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POSITIVE PERIODIC SOLUTIONS OF NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH A PARAMETER AND IMPULSE

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Abstract. In this paper, we consider first-order neutral differential equations with a parameter and impulse in the form of

$$\begin{gathered}\frac{d}{dt}[x(t)-cx(t-\gamma)]=-a(t)g(x(h_1(t)))x(t)+\lambda b(t)f\big(x(h_2(t))\big),\quad t\neq t_j;\\ \Delta\big[x(t)-cx(t-\gamma)\big]=I_j\big(x(t)\big),\quad t=t_j,\ j\in\mathbb{Z}^+.\end{gathered}$$

Leggett-Williams fixed point theorem, we prove the existence of three positive periodic solutions.

1. Introduction

The existence of periodic solutions of delay differential equations with or without impulses has been a focus of theoretical and practical importance because time-delay occurs areas such as mechanics, physics, biology, economy, population dynamic models, large-scale systems, automatic control systems, neural networks, chaotic systems, and so on. The existence of periodic solutions of time-delay systems with or without impulses has been extensively studied by Many researchers have studied this problem; se for example [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 17, 18, 19, 20, 21] and references therein. However, relatively few papers have been published on the existence of periodic solutions for neutral functional differential equations.

In this paper, we are concerned with the neutral differential equation

$$\begin{gathered}\frac{d}{dt}\big[x(t)-cx(t-\gamma)\big]=-a(t)g\big(x(h_1(t))\big)x(t)+\lambda b(t)f\big(x(h_2(t))\big),\quad t\neq t_j,\\ \Delta\big[x(t)-cx(t-\gamma)\big]=I_j\big(x(t)\big),\quad t=t_j,\ j\in\mathbb{Z}^+,\end{gathered}\tag{1.1}$$

where $\lambda>0$ is a positive parameter and there exists a positive constant q such that $t_{j+q}=t_j+\omega$, $I_{j+q}(x(t_{j+q}))=I_j(x(t_j))$, $j\in\mathbb{Z}^+$. Without loss of generality, we assume that $[0,\omega]\cap\{t_j,j\in\mathbb{Z}^+\}=\{t_1,t_2,\cdots,t_q\}$.

In this paper, we will use the following assumptions:

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- (H1) $a \in C(\mathbb{R}, [0, +\infty))$ is ω-periodic and there exists $t_1^* \in (0, \omega)$ such that $a(t_1^*) > 0$;
- (H2) $b \in C(\mathbb{R}, [0, +\infty))$ is ω-periodic and there exists $t_2^* \in (0, \omega)$ such that $b(t_2^*) > 0$;
- (H3) $h_1(t)$, $h_2(t) \in C(\mathbb{R}, \mathbb{R})$ are ω-periodic;
- (H4) $g \in C([0, \infty), [0, \infty))$, $I_j \in C([0, \infty), [0, \infty))$ and $f \in C([0, \infty), [0, \infty))$ are continuous, $0 < l \leq g(u) < L < \infty$ for all $u > 0$, l, L are two positive constants.

Using Leggett-Williams fixed point theorem [5] to show the existence of at least three positive periodic solutions of (1.1). To the best of the authors' knowledge, few authors discuss at least three positive periodic solutions for neutral functional differential equations with impulses and parameters.

2. Preliminaries

To obtain the existence of periodic solutions of system (1.1), we first make the following preparations.

Let $\beta = \int_0^\omega a(s)\, ds$, where a is a continuous ω-periodic function. In what follows, we set

$$\mathbb{X} = \big\{x \in C(\mathbb{R}, \mathbb{R}) : x(t + \omega) = x(t)\big\}$$

and define $\|x\| = \max\{|x(t)| : t \in [0, \omega]\}$. Then $(\mathbb{X}, \|\cdot\|)$ is a Banach space. Let $A : \mathbb{X} \to \mathbb{X}$ be defined by

$$(Ax)(t) = x(t) - cx(t - \gamma).$$

Lemma 2.1 ([13]). *If $|c| \neq 1$, then A has continuous bounded inverse A^{-1} and for all $x \in \mathbb{X}$,*

$$(A^{-1}x)(t) = \begin{cases} \sum_{j\geq 0} c^j x(t - j\gamma), & \text{if } |c| < 1, \\ -\sum_{j\geq 1} c^{-j} x(t + j\gamma), & \text{if } |c| > 1\,. \end{cases} \tag{2.1}$$

Then

$$\|A^{-1}x\| \leq \frac{\|x\|}{|1 - |c||}.$$

To establish the existence of periodic solutions of (1.1), we first consider the system

$$\begin{gathered} \frac{d}{dt}u(t) = -a(t)g\big((A^{-1}u)(h_1(t))\big)(A^{-1}u)(t) + \lambda b(t) f\big((A^{-1}u)(h_2(t))\big), \quad t \neq t_j, \\ \Delta u(t) = I_j\big((A^{-1}u)(t)\big), \quad t = t_j,\, j \in \mathbb{Z}^+, \end{gathered} \tag{2.2}$$

where A^{-1} is defined by (2.1). By Lemma 2.1, we conclude the following result.

Lemma 2.2. *$u(t)$ is an ω-periodic solution of (2.2) if and only if $(A^{-1}u)(t)$ is an ω-periodic solution of (1.1).*

Let $\mathbb{X}$ be a Banach space and K is a closed, nonempty subset of $\mathbb{X}$. K is a cone provided that

- (i) $\alpha_1 u + \beta_1 y \in K$ for all $u, v \in K$ and $\alpha_1, \beta_1 \geq 0$;
- (ii) $u, -u \in K$ imply $u = 0$.

Define $K_r = \{x \in K : \|x\| \leq r\}$. Let $\alpha(x)$ denote the positive continuous concave functional on K, that is $\alpha : K \to [0,\infty)$ is continuous and satisfies

$$\alpha(\lambda x + (1-\lambda)y) \geq \lambda\alpha(x) + (1-\lambda)\alpha(y) \quad \text{for all } x, y \in K, 0 \leq \lambda \leq 1,$$

and we denote the set $K(\alpha, a_1, b_1) = \{x \in K : a_1 \leq \alpha(x), \|x\| \leq b_1\}$.

Lemma 2.3 ([5]). *Let K be a cone of the real Banach space $\mathbb{X}$ and $\Phi : K_{c_3} \to K_{c_3}$ be a completely continuous operator, and suppose that there exists a concave positive functional α with that $\alpha(x) \leq \|x\|$ for $x \in K$ and numbers c_1, c_2, c_3, c_4 with $0 < c_4 < c_1 < c_2 \leq c_3$, satisfying the following conditions:*

(1) *$\{x \in K(\alpha, c_1, c_2) : \alpha(x) > c_1\} \neq \emptyset$ and $\alpha(\Phi x) > c_1$ if $x \in K(\alpha, c_1, c_2)$;*
(2) *$\|\Phi x\| < c_4$ if $x \in K_{c_4}$;*
(3) *$\alpha(\Phi x) > c_1$ for all $x \in K(\alpha, c_1, c_2)$ with $\|\Phi x\| > c_2$.*

Then Φ has at least three fixed points in K_{c_3}.

Aiming to apply Lemma 2.2 to (2.2), we rewrite (2.2) as

$$\begin{aligned} \frac{d}{dt}u(t) &= -a(t)g\big((A^{-1}u)(h_1(t))\big)u(t) + \big[a(t)\widehat{G}\big(u(t), u(h_1(t))\big) \\ &\quad + \lambda b(t)f\big((A^{-1}u)(h_2(t))\big)\big], \quad t \neq t_j, \\ \Delta u(t) &= I_j\big((A^{-1}u)(t)\big), \quad t = t_j,\, j \in \mathbb{Z}^+, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} \widehat{G}(u(t), u(h_1(t))) &= g\big((A^{-1}u)(h_1(t))\big)\big[u(t) - (A^{-1}u)(t)\big] \\ &= -cg\big((A^{-1}u)(h_1(t))\big)(A^{-1}u)(t-\gamma). \end{aligned}$$

The following lemma is fundamental in our discussion. Since the method is similar to that in the literature [15], we omit the proof.

Lemma 2.4. *$x(t)$ is an ω-periodic solution of* (1.1) *is equivalent to $u(t)$ is an ω-periodic solution of*

$$\begin{aligned} u(t) &= \int_t^{t+\omega} G(t,s)\Big[a(t)\widehat{G}\big(u(t), u(h_1(t))\big) + \lambda b(t)f\big((A^{-1}u)(h_2(t))\big)\Big]\,ds \\ &\quad + \sum_{j:t_j\in[0,\omega]} G(t,t_j)I_j\big((A^{-1}u)(t_j)\big), \end{aligned} \tag{2.4}$$

where

$$G(t,s) = \frac{e^{\int_t^s a(\theta)g((A^{-1}x)(h_1(\theta)))\,d\theta}}{e^{\int_0^\omega a(\theta)g((A^{-1}x)(h_1(\theta)))\,d\theta} - 1}, \quad s \in [t, t+\omega].$$

For $u \in \mathbb{X}$ and $t \in \mathbb{R}$, let the map Φ be defined by

$$\begin{aligned} (\Phi u)(t) &= \int_t^{t+\omega} G(t,s)\Big[a(t)\widehat{G}\big(u(t), u(h_1(t))\big) + \lambda b(t)f\big((A^{-1}u)(h_2(t))\big)\Big]\,ds \\ &\quad + \sum_{j:t_j\in[0,\omega]} G(t,t_j)I_j\big((A^{-1}u)(t_j)\big) \end{aligned} \tag{2.5}$$

It is easy to see that $G(t+\omega, s+\omega) = G(t,s)$ and

$$\frac{1}{\sigma^L - 1} \leq G(t,s) \leq \frac{\sigma^l}{\sigma^l - 1}, \quad s \in [t, t+\omega], \tag{2.6}$$

where $\sigma = \exp\big(-\int_0^\omega a(\theta)\,d\theta\big)$. Define the cone K in $\mathbb{X}$ by
$$K = \Big\{u \in \mathbb{X} : u(t) \ge \delta\|u\|,\ t \in [0,\omega]\Big\},$$
where $0 < \delta = \frac{(\sigma^l - 1)}{\sigma^l(\sigma^L - 1)} < 1$.

The following lemma is useful in the proofs of our main results. Since the method is similar to that in the literature [9], we omit the proof.

Lemma 2.5. *If $c \in (-\delta, 0]$ and $u \in K$. Then*
$$l|c|\frac{\delta - |c|}{1 - c^2}\|u\| \le \widehat{G}\big(u(t), u(h_1(t))\big) \le L\frac{|c|}{1 - |c|}\|u\|.$$

Lemma 2.6. *Assume that* (H1)–(H4) *and $c \in (-\delta, 0]$ hold, then Φ maps K into K.*

Proof. For any $u \in K$, it is clear that $\Phi u \in C(\mathbb{R},\mathbb{R})$, we have
$$\begin{aligned}
(\Phi u)(t+\omega) &= \int_t^{t+\omega} G(t+\omega, s+\omega)\Big[a(t+\omega)\widehat{G}\big(u(t+\omega), u(h_1(t+\omega))\big) \\
&\quad + \lambda b(t+\omega) f\big((A^{-1}u)(h_2(t+\omega))\big)\Big]\,ds \\
&\quad + \sum_{j:t_j\in[0,\omega]} G(t+\omega, t_j+\omega) I_j\big((A^{-1}u)(t_j+\omega)\big) \\
&= \int_t^{t+\omega} G(t,s)\Big[a(t)\widehat{G}\big(u(t), u(h_1(t))\big) + \lambda b(t) f\big((A^{-1}u)(h_2(t))\big)\Big]\,ds \\
&\quad + \sum_{j:t_j\in[0,\omega]} G(t,t_j) I_j\big((A^{-1}u)(t_j)\big) \\
&= (\Phi u)(t).
\end{aligned}$$
Thus, $(\Phi u)(t+\omega) = (\Phi u)(t)$, $t \in \mathbb{R}$. So that $\Phi u \in \mathbb{X}$. Since $c \in (-\delta, 0]$, it follows that $G(u(t), u(h_1(t))) \ge 0$ for $t \in \mathbb{R}$. In view of (2.5), (2.6), for $u \in K$, $t \in [0,\omega]$, we have
$$\begin{aligned}
\|(\Phi u)\| &\le \frac{\sigma^l}{\sigma^l - 1}\Big(\int_t^{t+\omega} \Big[a(t)\widehat{G}\big(u(t), u(h_1(t))\big) + \lambda b(t) f\big((A^{-1}u)(h_2(t))\big)\Big]\,ds \\
&\quad + \sum_{j:t_j\in[0,\omega]} I_j\big((A^{-1}u)(t_j)\big)\Big)
\end{aligned}$$
and
$$\begin{aligned}
|(\Phi u)(t)| &\ge \frac{1}{\sigma^L - 1}\Big(\int_t^{t+\omega} \Big[a(t)\widehat{G}\big(u(t), u(h_1(t))\big) + \lambda b(t) f\big((A^{-1}u)(h_2(t))\big)\Big]\,ds \\
&\quad + \sum_{j:t_j\in[0,\omega]} I_j\big((A^{-1}u)(t_j)\big)\Big) \\
&\ge \delta\|\Phi u\|.
\end{aligned}$$
Hence, $\Phi x \in K$. The proof is complete. $\square$

It is easy to see that Φ is continuous and bounded. From Lemma 2.5, we know that Φ maps bounded sets into relatively compact sets. Furthermore, by the theorem of Ascoli-Arzela [16], it is easy to prove that the function Φ is completely continuous.

For convenience in the following discussion, we introduce the following notation:

$$f^0 = \limsup_{v\to 0} \frac{f(v)}{v}, \quad I^0 = \limsup_{v\to 0} \sum_{j=1}^{q} \frac{I_j(v)}{v},$$
$$f^\infty = \limsup_{v\to \infty} \frac{f(v)}{v}, \quad I^\infty = \limsup_{v\to \infty} \sum_{j=1}^{q} \frac{I_j(v)}{v}$$

and for $c_2 > 0$,

$$I_{(c_2)} = \min_{\delta c_2 \le v \le c_2} \sum_{j=1}^{q} I_j(v).$$

3. Main Result

Our main result of this paper is stated as follows.

Theorem 3.1. *Assume that* (H1)-(H4) *and* $c \in (-\eta, 0]$, *where*

$$\eta := \min\big\{\delta,\, 1 - \frac{l\sigma^l\beta}{(\sigma^l - 1) + L\sigma^l\beta}\big\}.$$

Then there exists a number $c_2 > 0$ *such that*

(i) *For* $\delta c_2 \le u \le c_2$, $t \in R$,

$$f\big((A^{-1}u)(h_2(s))\big) > \frac{\sigma^l(\sigma^L - 1)}{\sigma^l - 1} u - \frac{\sigma^l}{\sigma^l - 1} I_{(c_2)} - \frac{\sigma^l}{\sigma^l - 1} l|c| \frac{\delta - |c|}{1 - c^2} \beta u\,;$$

(ii)

$$f^0 + I^0 < \frac{(1 - |c|)(\sigma^l - 1)}{L\sigma^l |c|} - \beta, \quad f^\infty + I^\infty < \frac{(1 - |c|)(\sigma^l - 1)}{L\sigma^l |c|} - \beta\,.$$

Then system (1.1) *has at least three positive* ω*-periodic solutions for*

$$\frac{\sigma^l - 1}{\sigma^l \int_t^{t+\omega} b(s)\,\mathrm{d}s} < \lambda < \frac{1}{\int_t^{t+\omega} b(s)\,\mathrm{d}s}.$$

Proof. By the condition $f^\infty + I^\infty < \frac{(1-|c|)(\sigma^l-1)}{L\sigma^l|c|} - \beta$ of (ii), one can find that for

$$\frac{\frac{(1-|c|)(\sigma^l-1)}{L\sigma^l|c|} - \beta - (f^\infty + I^\infty)}{2} > \varepsilon > 0,$$

there exists a $C_0 > c_2$ such that

$$\limsup_{v\to\infty} f(v) \le (f^\infty + \varepsilon)v, \quad \limsup_{v\to\infty} \sum_{j=1}^{q} I_j(v) \le (I^\infty + \varepsilon)v,$$

where $u > C_0$. Let $C_1 = C_0/\delta$, if $u \in K$, $\|u\| > C_1$, thus $u > C_0$ and we have

$$
\begin{aligned}
(\Phi u)(t) &= \int_t^{t+\omega} G(t,s)\Big[a(t)\widehat{G}\big(u(t),u(h_1(t))\big) + \lambda b(t) f\big((A^{-1}u)(h_2(t))\big)\Big]\,ds \\
&\quad + \sum_{j:t_j\in[0,\omega]} G(t,t_j) I_j\big((A^{-1}u)(t_j)\big) \\
&\le \frac{\sigma^l}{\sigma^l-1}\Big\{L\frac{|c|}{1-|c|}\|u\|\int_t^{t+\omega} a(t)ds \\
&\quad + (f^\infty+\varepsilon)L\frac{|c|}{1-|c|}\|u\|\int_t^{t+\omega}\lambda b(s)\,\mathrm{d}s + (I^\infty+\varepsilon)L\frac{|c|}{1-|c|}\|u\|\Big\} \\
&= \frac{L\sigma^l|c|}{(1-|c|)(\sigma^l-1)}\Big\{\beta + (f^\infty+\varepsilon)\int_t^{t+\omega}\lambda b(s)\,\mathrm{d}s + (I^\infty+\varepsilon)\Big\}\|u\| \\
&< \|u\|.
\end{aligned} \tag{3.1}
$$

Take $K_{C_1} = \{u \in K : \|u\| \le C_1\}$, then the set K_{C_1} is a bounded set. Since Φ is completely continuous, Φ maps bounded sets into bounded sets and there exists a number C_2 such that

$$\|\Phi u\| \le C_2 \quad \text{for all } u \in K_{C_1}.$$

If $C_2 \le C_1$, we obtain that $\Phi : K_{C_1} \to K_{C_1}$ is completely continuous. If $C_1 < C_2$, then from (3.1), we know that for any $u \in K_{C_2}\backslash K_{C_1}$, $\|u\| > C_1$ and $\|\Phi u\| < \|u\| < C_2$ hold. Then we obtain $\Phi : K_{C_2} \to K_{C_2}$ is completely continuous. Now, take $c_3 = \max\{C_1, C_2\}$, obviously $c_3 > c_2$ and $\Phi : K_{c_3} \to K_{c_3}$ is completely continuous.

Define the positive continuous concave functional $\alpha(u) = \min_{t\in[0,\omega]}\big\{|u(t)|\big\}$. First, we let $c_1 = \delta c_2$ and take $u \equiv \frac{c_1+c_2}{2}$, $u \in K(\alpha, c_1, c_2)$, $\alpha(u) > c_1$, then the set $\{u \in K(\alpha, c_1, c_2)\} \ne \emptyset$. And by (i), if $u \in K(\alpha, c_1, c_2)$, then $\alpha(u) \ge c_1$, and we have

$$
\begin{aligned}
&\alpha(\Phi u) \\
&= \min_{t\in[0,\omega]}\Big\{\int_t^{t+\omega} G(t,s)\Big[a(t)\widehat{G}\big(u(t),u(h_1(t))\big) + \lambda b(t) f\big((A^{-1}u)(h_2(t))\big)\Big]\,ds \\
&\quad + \sum_{j:t_j\in[0,\omega]} G(t,t_j) I_j\big((A^{-1}u)(t_j)\big)\Big\} \\
&\ge \frac{1}{\sigma^L-1}\Big\{l|c|\frac{\delta-|c|}{1-c^2}u\beta + \min_{t\in[0,\omega]}\Big\{\int_t^{t+\omega}\lambda b(s) f\big((A^{-1}u)(h_2(s))\big)\,ds\Big\} + I_{(c_2)}\Big\} \\
&> \frac{1}{\sigma^L-1}\Big\{l|c|\frac{\delta-|c|}{1-c^2}u\beta + \Big[\frac{\sigma^l(\sigma^L-1)}{\sigma^l-1}a(u) - \frac{\sigma^l}{\sigma^l-1}I_{(c_2)} \\
&\quad - \frac{\sigma^l}{\sigma^l-1}l|c|\frac{\delta-|c|}{1-c^2}\beta u\Big]\lambda\int_t^{t+\omega} b(s)ds + I_{(c_2)}\Big\} \\
&= \alpha(x) \ge c_1.
\end{aligned}
$$

Thus condition (1) of Lemma 2.3 holds.

Secondly, by the inequality $f^0 + I^0 < \frac{(1-|c|)(\sigma^l-1)}{L\sigma^l|c|} - \beta$ in condition of (ii), one can find that for

$$\frac{\frac{(1-|c|)(\sigma^l-1)}{L\sigma^l|c|} - \beta - (f^0+I^0)}{2} > \varepsilon > 0,$$

there exists c_4, with $0 < c_4 < c_1$ such that

$$\limsup_{v\to 0} f(v) \le (f^0+\varepsilon)v,\ \limsup_{v\to 0}\sum_{j=1}^{q} I_j(v) \le (I^0+\varepsilon)v,$$

where $0 \le v \le c_4$. If $u \in K_{c_4} = \Big\{u \Big| \|u\| \le c_4\Big\}$, then we have

$$\begin{aligned}
(\Phi u)(t) &= \int_t^{t+\omega} G(t,s)\Big[a(t)\widehat{G}\big(u(t),u(h_1(t))\big) + \lambda b(t) f\big((A^{-1}u)(h_2(t))\big)\Big]\,ds \\
&\quad + \sum_{j:t_j\in[0,\omega]} G(t,t_j) I_j\big((A^{-1}u)(t_j)\big) \\
&\le \frac{\sigma^l}{\sigma^l-1}\Big\{L\frac{|c|}{1-|c|}\|u\| \int_t^{t+\omega} a(t)ds \\
&\quad + (f^0+\varepsilon)L\frac{|c|}{1-|c|}\|u\| \int_t^{t+\omega} \lambda b(s)\,\mathrm{d}s + (I^0+\varepsilon)L\frac{|c|}{1-|c|}\|u\|\Big\} \\
&= \frac{L\sigma^l|c|}{(1-|c|)(\sigma^l-1)}\Big\{\beta + (f^0+\varepsilon)\int_t^{t+\omega} \lambda b(s)\,\mathrm{d}s + (I^0+\varepsilon)\Big\}\|u\| \\
&< \|u\| \le c_4.
\end{aligned}$$

That is, condition (2) of Lemma 2.3 holds.

Finally, if $x \in K(\alpha, c_1, c_3)$ with $\|\Phi u\| > c_2$, by the definition of the cone K, we have

$$\Phi u \ge \delta\|\Phi u\| > \delta c_2 = c_1.$$

Thus condition (3) of Lemma 2.3 holds. Therefore, by Lemma 2.3, we obtain that the operator Φ has at least three fixed points in K_{c_3}. From Lemma 2.2, we know that (1.1) has at least three fixed points in K_{c_3}. The proof of Theorem 3.1 is complete. □

Corollary 3.2. *The conclusion in Theorem 3.1, sis still true when (ii) is replaced by*

(ii*) $f^0 = 0,\ \widehat{f}^0 = 0,\ f^\infty = 0,\ \widehat{f}^\infty = 0.$

4. An example

Consider the problem

$$\begin{gathered}
\frac{d}{dt}\big[x(t) - \frac{1}{3}x(t-\frac{\pi}{2})\big] = -\frac{1}{2\pi}(\frac{1}{3} + e^{-x(t)})x(t) + \lambda(1-\sin t)x^2(t)e^{-x(t)}, \quad t \ne t_j, \\
\Delta\big[x(t) - \frac{1}{3}x(t-\frac{\pi}{2})\big] = 0.1x^3(t_j)e^{-3x(t_j)}, \quad t = t_j,\ j\in\mathbb{Z}^+,
\end{gathered} \tag{4.1}$$

where λ is nonnegative parameter. Take $\gamma = \frac{\pi}{2}$, $c = \frac{1}{3}$, $a(t) = \frac{1}{2\pi}$, $b(t) = 1-\sin t$, $j = 2k$, $k = 1, 2, \dots$, $g(x(h_1(t))) = \frac{1}{3} + e^{-x(t)}$, and $f(x(h_2(t))) = x^2(t)e^{-x(t)}$. Clearly, $L = \frac{4}{3}$, $l = \frac{1}{3}$ and $\beta = 1$. According to Corollary 3.2, Equation (4.1) has at least three positive periodic solutions.

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