Lions, in [17], regularized the problem by adding a "corrector" to the main equation. They considered the problem

$$u_t + Au - \epsilon A^* Au = 0, \quad 0 < t < T,$$
$$u(T) = \varphi.$$

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Gajewski and Zaccharias [10] considered a similar problem. Their error estimate for the approximate solutions is

$$\|u^\epsilon(t) - u(t)\|^2 \le \frac{2}{t^2}(T-t)\|u(0)\|.$$

Note that these estimate can not be used at time $t = 0$.

In 1983, Showalter, presented a different method called the quasiboundary value (QBV) method to regularize that linear homogeneous problem which gave a stability estimate better than the one of discussed method. The main ideas of the method is of adding an appropriate "corrector" into the final data. Using the method, Clark and Oppenheimer, in [4], and Denche-Bessila, very recently in [5], regularized the backward problem by replacing the final condition by

$$u(T) + \epsilon u(0) = g \tag{1.4}$$

and

$$u(T) - \epsilon u'(0) = g \tag{1.5}$$

respectively. Although there are many papers on the linear homogeneous case of the backward problem, we only find a few papers on the nonhomogeneous case, such in [28, 29].

In 2006, Trong and Tuan [28], approximated the problem (1.1)–(1.3) by the quasi-reversibility method. However, the stability magnitude of the method is of order $e^{\frac{T}{\epsilon}}$. Moreover, the error between the approximate problem and the exact solution is

$$\epsilon(T-t)\sqrt{\frac{8}{t^4}\|u(.,0)\|^2 + t^2\|\frac{\partial^4 f(x,t)}{\partial x^4}\|^2_{L^2(0,T;L^2(0,\pi))}},$$

(see [5, page 5]) which is very large when ϵ fixed and t is small (tend to zero).

Very recently, in [29], the authors used an improved version of QBV method to regularize problem in one dimensional of (1.1)–(1.3) in the nonlinear case of function f. However, in [29], the authors can only estimate the error in the case which, the final value g satisfies the condition

$$\sum_{k=1}^{\infty} e^{2Tk^2} g_k^2 < \infty \tag{1.6}$$

(see [29, page 242]). The functions satisfying this condition are quite scarce and so this method is not useful to consider many nonhomogeneous backward problem in the another case of final value g, which the condition (1.6) is not satisfied for functions such as $g(x) = a$, where a is constant. We also note that the error between the approximate problem and the exact solution is $C\epsilon^{\frac{t}{T}}$, which is not near to zero, if ϵ fixed and t tend to zero. Hence, the convergence of the approximate solution is very slow when t is near to the original time.

In the present paper, we shall regularize this problem (1.1)–(1.3)by perturbing the final value g with new way, which is different the ways in (1.4)and (1.5). We approximate problem by the following problem

$$u_t^\epsilon - u_{xx}^\epsilon = \sum_{p=1}^{\infty} \frac{e^{-Tp^2}}{\epsilon p^2 + e^{-Tp^2}} f_p(t)\sin(px), \quad (x,t) \in (0,\pi)\times(0,T), \tag{1.7}$$

$$u^\epsilon(0,t) = u^\epsilon(\pi,t) = 0 \quad (x,t)\in(0,\pi)\times[0,T] \tag{1.8}$$

$$u^\epsilon(x,T) = \sum_{p=1}^{\infty} \frac{e^{-Tp^2}}{\epsilon p^2 + e^{-Tp^2}} g_p \sin(px), \quad x\in(0,\pi) \tag{1.9}$$

where $0<\epsilon<1$,

$$f_p(t) = \frac{2}{\pi}\int_0^\pi f(x,t)\sin(px)dx,\ \ g_p = \frac{2}{\pi}\int_0^\pi g(x)\sin(px)dx \tag{1.10}$$

and $\langle\cdot,\cdot\rangle$ is the inner product in $L^2((0,\pi))$.

We shall prove that, the (unique) solution u^ϵ of (1.7)–(1.9) satisfies the following equality

$$u^\epsilon(x,t) = \sum_{p=1}^{\infty}\Big(\frac{e^{-tp^2}}{\epsilon p^2 + e^{-Tp^2}} g_p - \int_t^T \frac{e^{(s-t-T)p^2}}{\epsilon p^2 + e^{-Tp^2}} f_p(s)ds\Big)\sin(px) \tag{1.11}$$

where $0\le t\le T$.

Note that our method give a better approximation than the quasi-reversibility method in [28], and the final value $g(x)$ is not essential to satisfy the condition (*), which only in $L^2(0,\pi)$. Especially,the convergence of the approximate solution at $t=0$ is also proved (In [29], the error $\|u(.,0)-u^\epsilon(.,0)\|$ is not given). This is an improvement of many known results in [1, 4, 5, 9, 10, 28, 29, 30, 31].

The remainder of the paper is divided into three sections. In Section 1, we shall show that (1.7)–(1.9) is well posed and that the solution $u^\epsilon(x,t)$ satisfies (1.11). Then, in Section 2, we estimate the error between an exact solution u_0 of Problem (1.1)–(1.3) and the approximation solution u^ϵ. In fact, we shall prove that

$$\|u^\epsilon(.,t) - u_0(.,t)\| \le \frac{C}{1+\ln(\frac{T}{\epsilon})} \tag{1.12}$$

where $\|\cdot\|$ is norm in $L^2(0,\pi)$ and C depends on u_0 and f.

Finally, a numerical experiment will be given in Section 3.

2. Well-posedness of Problem (1.7)–(1.9)

In this section, we shall study the existence, the uniqueness and the stability of a (weak) solution of Problem (1.7)–(1.9).

Theorem 2.1. *Let $f(x,t)\in L^2(0,T);L^2(0,\pi))$ and $g(x)\in L^2(0,\pi)$. Let a given $\epsilon\in(0,eT)$. Then (1.7)–(1.9) has a unique weak solution $u^\epsilon \in C([0,T];L^2(0,\pi)\cap L^2(0,T;H_0^1(0,\pi))\cap C^1(0,T;H_0^1(0,\pi))$ satisfying (1.11). The solution depends continuously on g in $C([0,T];L^2(0,\pi))$.*

Proof. The proof is divided into two steps. In Step 1, we prove the existence and the uniqueness of a solution of (1.7)–(1.9). In Step 2, the stability of the solution is given.

Step 1. The existence and the uniqueness of a solution of (1.7)–(1.9) We divide this step into two parts.

Part A If $u^\epsilon \in C([0,T];L^2(0,\pi)) \cap L^2(0,T;H_0^1(0,\pi)) \cap C^1(0,T;H_0^1(0,\pi))$ satisfies (1.12) then u^ϵ is solution of (1.7)–(1.9). We have

$$u^\epsilon(x,t) = \sum_{p=1}^{\infty} \Big(\frac{e^{-tp^2}}{\epsilon p^2 + e^{-Tp^2})} g_p - \int_t^T \frac{e^{(s-t-T)p^2}}{\epsilon p^2 + e^{-Tp^2}} f_p(s) ds \Big) \sin(px) \tag{2.1}$$

for $0 \le t \le T$. We can verify directly that

$$u^\epsilon \in C([0,T];L^2(0,\pi) \cap C^1((0,T);H_0^1(0,\pi)) \cap L^2(0,T;H_0^1(0,\pi))).$$

In fact, $u^\epsilon \in C^\infty((0,T];H_0^1(0,\pi)))$. Moreover, one has

$$\begin{aligned}
&u_t^\epsilon(x,t)\\
&= \sum_{p=1}^{\infty} \Big(\frac{-p^2 e^{-tp^2}}{\epsilon p^2 + e^{-Tp^2}} g_p - \int_t^T \frac{p^2 e^{(s-t-T)p^2}}{\epsilon p^2 + e^{-Tp^2}} f_p(s) ds + \frac{e^{-Tp^2}}{\epsilon p^2 + e^{-Tp^2}} f_p(t) \Big) \sin(px)\\
&= -\frac{2}{\pi} \sum_{p=1}^{\infty} p^2 \langle u^\epsilon(x,t), \sin px \rangle \sin(px) + \sum_{p=1}^{\infty} \frac{e^{-Tp^2}}{\epsilon p^2 + e^{-Tp^2}} f_p(t) \sin(px)\\
&= u_{xx}^\epsilon(x,t) + \sum_{p=1}^{\infty} \frac{e^{-Tp^2}}{\epsilon p^2 + e^{-Tp^2}} f_p(t) \sin(px)
\end{aligned}$$

and

$$u^\epsilon(x,T) = \sum_{p=1}^{\infty} \frac{e^{-Tp^2}}{\epsilon p^2 + e^{-Tp^2}} g_p \sin(px)$$

So u^ϵ is the solution of (1.7)–(1.9).

Part B The Problem (1.7)–(1.9) has at most one solution $C([0,T];H_0^1(0,\pi)) \cap C^1((0,T);L^2(0,\pi))$. A proof of this statement can be found in [3, Theorem 11]. Since Part A and Part B are proved, we complete the proof of Step 1.

Step 2. The solution of the problem (1.7)–(1.9) depends continuously on g in $L^2(0,\pi)$. Let u and v be two solutions of (1.7)–(1.9) corresponding to the final values g and h. From we have

$$u(x,t) = \sum_{p=1}^{\infty} \Big(\frac{e^{-tp^2}}{\epsilon p^2 + e^{-Tp^2}} g_p - \int_t^T \frac{e^{(s-t-T)p^2}}{\epsilon p^2 + e^{-Tp^2}} f_p(s) ds \Big) \sin(px) \quad 0 \le t \le T, \tag{2.2}$$

$$v(x,t) = \sum_{p=1}^{\infty} \Big(\frac{e^{-tp^2}}{\epsilon p^2 + e^{-Tp^2}} h_p - \int_t^T \frac{e^{(s-t-T)p^2}}{\epsilon p^2 + e^{-Tp^2}} f_p(s) ds \Big) \sin(px) \quad 0 \le t \le T, \tag{2.3}$$

where

$$g_p = \frac{2}{\pi} \int_0^\pi g(x) \sin(px) dx, \quad h_p = \frac{2}{\pi} \int_0^\pi h(x) \sin(px) dx.$$

For $\lambda > 0$, we define the function

$$h(\lambda) = \frac{1}{\epsilon\lambda + e^{-\lambda T}}.$$

Then

$$h(\lambda) \le h\Big(\frac{\ln(T/\epsilon)}{T}\Big) = \frac{T}{\epsilon\big(1 + \ln(T/\epsilon)\big)} \quad \epsilon \in (0, eT).$$

This follows that

$$
\begin{aligned}
\|u(.,t)-v(.,t)\|^2 &= \frac{\pi}{2}\sum_{p=1}^{\infty}\Big|\frac{e^{-tp^2}}{\epsilon p^2+e^{-Tp^2}}(g_p-h_p)\Big|^2 \\
&\le \frac{\pi}{2}\Big(\frac{T}{\epsilon\,(1+\ln(T/\epsilon))}\Big)^2\sum_{p=1}^{\infty}|g_p-h_p|^2 \\
&= \Big(\frac{T}{\epsilon\,(1+\ln(T/\epsilon))}\Big)^2\|g-h\|^2.
\end{aligned}
\tag{2.4}
$$

Hence

$$
\|u(.,t)-v(.,t)\| \le \frac{T}{\epsilon\,(1+\ln(T/\epsilon))}\|g-h\|.
$$

This completes the proof of Step 2 and the proof of our theorem. □

Remark 2.2. In [9, 10, 28], the stability magnitude is $e^{\frac{T}{\epsilon}}$ (see [5, Theorem 2.1], it is ϵ^{-1}. One advantage of this method of regularization is that the order of the error, introduced by small changes in the final value g, is less than the order given in [28].

Theorem 2.3. *For any $g(x)\in L^2(0,\pi)$, the approximation $u^\epsilon(x,T)$ converges to $g(x)$ in $L^2(0,\pi)$ as ϵ tends to zero.*

Proof. We have $g(x)=\sum_{p=1}^{\infty} g_p\sin(px)$, where g_p is defined in (1.10). Let $\alpha>0$, choose some N for which $\frac{\pi}{2}\sum_{p=N+1}^{\infty} g_p^2<\alpha/2$. We have

$$
\|u^\epsilon(x,T)-g(x)\|^2 = \frac{\pi}{2}\sum_{p=1}^{\infty}\frac{\epsilon^2p^4g_p^2}{(\epsilon p^2+e^{-Tp^2})^2}\,. \tag{2.5}
$$

Then

$$
\|u^\epsilon(x,T)-g(x)\|^2 \le \epsilon^2\frac{\pi}{2}\sum_{p=1}^{N}p^4g_p^2e^{2Tp^2}+\frac{\alpha}{2}
$$

By taking ϵ such that $\epsilon<\sqrt{\alpha}\big(\pi\sum_{p=1}^{N}p^4g_p^2e^{2Tp^2}\big)^{-1/2}$, we get

$$
\|u^\epsilon(x,T)-g(x)\|^2<\alpha
$$

which completes the proof.

In the case $\frac{d^2g}{dx^2}\in L^2(0,\pi)$, we have the error estimate

$$
\begin{aligned}
\|u^\epsilon(x,T)-g(x)\|^2 &= \frac{\pi}{2}\sum_{p=1}^{\infty}\Big(\frac{e^{-Tp^2}}{\epsilon p^2+e^{-Tp^2}}-1\Big)^2g_p^2 \\
&= \frac{\pi}{2}\sum_{p=1}^{\infty}\frac{\epsilon^2p^4g_p^2}{(\epsilon p^2+e^{-Tp^2})^2} \\
&\le \frac{\pi}{2}\frac{T}{\big(1+\ln(\frac{T^2}{\epsilon})\big)^2}\sum_{p=1}^{\infty}p^4g_p^2 = \frac{T^2}{(1+\ln(T/\epsilon))^2}\|g_{xx}\|^2
\end{aligned}
$$

Then, we get

$$
\|u(x,T)-g(x)\| \le \frac{T}{1+\ln(T/\epsilon)}\|g_{xx}\|.
$$

This completes the proof. □

Theorem 2.4. *Let $g(x), \epsilon \in L^2(0,\pi)$ be as in Theorem 2.3, and let f_{xx} be in $L^2(0,T;L^2(0,\pi))$. If the sequence $u^\epsilon(x,0)$ converges in $L^2(0,\pi)$, then the problem (1.1)–(1.3) has a unique solution u. Furthermore,we then have that $u^\epsilon(x,t)$ converges to $u(t)$ as ϵ tends to zero uniformly in t.*

Proof. Assume that $\lim_{\epsilon\to 0} u^\epsilon(x,0) = u_0(x)$ exists. Let

$$u(x,t) = \sum_{p=1}^{\infty}\Big(e^{-tp^2}u_{0p} - \int_0^t e^{(s-t)p^2} f_p(s)ds\Big)\sin(px)$$

where $u_{0p} = \frac{2}{\pi}\int_0^\pi u_0(x)\sin(px)dx$. It is clear to see that $u(x,t)$ satisfies (1.1)–(1.2). We have the formula of $u^\epsilon(x,t)$

$$u^\epsilon(x,t) = \sum_{p=1}^{\infty}\Big(e^{-tp^2}u^\epsilon_{0p} - \int_0^t \frac{e^{(s-t-T)p^2}}{\epsilon p^2 + e^{-Tp^2}} f_p(s)ds\Big)\sin(px)$$

where $u^\epsilon_{0p} = \frac{2}{\pi}\int_0^\pi u^\epsilon(x,0)\sin(px)dx$. In view of the inequality $(a+b)^2 \le 2(a^2+b^2)$, we have

$$\begin{aligned}
&\|u^\epsilon(x,t) - u(x,t)\|^2\\
&\le \frac{\pi}{2}\sum_{p=1}^{\infty}(u^\epsilon_{0p} - u_{0p})^2 + \frac{\pi}{2}t^2\sum_{p=1}^{\infty}\Big(\int_0^t e^{(2s-2t)p^2}\frac{\epsilon^2p^4}{(\epsilon p^2 + e^{-Tp^2})^2} f_p^2 ds\Big)\\
&\le \|u^\epsilon(x,0) - u_0(x)\|^2 + T^2\frac{T^2}{(1+\ln(T/\epsilon))^2}\int_0^t\sum_{p=1}^{\infty}p^4 f_p^2 ds\\
&= \|u^\epsilon(x,0) - u_0(x)\|^2 + \frac{T^4}{(1+\ln(T/\epsilon))^2}\int_0^t \|f_{xx}\|^2 ds\\
&\le \|u^\epsilon(x,0) - u_0(x)\|^2 + \frac{T^4}{(1+\ln(T/\epsilon))^2}\|f_{xx}\|^2_{L^2(0,T;L^2(0,\pi))}
\end{aligned}$$

Hence, $\lim_{\epsilon\to 0} u^\epsilon(x,t) = u(x,t)$. Thus $\lim_{\epsilon\to 0} u^\epsilon(x,T) = u(x,T)$. Using theorem 2.3, we have $u(x,T) = g(x)$. Hence, $u(x,t)$ is the unique solution of the problem (1.1)–(1.3). We also see that $u^\epsilon(x,t)$ converges to $u(x,t)$ uniformly in t. $\square$

Theorem 2.5. *Let $f(x,t), g(x), \epsilon$ be as Theorem 2.4. If the sequence $u^\epsilon_t(x,0)$ converges in $L^2(0,\pi)$, then the problem (1.1)–(1.3) has a unique solution u. Furthermore,we have that $u^\epsilon(x,t)$ converges to $u(x,t)$as ϵ tends to zero in $C^1(0,T;L^2(0,\pi))$.*

Proof. Assume that $\lim_{\epsilon\to 0} u^\epsilon_t(x,0) = v(x)$ in $L^2(0,\pi)$. Let $v(x) = \sum_{p=1}^\infty v_p\sin(px)$ where $v_p = \frac{2}{\pi}\int_0^\pi v(x)\sin(px)dx$. Denote by $w_p = -\frac{v_p}{p^2}$ and $w(x) = \sum_{p=1}^\infty w_p\sin(px)$. It is easy to show that the function $u(x,t)$ defined by

$$u(x,t) = \sum_{p=1}^{\infty}\Big(e^{-tp^2}w_p - \int_0^t e^{(s-t)p^2} f_p ds\Big)\sin(px)$$

is a solution of the problem

$$\begin{gathered}
u_t(x,t) - u_{xx} = f(x,t),\\
u(x,0) = w(x)
\end{gathered}$$

Since $u^\epsilon(x,t)$ is the solution of (1.7)–(1.9), we have

$$u^\epsilon_{tp}(t) = p^2 u^\epsilon_p(t) + \frac{e^{-Tp^2}}{\epsilon p^2 + e^{-Tp^2}} f_p(t),$$
$$u_{tp}(t) = p^2 u_p(t) + f_p(t)$$

where

$$u^\epsilon_p(t) = \frac{2}{\pi}\int_0^\pi u^\epsilon(x,t)\sin(px)dx, \quad u_p(t) = \frac{2}{\pi}\int_0^\pi u(x,t)\sin(px)dx$$
$$u^\epsilon_{tp}(t) = \frac{2}{\pi}\int_0^\pi u^\epsilon_t(x,t)\sin(px)dx, \quad u_{tp}(t) = \frac{2}{\pi}\int_0^\pi u_t(x,t)\sin(px)dx$$

So that

$$u^\epsilon_p(t) - u_p(t) = \frac{1}{p^2}(u^\epsilon_{tp}(t) - u_{tp}(t)) + \frac{\epsilon}{\epsilon p^2 + e^{-Tp^2}} f_p(t) \tag{2.6}$$

By a direct computation,

$$\begin{aligned}
\|u^\epsilon(.,t) - u(.,t)\|^2 &= \frac{\pi}{2}\sum_{p=1}^\infty |u^\epsilon_p(t) - u_p(t)|^2 \\
&\le \sum_{p=1}^\infty \pi|u^\epsilon_{tp}(t) - u_{tp}(t)|^2 + \pi\sum_{p=1}^\infty \frac{\epsilon^2}{(\epsilon p^2 + e^{-Tp^2})^2} f_p^2(t) \\
&\le 2\|u^\epsilon_t(x,t) - u_t(x,t)\|^2 + 2\frac{T^2}{(1+\ln(T/\epsilon)^2}\|f(.,t)\|^2
\end{aligned}$$

Hence

$$\|u^\epsilon(.,0) - u(.,0)\|^2 \le 2\|u^\epsilon_t(x,0) - u_t(x,0)\|^2 + \frac{2T^2}{(1+\ln(T/\epsilon))^2}\|f(.,t)\|^2$$

Using $\lim_{\epsilon\to 0} u^\epsilon_t(x,0) = v(x) = u_t(x,0)$, we get $\lim_{\epsilon\to 0}\|u^\epsilon(x,0) - u(x,0)\| = 0$. On the other hand, we have

$$u^\epsilon(x,T) = \sum_{p=1}^\infty \Big(e^{-Tp^2}u^\epsilon_p(0) + \int_0^T \frac{e^{(s-2T)p^2}}{\epsilon p^2 + e^{-Tp^2}} f_p(s)ds\Big)\sin(px)$$
$$u(x,T) = \sum_{p=1}^\infty \Big(e^{-Tp^2}u_p(0) + \int_0^T e^{(s-T)p^2} f_p(s)ds\Big)\sin(px)$$

It follows that

$$\begin{aligned}
&\|u^\epsilon(.,T) - u(.,T)\|^2 \\
&\le 2\sum_{p=1}^\infty e^{-2Tp^2}|u^\epsilon_p(0) - u_p(0)|^2 + 2T^2\sum_{p=1}^\infty\int_0^T \frac{\epsilon^2 p^4}{(\epsilon p^2 + e^{-Tp^2})^2} f_p^2(s)ds \\
&\le \|u^\epsilon(.,0) - u(.,0)\|^2 + 2\frac{T^2}{(1+\ln(T/\epsilon))^2}\|f_{xx}(.,t)\|^2
\end{aligned}$$

Hence, $\lim_{\epsilon\to 0}\|u^\epsilon(x,T) - u(x,T)\| = 0$. Using the Theorem 2.3, we obtain $u(x,T) = g(x)$. This implies that $u(x,t)$ is the unique solution of (1.1)–(1.3). □

Theorem 2.6. *If there exists $m \in (0,2)$ so that $\sum_{p=1}^\infty p^{2m}e^{mTp^2}g_p^2$ converges, then*

$$\|u^\epsilon(x,T) - g(x)\| \le \frac{\sqrt{C_1\epsilon^m}}{m}$$

where $C_1 = 4\sum_{p=1}^{\infty} p^{2m} e^{mTp^2} g_p^2$.

Proof. Let m be in $(0,2)$ such that $\sum_{p=1}^{\infty} p^{2m} e^{mTp^2} g_p^2$ converges, and let n be in $(0,2)$. Fix a natural integer p, and define

$$g_p(\epsilon) = \frac{\epsilon^n}{(\epsilon p^2 + e^{-Tp^2})^2}.$$

It can be shown that $g_p(\epsilon) \le g_p(\epsilon_0)$, for all $\epsilon > 0$ where $\epsilon_0 = \frac{ne^{-Tp^2}}{(2-n)p^2}$. Furthermore, from (2.5), we have

$$\|u^\epsilon(x,T) - g(x)\|^2 = \sum_{p=1}^{\infty} \frac{\epsilon^2 p^4 g_p^2}{(\epsilon p^2 + e^{-Tp^2})^2} = \epsilon^{2-n} \sum_{p=1}^{\infty} p^4 g_p^2 g_p(\epsilon) \tag{2.7}$$

It follows that

$$\|u^\epsilon(x,T) - g(x)\|^2 \le \epsilon^{2-n} (\frac{n}{2-n})^n \sum_{p=1}^{\infty} p^{4-2n} g_p^2 e^{(2-n)Tp^2} \tag{2.8}$$

If we choose $n = 2-m$, we obtain $\|u^\epsilon(x,T) - g(x)\|^2 \le C_1 \epsilon^m m^{-2}$. □

Theorem 2.7. *Let $f \in L^2(0,T;L^2(0,\pi))$ and $g \in L^2(0,\pi)$ and $\epsilon \in (0,eT)$. Suppose that Problem (1.1)–(1.3) has a unique solution $u(x,t)$ in $C([0,T];H_0^1(0,\pi)) \cap C^1((0,T);L^2(0,\pi))$ which satisfies $\|u_{xx}(.,t)\| < \infty$. Then*

$$\|u(.,t) - u^\epsilon(.,t)\| \le \frac{C}{1+\ln(T/\epsilon)}$$

for every $t \in [0,T]$, where $C = T\sup_{t\in[0,T]} \|u_{xx}(.,t)\|$ and u^ϵ is the unique solution of (1.7)–(1.9).

Proof. Suppose (1.1)–(1.3) has an exact solution u in the space $C([0,T];H_0^1(I)) \cap C^1((0,T);L^2(I))$, we get the formula

$$u(x,t) = \sum_{p=1}^{\infty} (e^{-(t-T)p^2} g_p - \int_t^T e^{-(t-s)p^2} f_p(s) ds) \sin(px) \tag{2.9}$$

From (1.11) and (2.9), we obtain

$$\begin{aligned}
|u_p(t) - u_p^\epsilon(t)| &= \Big|\Big(e^{-(t-T)p^2} - \frac{e^{-tp^2}}{\epsilon p^2 + e^{-Tp^2}}\Big)\Big(g_p - \int_t^T e^{-(T-s)p^2} f_p(s) ds\Big)\Big| \\
&= \epsilon p^2 \frac{e^{-(t-T)p^2}}{\epsilon p^2 + e^{-Tp^2}} \Big|\Big(g_p - \int_t^T e^{-(T-s)p^2} f_p(s) ds\Big)\Big| \\
&\le \frac{T}{1+\ln(T/\epsilon)} \Big|\Big(p^2 e^{-(t-T)p^2} g_p - \int_t^T p^2 e^{-(t-s)p^2} f_p(s) ds\Big)\Big|
\end{aligned} \tag{2.10}$$

It follows that

$$
\begin{aligned}
&\|u(.,.,t) - u^\epsilon(.,.,t)\|^2 \\
&= \frac{\pi}{2}\sum_{p=1}^{\infty} |u_p(t) - u_p^\epsilon(t)|^2 \\
&\le \frac{\pi}{2}\Big(\frac{T}{1+\ln(T/\epsilon)}\Big)^2 \sum_{p=1}^{\infty}\Big(p^2 e^{-(t-T)p^2} g_p - \int_t^T p^2 e^{-(t-s)p^2} f_p(s)ds\Big)^2 \\
&= \Big(\frac{T}{1+\ln(T/\epsilon)}\Big)^2 \|u_{xx}(.,t)\|^2 \le \Big(\frac{C}{1+\ln(T/\epsilon)}\Big)^2
\end{aligned}
$$

Hence

$$
\|u(.,t) - u^\epsilon(.,t)\| \le \frac{C}{1+\ln(T/\epsilon)}
$$

where $C = T\sup_{t\in[0,T]}\|u_{xx}(.,t)\|$. This completes the proof □

Remark 2.8. Note that in [28, Theorem 3.3], the exact solution u satisfies the condition $\Delta^2 u(x,t) \in L^2(0,\pi)$, while the condition of its in this theorem is $\Delta u \in L^2(0,\pi)$. So, this also implies that the final value g in our theorem is only in $L^2(0,\pi)$, not satisfying the condition (*) given in [29] (see Introduction). Further more, we also have the error estimate $\|u_t(.,t) - u_t^\epsilon(.,t)\|$ which is not given in [28, 29]. Hence, this result is an improvement of known result in [28, 29].

Theorem 2.9. *Let $f \in L^2(0,T;L^2(0,\pi))$ and $g \in L^2(0,\pi)$ and $\epsilon \in (0,eT)$. Suppose that Problem (1.1)–(1.3) has a unique solution $u(x,t)$ in $C([0,T];H_0^1(0,\pi)) \cap C^1((0,T);L^2(0,\pi))$ which satisfies $\|u_{xxxx}(.,t)\| < \infty$. Then*

$$
\|u_t(.,t) - u_t^\epsilon(.,t)\| \le \frac{D}{1+\ln(T/\epsilon)}
$$

for every $t \in [0,T]$, where

$$
D = T\Big(2\sup_{t\in[0,T]}(\|u_{xxxx}(.,t)\|^2 + \|f_{xx}(.,t)\|^2)\Big)^{1/2}
$$

and u^ϵ is the unique solution of (1.7)–(1.9).

Proof. In view of (2.6), we have

$$
\begin{aligned}
u_{tp}^\epsilon(t) - u_{tp}(t) &= p^2(u_p^\epsilon(t) - u_p(t)) - \frac{\epsilon p^2}{\epsilon p^2 + e^{-Tp^2}} f_p(t) \\
&= \frac{\epsilon p^4 e^{-(t-T)p^2}}{\epsilon p^2 + e^{-Tp^2}}\Big(g_p - \int_t^T e^{-(T-s)p^2} f_p(s)ds\Big) - \frac{\epsilon p^2}{\epsilon p^2 + e^{-Tp^2}} f_p(t) \\
&= \frac{\epsilon p^4}{\epsilon p^2 + e^{-Tp^2}} u_p(t) - \frac{\epsilon p^2}{\epsilon p^2 + e^{-Tp^2}} f_p(t) \\
&= \frac{\epsilon p^2}{\epsilon p^2 + e^{-Tp^2}}(p^2 u_p(t) - f_p(t))
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
\|u_t(.,t)-u_t^\epsilon(.,t)\|^2 &= \frac{\pi}{2}\sum_{p=1}^{\infty}|u_{tp}^\epsilon(t)-u_{tp}(t)|^2\\
&\le \pi\frac{\epsilon^2}{(\epsilon p^2+e^{-Tp^2})^2}\sum_{p=1}^{\infty}(p^8u_p^2(t)+p^4f_p^2(t))\\
&= \frac{2T^2}{(1+\ln(T/\epsilon))^2}(\|u_{xxxx}(x,t)\|^2+\|f_{xx}(x,t)\|^2)
\end{aligned}
$$

This completes the proof. □

In the case of nonexact data, one has the following result.

Theorem 2.10. *Let f, g, ϵ be as in Theorem 2.7. Assume that the exact solution u of (1.1)–(1.3) corresponding to g satisfies*

$$u\in C([0,T];L^2(0,\pi))\cap L^2(0,T;H_0^1(0,\pi))\cap C^1((0,T);L^2(0,\pi)),$$

and $\|u_{xx}(.,t)\|<\infty$. Let $g_\epsilon\in L^2(0,\pi)$ be a measured data such that

$$\|g_\epsilon-g\|\le\epsilon.$$

Then there exists a function u^ϵ satisfying

$$\|u(.,t)-u^\epsilon(.,t)\|\le\frac{C+T}{1+\ln(T/\epsilon)}$$

for every $t\in[0,T]$ and C is defined in Theorem 2.7.

Proof. Let v^ϵ be the solution of problem (1.7)–(1.9) corresponding to g and let again u^ϵ be the solution of problem (1.7)–(1.9) corresponding to g_ϵ where g, g_ϵ are in right hand side of (1.7). Using Theorem 2.7 and Step 2 in Theorem 2.1, we get

$$
\begin{aligned}
\|u^\epsilon(.,t)-u(.,t)\| &\le \|u^\epsilon(.,t)-v^\epsilon(.,t)\|+\|v^\epsilon(.,t)-u(.,t)\|\\
&\le \frac{T}{\epsilon(1+\ln(T/\epsilon)}\|g_\epsilon-g\|+\frac{T}{(1+\ln(T/\epsilon)}\|u_{xx}(.,t)\|\\
&\le \frac{C+T}{1+\ln(T/\epsilon)}
\end{aligned}
$$

for every $t\in(0,T)$ and where C is defined in Theorem 2.7. This completed the proof. □

3. A numerical example

We consider

$$
\begin{gathered}
u_t-u_{xx}=f(x,t)\equiv 2e^t\sin x,\\
u(x,1)=g(x)\equiv e\sin x.
\end{gathered}\tag{3.1}
$$

The exact solution to this problem is

$$u(x,t)=e^t\sin x$$

Note that $u(x,1/2)=\sqrt{e}\sin(x)\approx 1.648721271\sin(x)$. Let g_n be the measured final data

$$g_n(x)=e\sin(x)+\frac{1}{n}\sin(nx).$$

So that the data error, at the final time, is

$$F(n) = \|g_n - g\|_{L^2(0,\pi)} = \sqrt{\int_0^\pi \frac{1}{n^2}\sin^2(nx)dx} = \frac{\pi}{2n}.$$

The solution of (3.4), corresponding the final value g_n, is

$$u^n(x,t) = e^t \sin(x) + \frac{1}{n}e^{n^2(1-t)}\sin(nx),$$

The error at the original time is

$$O(n) := \|u^n(.,0) - u(.,0)\|_{L^2(0,\pi)} = \sqrt{\int_0^\pi \frac{e^{2n^2}}{n^2}\sin^2(nx)\,dx} = \frac{e^{n^2}}{n}\frac{\pi}{2}.$$

Then, we notice that

$$\lim_{n\to\infty} F(n) = \lim_{n\to\infty} ||g_n - g||_{L^2(0,\pi)} = \lim_{n\to\infty}\frac{1}{n}\frac{\pi}{2} = 0, \tag{3.2}$$

$$\lim_{n\to\infty} O(n) = \lim_{n\to\infty}\|u^n(.,0) - u(.,0)\|_{L^2(0,\pi)} = \lim_{n\to\infty}\frac{e^{n^2}}{n}\frac{\pi}{2} = \infty. \tag{3.3}$$

From the two equalities above, we see that (3.1) is an ill-posed problem. Approximating the problem as in (1.1)–(1.3), the regularized solution is

$$u^\epsilon(x,t) = \sum_{p=1}^\infty \Big(\frac{e^{-tp^2}}{\epsilon p^2 + e^{-p^2}}g_p - \int_t^1 \frac{e^{(s-t-1)p^2}}{\epsilon p^2 + e^{-p^2}}f_p(s)ds\Big)\sin(px) \tag{3.4}$$

for $0 \le t \le 1$. Hence, we have

$$u^\epsilon(x,t) = \frac{e^{1-t}}{\epsilon + e^{-1}}\sin x - 2\Big(\int_t^1 \frac{e^{2s-t-1}}{\epsilon + e^{-1}}ds\Big)\sin x + \frac{1}{n}\frac{e^{-tn^2}}{\epsilon n^2 + e^{-n^2}}\sin(nx)\,. \tag{3.5}$$

It follows that

$$u^\epsilon(x,\frac{1}{2}) = \Big(\frac{e^{1/2}}{\epsilon + e^{-1}} - 2\int_{\frac{1}{2}}^1 \frac{e^{2s-\frac{3}{2}}}{\epsilon + e^{-1}}ds\Big)\sin x + \frac{1}{n}\frac{e^{-\frac{1}{2}n^2}}{\epsilon n^2 + e^{-n^2}}\sin(nx) \tag{3.6}$$

Let $a_\epsilon = \|u_\epsilon(.,\frac{1}{2}) - u(.,\frac{1}{2})\|$ be the error between the regularized solution u_ϵ and the exact solution u in the time $t = \frac{1}{2}$. Let $n = 300$ and

$$\epsilon = \epsilon_1 = 10^{-2}\sqrt{\frac{\pi}{2}}, \epsilon = \epsilon_2 = 10^{-4}\sqrt{\frac{\pi}{2}},$$
$$\epsilon = \epsilon_3 = 10^{-10}\sqrt{\frac{\pi}{2}}, \epsilon = \epsilon_4 = 10^{-15}\sqrt{\frac{\pi}{2}}.$$

We note that the new method in this article give a better approximation than the previous method in [28]. To prove this, we have in view of the error table in [28, p. 9].

Furthermore, we continue to approximate this problem by the method given in [28], which gives regularized solution

$$v^\epsilon(x,t) = \sum_{p=1}^\infty \Big(\frac{e^{-tp^2}}{\epsilon + e^{-p^2}}g_p - \int_t^1 \frac{e^{-tp^2}}{\epsilon^s + e^{-sp^2}}f_p(s)ds\Big)\sin(px).$$

TABLE 1.

ϵ	u_ϵ	a_ϵ
$\epsilon_1 = 10^{-2}\sqrt{\frac{\pi}{2}}$	$1.59440220314355\sin(x)$ $+4.636337144\times 10^{-39093}\sin(300x)$	0.06807885585
$\epsilon_2 = 10^{-4}\sqrt{\frac{\pi}{2}}$	$1.64815976557002\sin(x)$ $+4.636337144\times 10^{-39091}\sin(300x)$	0.0007037421545
$\epsilon_3 = 10^{-10}\sqrt{\frac{\pi}{2}}$	$1.64872127013843\sin x$ $+4.636337144\times 10^{-39084}\sin(300x)$	$1.253314137\times 10^{-9}$
$\epsilon_4 = 10^{-16}\sqrt{\frac{\pi}{2}}$	$1.64872127070011\sin(x)$ $+4.636337144\times 10^{-39079}\sin(300x)$	$5.810786885\times 10^{-39079}$

TABLE 2.

ϵ	u_ϵ	$\|u-u_\epsilon\|$
$10^{-2}\sqrt{\frac{\pi}{2}}$	$1.643563444\sin(x)+0.8243606355\sin 200x$	0.1462051256
$10^{-4}\sqrt{\frac{\pi}{2}}$	$1.648617955\sin(x)+0.1648721271\sin 10000x$	0.02066391506
$10^{-10}\sqrt{\frac{\pi}{2}}$	$1.648721271\big(\sin(x)+10^{-10}\sin(10^{10}x)\big)$	0.00002066365678
$10^{-16}\sqrt{\frac{\pi}{2}}$	$1.648721271\big(\sin(x)+10^{-16}\sin(10^{16}x)\big)$	$2.066365678\times 10^{-8}$
$10^{-30}\sqrt{\frac{\pi}{2}}$	$1.648721271\big(\sin(x)+10^{-30}\sin(10^{30}x)\big)$	$2.066365678\times 10^{-15}$

TABLE 3.

ϵ	v_ϵ	a_ϵ
$\epsilon_1 = 10^{-2}\sqrt{\frac{\pi}{2}}$	$1.701714206\sin(x)$ $+4.172703428\times 10^{-39088}\sin(300x)$	0.06641679460
$\epsilon_2 = 10^{-4}\sqrt{\frac{\pi}{2}}$	$1.656775314\sin(x)$ $+4.172703428\times 10^{-39086}\sin(300x)$	0.01009424595
$\epsilon_3 = 10^{-10}\sqrt{\frac{\pi}{2}}$	$1.648724344\sin x$ $+4.172703428\times 10^{-39080}\sin(300x)$	0.000003851434344
$\epsilon_4 = 10^{-16}\sqrt{\frac{\pi}{2}}$	$1.648721273\sin(x)$ $+4.172703428\times 10^{-39074}\sin(300x)$	$2.506628275\times 10^{-9}$

Hence, we have

$$v^\epsilon(x,t) = \frac{e^{1-t}}{\epsilon+e^{-1}}\sin x - 2\Big(\int_t^1 \frac{e^{s-t}}{\epsilon^s+e^{-s}}ds\Big)\sin x + \frac{1}{n}\frac{e^{-tn^2}}{\epsilon+e^{-n^2}}\sin(nx) \tag{3.7}$$

for $0\le t\le 1$. It follows that

$$v^\epsilon(x,\frac{1}{2}) = \frac{e^{1/2}}{\epsilon+e^{-1}}\sin x - 2\Big(\int_{\frac{1}{2}}^1 \frac{e^{s-\frac{1}{2}}}{\epsilon^s+e^{-s}}ds\Big)\sin x + \frac{1}{n}\frac{e^{-\frac{1}{2}n^2}}{\epsilon+e^{-n^2}}\sin(nx). \tag{3.8}$$

Looking at Tables 1,2,3, a comparison between the three methods, we can see the error results of in Table 1 are smaller than the errors in Tables 2 and 3. This shows that our approach has a nice regularizing effect and give a better approximation with comparison to the previous method in, for example [28, 29].

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