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REMARKS ON VACUUM STATE AND UNIQUENESS OF CONCENTRATION PROCESS

VLADIMIR G. DANILOV

Abstract. We give two examples of nonuniqueness of generalized solutions of pressureless gas dynamics systems. In both of these examples, the presence of the Dirac δ-function leads to nonuniqueness.

1. Introduction

In this note, we present two examples of nonuniqueness of the solution of the pressureless gas dynamics system. This system has the form

$$\begin{aligned} \partial_t\rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) = 0, \end{aligned} \tag{1.1}$$

and, as is well known, in the domain, where the solution belongs to C^1, it is equivalent to the system

$$\begin{aligned} \partial\rho_t + \partial_x(\rho u) = 0, \\ \partial_t u + \frac{1}{2}\partial_x u^2 = 0. \end{aligned} \tag{1.2}$$

However, these systems are quite different if one considers generalized solutions.

In both cases, the nonuniqueness of the solution originates from the fact that the initial condition contains the Dirac δ-function. We also note that the nonuniqueness of the solution of system (1.1) in the examples presented here arises because of a quite different mechanism than that found in [7, p. 145].

This type of nonuniqueness for system (1.1) arises because of the properties of the conditions posed on the discontinuity curve, which are analogs of the Rankine–Hugoniot conditions for shock waves. (These conditions are usually implicit conditions in works concerning the study of (1.1), (1.2) in general functional spaces).

Moreover, our solutions for system (1.1) is an entropy solution in the sense of [7, Definition 2]. Such a solution must be unique due to the result of [7].

Apparently, this can be explained by the fact that there are different definitions of the generalized solution. In this paper, we use the definitions given in [3]. These

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definitions are direct analogies of the definitions (in the form of integral identities) of generalized solutions of conservation laws belonging to $L^1 \cap L^\infty$.

A definition of δ-shock wave type generalized solutions in the form of integral identities was given in [3], where the integral identities were obtained as the limits of the results obtained by substituting approximate solutions (weak asymptotic solutions in [3]) into the original equations. In what follows, we present another (heuristic) method for obtaining these integral identities.

All of this shows that these definitions based on integral identities are quite natural. We note that, in the literature, another approach is well known in the definition of δ-shock wave type solutions for systems (1.1) and (1.2), see, e.g., [1, 2, 7, 12].

In these works, the generalized solution of the continuity equation in systems (1.1) and (1.2) is determined in the form of an integral identity over the measure determined by the function ρ. Since the Dirac δ-function on the trajectory of the discontinuity is considered as a term in the density of this measure and the functions depending on the velocity u must be integrated, it follows from formal considerations that the value of u must be determined on the trajectory of the discontinuity. It is clear that such an approach cannot be uniquely possible. For example, in system (1.2), the definition of the generalized solution for the second equation (for u) is in no way related to ρ and it is defined by a general definition of the $L' \cap L^\infty$ solution of the conservation law. But it turns out that this definition must take account of the second equation (about which the first equation does not know anything).

Apparently, all this originates from the attempts to define the product $\delta(z)H(z)$ (of the Dirac δ-function by the Heaviside function), which formally appears in substituting the δ-shock wave type solution into the system of equations.

Nevertheless, it is well known that such a definition is not unique. In [6], a new method for constructing integral identities determining the δ-shock wave type solutions was proposed.

In this method, we priorly do not assume that there is some fixed definition of the product $\delta(z)H(z)$. We only assume that the equation holds in the sense of the space $\mathcal{D}'(\mathbb{R}^{n+1})$. As a rule, such an assumption in the case of conservation laws implies a definition of the usual generalized solution in the form of an integral identity. Absolutely the same was obtained in [6]. Moreover, we present these considerations and construct the corresponding definition for the system of equations (1.1), see Sec. 2. Thus, our remark about the nonuniqueness of the δ-shock wave type solution to system (1.1) has one more explanation: this nonuniqueness is related to the nonuniqueness of the definition of the product of generalized functions. If we fix such a definition, then, of course, the nonuniqueness disappears. We describe this in more detail in Sec. 2.

The nonuniqueness for system (1.2) arises when we consider an unstable step in the initial data for u and the δ-function for ρ at the point of jump of u. This means that the mass concentrated at the origin of the rarefaction domain fills the vacuum "nonuniquely."

Of course, the examples given below cast a shadow on the physical consistency of the models related to systems (1.1) and (1.2). In any case, one must attentively examine the conclusions about the real processes obtained using these models.

2. Results

Definitions of generalized δ-shock wave type solutions to systems (1.1) *and* (1.2).

Definition 2.1. Let $\Gamma = \{\gamma_i, i \in I\}$ be a graph in the half-plane $\{x \in \mathbb{R}^1, t \geq 0\}$ containing C^1 arcs γ_i, and let I be a finite set. By $I_0 \subset I$ we denoted the arcs starting from the point $x_k^0 \in \mathbb{R}^1$. A distribution $\rho(x,t)$ and a graph Γ, where

$$\begin{gathered} \rho(x,t) = R(x,t) + E(t)\delta(\Gamma), \quad E(t)\delta(\Gamma) = \sum_{i\in I} e_i(t)\delta(\gamma_i), \\ e_i(t) \in C^1(\gamma_i), \quad \gamma_i = \{x = \varphi_i(t)\}, \end{gathered} \tag{2.1}$$

$R(x,t) \in C^1((\mathbb{R}^1 \times \mathbb{R}^+) \setminus \Gamma)$ and a function $u = u(x,t) \in L^\infty(\mathbb{R}^1 \times \mathbb{R}^+) \cap C^1((\mathbb{R}^1 \times \mathbb{R}^+) \setminus \Gamma)$. is called a *generalized δ-shock wave type solution to* (1.2) if the integral identities

$$\int_0^\infty \int_{\mathbb{R}^1} (u\zeta_t + \frac{1}{2}u^2\zeta_x)\,dxdt + \int_{\mathbb{R}^1} (u\zeta)\Big|_{t=0} dx = 0, \tag{2.2}$$

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^1} (R\zeta_t + uR\zeta_x)\,dxdt + \sum_{i\in I}\int_{\gamma_i} e_i(t)\frac{d\zeta}{dt_i}\,dt \\ &+ \int_{\mathbb{R}^1} R\zeta\Big|_{t=0} dx + \sum_{k\in I_0} e_k(0)\zeta(\varphi_k(0),0) = 0, \end{aligned} \tag{2.3}$$

hold for all test functions $\zeta(x,t) \in \mathcal{D}(\mathbb{R}^1 \times \mathbb{R}^1_+)$ and $\frac{d}{dt_i} = \frac{\partial}{\partial t} + \varphi_{it}\frac{\partial}{\partial x}$.

The appearance of the summand

$$\sum_{i\in I}\int_{\gamma_i} e_i(t)\frac{d\zeta}{dt_i}\,dt$$

in (2.3) can easily be explained. Indeed, let ρ have the form (2.1), then differentiating in t, we obtain (see [6])

$$\rho_t = \sum_{i\in I} e_i(t)(-\varphi_{it})\delta'(\gamma_i) + \text{smoother summands}.$$

Hence it is clear that we must have

$$(\rho u)_x = -\sum_{i\in I} e_i(-\varphi_{it})\delta'(\gamma_i) + \text{smoother summands}. \tag{2.4}$$

Now, for any test function $\zeta(x,t)$ such that $\zeta(x,0) = 0$, we have

$$\begin{aligned} \langle \rho_t + (u\rho)_x, \zeta\rangle &= -\langle \rho, \zeta_t(x,t)\rangle - \langle \rho u, \zeta_x(x,t)\rangle \\ &= -\langle R, \zeta_t(x,t)\rangle - \langle E(t)\delta(\Gamma), \zeta_t(x,t)\rangle - \langle Ru, \zeta_x\rangle \\ &\quad + \sum_{i\in I}\langle e_i(t)(-\varphi_{it})\delta(\gamma_i), \zeta_x(x,t)\rangle \\ &= -\langle R, \zeta_t\rangle - \langle Ru, \zeta_x\rangle - \sum_{i\in I}\int_{\gamma_i} e_i(t)(\zeta_t + \varphi_{it}\zeta_x)\,dt. \end{aligned}$$

Here $\langle, \zeta\rangle$ denotes the action of a generalized function on a test function ζ.

Of course, these calculations are not a proof, this is only a motivation.

Definition 1 gives a method for calculating the functions contained in (2.1). Suppose that

$$u = u_0 + \sum_{i\in I} H(x-\varphi_i)u_i, \tag{2.5}$$

where $\varphi_i(t)$, $u_0(x,t)$, and $u_i(x,t)$ are smooth functions (i.e., the velocities have jumps on the curves $x=\varphi_i$). Then, integrating by parts, we obtain

$$\begin{gathered}\int_0^\infty \int_{\mathbb{R}^1\setminus\bigcup\{x=\varphi_i\}} (R_t + (uR)_x)\zeta\,dxdt \\ -\sum_{i\in I}\int_{x=\varphi_i} \{[R]\varphi_{it} - [uR]\}\zeta\,dt + \sum_{i\in I}\int_{\gamma^i} e_{it}\zeta\,dt = 0,\end{gathered}$$

where $[g]$ is a jump of the function g across the discontinuity curve $x=\varphi_i(t)$, $[g] = g(\varphi_i+0) - g(\varphi_i-0)$.

This and (2.2) imply the system of equations

$$\begin{gathered} u_t + \frac{1}{2}(u^2)_x = 0, \\ R_t + (uR)_x = 0 \quad (x,t)\in\mathbb{R}^1\times\mathbb{R}^1_+ \setminus \bigcup_{i\in I}\{x=\varphi_i\}, \end{gathered} \tag{2.6}$$

$$\begin{gathered} \varphi_{it} = u(\varphi_i+0,t) + u(\varphi_i-0,t) = \frac{1}{2}\frac{[u^2]}{[u]}\Big|_{x=\varphi_i}, \\ e_{it} = \varphi_{it}[R]|_{x=\varphi_i} - [uR]|_{x=\varphi_i}, \quad i\in I. \end{gathered} \tag{2.7}$$

The signs of the summands in (2.7) differ from the signs of the similar summands in [3], since the jumps on the curves $x=\varphi_i$ are defined in different ways. Systems (2.6), (2.7) and (2.13), (2.14) are, in fact, known, see [1, 12].

In this case, at the nodes of the graph Γ lying above the axis $\{t=0\}$, the following “Kirchhoff laws” must be satisfied:

$$\sum_{i\in \operatorname{In} A_k} e_i(t_k^*-0) = \sum_{i\in \operatorname{Out} A_k} e_i(t_k^*+0), \tag{2.8}$$

where In and Out are the respective sets of incoming and outgoing arcs associated with a certain node $A_k = (x_k, t_k^*)$.

For system (1.1), we have the definition of the solution in the following form.

Definition 2.2. Let $\Gamma = \{\gamma_i, i\in I\}$ be a graph in the half-plane $\{x\in\mathbb{R}^1, t\ge 0\}$ containing C^1 arcs γ_i, and let I be a finite set. By $I_0\subset I$ we denoted the arcs starting from the point $x_k^0\in\mathbb{R}^1$. A functions $u=u(x,t)\in L^\infty(\mathbb{R}^1\times\mathbb{R}^+)\cap C^1((\mathbb{R}^1\times\mathbb{R}^+)\setminus\Gamma)$, a distribution $\rho(x,t)$, and a graph Γ, where

$$\begin{gathered} \rho(x,t) = R(x,t) + E(t)\delta(\Gamma), \quad E(t)\delta(\Gamma) = \sum_{i\in I} e_i(t)\delta(\gamma_i), \\ e_i(t)\in C^1(\gamma_i), \quad \gamma_i = \{x=\varphi_i(t)\}. \end{gathered} \tag{2.9}$$

The function $R(x,t) \in C^1((\mathbb{R}^1 \times \mathbb{R}^+) \setminus \Gamma)$ is called a *generalized δ-shock wave type solution* of (1.1) if the integral equalities

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^1} (Ru\zeta_t + Ru^2\zeta_x)\,dxdt + \sum_{i\in I}\int_{\gamma_i} \varphi_{it} e_i(t)\frac{d\zeta}{dt_i}\,dt \\
&+ \int_{\mathbb{R}^1} (Ru\zeta)\Big|_{t=0}\,dx + \sum_{k\in I_0} \varphi_{kt}(0)e_k(0)\zeta(\varphi_k(0),0) = 0,
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^1} (R\zeta_t + uR\zeta_x)\,dxdt + \sum_{i\in I}\int_{\gamma_i} e_i(t)\frac{d\zeta}{dt_i}\,dt \\
&+ \int_{\mathbb{R}^1} R\zeta\Big|_{t=0}\,dx + \sum_{k\in I_0} e_k(0)\zeta(\varphi_k(0),0) = 0,
\end{aligned} \tag{2.11}$$

hold for all test functions $\zeta(x,t) \in \mathcal{D}(\mathbb{R}^1 \times \mathbb{R}^1_+)$ and $\frac{d}{dt_i} = \frac{\partial}{\partial t} + \varphi_{it}\frac{\partial}{\partial x}$.

Relation (2.11) coincides exactly with relation (2.3). The second summand in the left-hand side of (2.10) can also be easily explained as well as the corresponding summand in (2.9). Indeed, in view of (2.4), we have

$$(\rho u^2)_x = \sum_{i\in I} e_i\varphi_{it}^2\delta'(\gamma_i) + \text{smoother summands}. \tag{2.12}$$

Now, just as above, for any test function $\zeta(x,t)$ such that $\zeta(x,0) = 0$, we have

$$\begin{aligned}
&\langle(\rho u)_t + (\rho u^2)_x, \zeta\rangle = -\langle\rho u, \zeta_t\rangle - \langle\rho u^2, \zeta_x\rangle \\
&= -\langle Ru, \zeta_t\rangle - \sum_{i\in I} e_i\varphi_{it}\langle\delta(\gamma_i),\zeta_t\rangle - \langle Ru^2,\zeta_x\rangle - \sum_{i\in I} e_i(\varphi_{it})^2\langle\delta(\gamma_i),\zeta_x\rangle \\
&= -\langle Ru, \zeta_t\rangle - \langle Ru^2,\zeta_x\rangle - \sum_{i\in I}\int_{\gamma_i} e_i\varphi_{it}(\zeta_t + \varphi_{it}\zeta_x)\,dt.
\end{aligned}$$

As in Definition 2.1, Definition 2.2 leads to a system of equations for the unknown functions u, R, e_i, φ_i contained in (2.10), (2.11):

$$\begin{gathered}
(Ru)_t + (Ru^2)_x = 0, \\
R_t + (Ru)_x = 0, \quad (x,t) \in (R^1 \times R^+_1) \setminus \bigcup\{x = \varphi_i\},
\end{gathered} \tag{2.13}$$

$$\begin{gathered}
(e_i\varphi_{it})'_t = \varphi_{it}[Ru]\Big|_{x=\varphi_i} - [Ru^2]\Big|_{x=\varphi_i}, \\
e_{it} = \varphi_{it}[R]\Big|_{x=\varphi_i} - [Ru]\Big|_{x=\varphi_i}, \quad i \in I.
\end{gathered} \tag{2.14}$$

Here an analog of the two "Kirchhoff laws" is given by the equations

$$\begin{gathered}
\sum_{i\in \mathrm{In}\,A_k} e_i(t_k^* - 0) = \sum_{i\in \mathrm{Out}\,A_k} e_i(t_k^* + 0), \\
\sum_{i\in \mathrm{In}\,A_k} e_i(t_k^* - 0)\varphi_{it}(t_k^* - 0) = \sum_{i\in \mathrm{Out}\,A_k} e_i(t_k^* + 0)\varphi_{it}(t_k^* + 0).
\end{gathered} \tag{2.15}$$

Obviously, a significant distinction of system (2.13) from Eqs. (2.7) is that system (2.13) consists of second-order equations and, formally, to solve this system, it is required to know the values $e_i(0)$, $\varphi_i(0)$, and $\varphi_{it}(0)$ (!). Obviously, these values cannot be found from the initial conditions for the original problem (we discuss this later in more detail).

Thus, we have the following theorem.

Theorem 2.3. *Suppose that system* (2.6)–(2.8) *(*(2.13)–(2.15)*) for* $t \in [0, T]$ *has a classical solution. Then system* (1.2) *(respectively,* (1.1)*) has a generalized* δ*-shock wave type solution in the sense of Definition 2.1 (Definition 2.2).*

Thus, just as in the case of classical shock waves, constructing generalized δ-shock type solutions is reduced to solving a system of ordinary differential equations [8, 9, 10].

The existence of the solution, for example, in the case of piecewise constant initial functions $u|_{t=0}$ and $\rho|_{t=0}$, can be proved easily.

As is easy to see, the solutions to systems (1.1) and (1.2) in the sense of the definitions given above satisfy the conservation laws in the following form.

Suppose that there exists a number A such that

$$\langle \rho(x,t), \eta(x)\rangle = 0, \quad t \in [0, T],$$

for any test function $\eta(x)$, $\sup \eta(x) \in \mathbb{R}^1 \setminus [-A, A]$, where $\langle \rho, \eta\rangle$ denotes the action of a generalized function $\rho(x,t)$ on the the test function $\eta(x)$, t is a parameter, and $\rho(x,t)$ is a component of the solution to system (1.1) or (1.2) in the sense of the above definitions constructed using the solutions of systems (2.13)–(2.15) or (2.6)–(2.8).

Lemma 2.4. *For any test function* $\zeta(x)$*,* $\zeta(x) = 1$ *for* $x \in [-A, A]$*,* $t \in [0, T]$*, the following relation holds:*

$$\langle \rho(x,t), \zeta(x)\rangle = \langle \rho(x,0), \zeta(x)\rangle.$$

For system (1.1), we can formulate one more conservation law.

Lemma 2.5. *The following relation holds:*

$$\langle \rho u, \zeta(x)\rangle = \langle \rho u|_{t=0}, \zeta(x)\rangle.$$

where $t \in [0, T]$*,* ζ *is a test function satisfying the assumption of Lemma* 1*,* ρ *and* u *are solutions of system* (1.1) *in the sense of Definition 2.2 constructed using solutions of system* (2.13)–(2.15)*.*

The proof of Lemma 1 can be found in [4]. Here we only prove Lemma 2 whose proof is similar to the proof of Lemma 1.

For simplicity, we consider the case in which the graph Γ contains a single arc $x = \varphi(t)$. Then we have

$$\begin{aligned}
\frac{d}{dt}\langle \rho u, \zeta\rangle &= \frac{d}{dt}\int_{\mathbb{R}^1} Ru\,dt + (e\varphi_t)_t \\
&= -\varphi_t[Ru]\Big|_{x=\varphi} + \int_{-\infty}^{\varphi} (Ru)_t\,dx + \int_{\varphi}^{\infty} (Ru)_t\,dx + (e\varphi_t')_t' \\
&= -\int_{-\infty}^{\varphi} (Ru^2)_x\,dx - \int_{\varphi}^{\infty} (Ru^2)_x\,dx + (e\varphi_t)_t - \varphi_t[Ru]\Big|_{x=\varphi} \\
&= [Ru^2]\Big|_{x=\varphi} - \varphi_t[Ru]\Big|_{x=\varphi} + (e\varphi_t)_t = 0.
\end{aligned}$$

The last equality is precisely the first equation in (2.14).

Now it is natural to pose the problem of the uniqueness of the solution.

Examples of nonuniqueness. In what follows, we give an answer to this question about the uniqueness in the form of examples. The general answer is the following: the solution may be nonunique if the initial conditions for ρ contain an atomic measure (the Dirac δ-function).

In particular, the solution of the Cauchy problem to system (1.2) is constructed in [2] in the case where the initial profile of velocity is an unstable step function. To construct the second component ρ of the solution, i.e., to solve the continuity equation, the authors [2] choose a class of functions invariant under the scaling transformation

$$x \to kx, \quad t \to kt.$$

Indeed, the group of scaling transformations acts on the solution of the system considered. But these are particular solutions. For example, in [11], such solutions are considered in a quite different context. In [2], the statement that a vacuum domain exists is derived from the assumption that such invariant solutions are unique. Such a statement cannot be made based only on the consideration of *particular* solutions.

The following natural question arises: Can solutions that are not contained in the class of solutions invariant under the action of the scaling group help to fill a vacuum?

More generally, the question can be formulated as follows: Do there exist any natural conditions ensuring the uniqueness of the Goursat problem solutions considered in [2]. In this small note, we give an affirmative answer to this question. Namely, for any initial regular distribution ρ with compact support (perhaps, with a first kind discontinuity), the solution of the Goursat problem is zero in the rarefaction domain.

In our considerations, we do not use the regularization procedure, which uniformly approximates the solution of the Cauchy problem for (1.1). This can be done using the simple formulas from [5], but the problem is very simple and does not require any special *technical* methods.

So the solution of the Cauchy problem for (1.2) with the initial conditions

$$\begin{gathered} u|_{t=0} = \begin{cases} u_l, & x < x_0, \\ u_r, & x > x_0, \end{cases} \quad u_{l,r} = \text{const}, \quad u_l < u_r, \\ \rho|_{t=0} = \begin{cases} \rho_l, & x < x_0, \\ \rho_r, & x > x_0, \end{cases} \quad \rho_{l,r} \geq 0, \end{gathered} \tag{2.16}$$

for $t > 0$ has the form

$$u = \begin{cases} u_l, & x < x_0 + u_l t, \\ u_l + \frac{x - u_l t - x_0}{t}, & x \in [x_0 + u_l t, x_0 + u_r t], \\ u_r, & x > x_0 + u_r t, \end{cases} \tag{2.17}$$

and, respectively,

$$\rho = \begin{cases} \rho_l(x - u_l t), & x < x_0 + u_l t, \\ \rho_0(\frac{x - x_0}{t}) t^{-1}, & x \in (x_0 + u_l t, x_0 + u_r t), \\ \rho_r(x - u_l t), & x > x_0 + u_r t, \end{cases} \tag{2.18}$$

where $\rho_0 = \rho_0(z)$ is an arbitrary C^1-function.

Formulas (2.17), (2.18) can be verified by a direct substitution. We only note that since the function u in (2.17) is continuous for $t > 0$, the Rankine-Hugoniot type conditions are identically satisfied on the lines $x = x_0 + u_j t$, $j = l, r$, because the equation for ρ is linear in ρ.

Calculating the integral $\int_{R^1} \rho(x,t)\,dt$ for $t > 0$ ($\rho(x,t)$ is defined in (2.18)), we obtain

$$\int_{R^1} \rho(x,t)\,dx = \int_{x<x_0} \rho_l\,dx + \int_{x>x_0} \rho_r\,dx + \int_{u_l}^{u_r} \rho_0(z)\,dz.$$

Hence, since $\rho|_{t=0}$ is nonnegative and $\langle \rho, \zeta \rangle$ is preserved by Lemma 1, we have

$$\int_{u_l}^{u_r} \rho_0(z)\,dz = 0.$$

Otherwise, we come to a contradiction, because from (2.16) and the mass conservation law we must have

$$\langle \rho|_{t=0}, \zeta \rangle = \int_{x<x_0} \rho_l\,dx + \int_{x>x_0} \rho_r\,dx = \int_{x<x_0} \rho_l\,dx + \int_{x>x_0} \rho_r\,dx + \int_{u_l}^{u_r} \rho_0(z)\,dz.$$

We point out that we derived this relation without any assumptions on the properties of *particular* solutions to system (1.2). We also note that a (more general than that in [2]) assumption ensuring the uniqueness of the solution of the Goursat problem in the case under study could be the assumption that ρ is bounded. However, if simultaneously with the rarefaction wave we consider shock waves in the u-component, then δ-shock type solutions arise, which is prohibited by the boundedness condition.

But if the initial condition for ρ is replaced by the condition

$$\rho|_{t=0} = \rho_l H(x_0 - x) + \rho_r H(x - x_0) + \hat{\rho}\delta(x - x_0),$$

then the choice of the function ρ_0 in (2.18) is restricted only by the condition

$$\int_{u_l}^{u_r} \rho_0(z)\,dz = \hat{\rho}$$

and the solution of this "singular" Goursat problem is not unique.

Nonuniqueness of the Cauchy problem solution in the case of system (1.1). It is proved in [3] that for system (1.1) to have a solution of the form

$$\begin{gathered} u = u_0(x,t) + u_1(x,t)H(\varphi(t) - x), \\ \rho = \rho_0(x,t) + \rho_1(x,t)H(\varphi(t) - x) + e(t)\delta(x - \varphi(t)) \end{gathered} \tag{2.19}$$

in the sense of the integral identity introduced in [3] (also see Definition 2 at the beginning), it is necessary that besides of other relations the following equations must be satisfied

$$e_t(t) = -([u\rho] - [\rho]\varphi_t)|_{x=\varphi}, \quad \frac{d}{dt}(e\varphi_t) + ([u^2\rho] - [u\rho]\varphi_t)|_{x=\varphi} = 0, \tag{2.20}$$

where $[f]|_{x=\varphi} = f(\varphi(t) + 0) - f(\varphi(t) - 0)$ as above.

We restrict ourselves to considering the case

$$u_0 = u_1 = \text{const}, \quad u_0 < 0, \quad u_0 + u_1 > 0, \quad \rho_0 = \rho_1 = \text{const} \geq 0. \tag{2.21}$$

Obviously, to construct the solution in this case, it suffices to construct the solution to system (1.2), which in this case is a system with constant coefficients.

We note that if conditions (2.20) are satisfied, then the inequality

$$\frac{u(x_2,t)-u(x_1,t)}{x_2-x_1} \le 0$$

holds for any small x_1, x_2 and hence the condition (see [7, p. 119])

$$\frac{u(x_2,t)-u(x_1,t)}{x_2-x_1} \le \frac{1}{t}$$

that the solution is an entropy solution is also satisfied.

Next, it follows from Definition 2.2 that

$$\rho u^2 = \rho_r u_r^2 + H(\varphi(t)-x)(\rho_1 u_l^2 - \rho_r u_r^2) + e(\varphi_t)^2\delta(x-\varphi(t))$$

and ρu^2 weakly converges to its initial value under the assumption that the functions $e(t)$ and $\varphi_t(t)$ are continuous for $t \ge 0$. This readily follows from the formulas for the solution of system (2.20) given below.

It is easy to see that (2.20) form a second-order system of equations for e, φ. The original system is a first-order system, hence the value $\varphi_t(0)$ remains undetermined. In [12], it is shown that, in the case of constant u_i, ρ_i, $i = 1, 2$, system (2.3) has a unique solution if $e(0) = 0$. In this case, the solution is independent of $\varphi_t(0)$.

The formula for $\varphi(t)$ obtained in [12, Theorem 4.3] has the form

$$\varphi(t) = \frac{e(0) + [u\rho]t - \sqrt{e(0)^2 + 2e(0)e_t(0)t + \rho_r\rho_l(u_r-u_l)^2t^2}}{[\rho]}, \quad [\rho] \ne 0,$$

It follows from the first equation in (2.20) that the values of the constants $e_t(0)$ and $\varphi_t(0)$ can be expressed linearly in terms of each other

$$e_t(0) = [u\rho]^0 - [\rho]^0\varphi_t(0),$$

where $[\;]^0 = [\;]\big|_{t=0}$ and hence the quantities $[\;]^0$ are functions of the argument $\varphi(0)$.

From these formulas it is easily seen that, in the case $e(0) = 0$, the expression $\varphi_t(0)$ $(e_t(0))$ is not contained in the formula for $\varphi(t)$.

Indeed, for $e(0) = 0$ and $[\rho] \ne 0$, relations (2.20) imply the following equation for $\varphi_t(0)$:

$$\varphi_t(0)^2[\rho]^0 - 2\varphi_t(0)[u\rho]^0 + [u^2\rho]^0 = 0.$$

Solving this equation under the additional condition $u_r|_{t=0} < \dot\varphi(0) < u_l|_{t=0}$, which is necessary for the existence of the desired δ-shock type solutions, we obtain

$$\varphi_t(0) = \Big([u\rho]^0 - \sqrt{([u\rho]^0)^2 - [\rho]^0[u^2\rho]^0}\Big)([\rho]^0)^{-1} =: G(\varphi(0)).$$

Thus, in the case $e(0) = 0$, the missing constant is determined by the natural initial data of the problem.

It is also easy to verify that $e_t(0) > 0$ in this case. Hence from the second equation in (21) we obtain $|\frac{d^2\varphi}{dt^2}(0)| < \infty$. Therefore, although the coefficient of the second derivative $\frac{d^2\varphi}{dt^2}$ vanishes for $t = 0$, system (2.20) has a smooth solution at least in the small in t. This can be proved as usual, by reducing the problem to an integral equation.

The case $\rho_1|_{t=0} = -[\rho]^0 = 0$ is considered similarly (see [12]) for $u_i, \rho_i = \text{const}$. In this case,

$$e(t) = e(0) - t[\rho u] = e(0) - t\rho[u]$$

and the problem is reduced to solving the ordinary differential equation

$$e_t(t)\varphi_t + e\varphi_{tt} = \varphi_t\rho[u] - \rho[u^2].$$

Hence we obtain

$$\varphi_t = \frac{[u^2]}{2[u]} + \Big(\varphi_t(0) - \frac{[u^2]}{2[u]}\Big)\frac{e(0)}{(e(0) - t\rho[u])^2}.$$

It is clear that for $e(0) = 0$, the solution is independent of the constant $\varphi_t(0)$, which cannot be determined from the Cauchy conditions. We note once more that (23) implies that $[u] = -u_1 < 0$.

We note that if we use the construction of the nonconservative Volpert-Khudaev product and the definition of the measure solution that follows from this construction, then we, as was already noted in the Introduction, fix the definition of the product $\delta(z)H(z)$. In this case, this means that we set $u|_{x=\varphi} = \varphi_t$ and, in particular, $\varphi_t(0) = u|_{\substack{x=\varphi\\t=0}}$. Thus, the last term in the first integral identity in Definition 3 seems to be determined by specifying the initial velocity and the solution of the entire system (1.1) is determined by specifying two initial conditions (for the velocity and density) in the form (21). But, as was already noted, this is a delusion: such a method for specifying the initial conditions is not necessary (unique).

Thus, we see that the "singular" Cauchy problem for system (1.1) does not have the property that the solution is unique and the problem whose initial conditions do not contain the Dirac function has a unique solution.

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Moscow Technical University of Communications and Informatics, Aviamotornaya, 8a, 111024, Moscow, Russia

E-mail address: danilov@miem.edu.ru