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OSCILLATION THEORY FOR A PAIR OF SECOND ORDER DYNAMIC EQUATIONS WITH A SINGULAR INTERFACE

PALLAV KUMAR BARUAH, DASU KRISHNA KIRAN VAMSI

Abstract. In this paper we consider a pair of second order dynamic equations defined on the time scale $I = [a, c] \cup [\sigma(c), b]$. We impose matching interface conditions at the singular interface c. We prove a theorem regarding the relationship between the number of eigenvalues and zeros of the corresponding eigenfunctions.

1. Introduction

The study of waves plays a important role in physical sciences. Waves of simple nature oscillate with a fixed frequency and wave-length. The study of these simple sinusoidal waves form the basis for the study of almost all forms of linear wave motion. The oscillation nature of waves can be modelled by differential equations specifically by ordinary Sturm-Liouville operators. In [1], the oscillatory nature of the self-adjoint second order dynamic equation

$$Lx(t) = (px^{\Delta})^{\Delta}(t) + q(t)x^{\sigma}(t)$$

is discussed. Also, Sturm's comparison and separation theorems have been proved for the self-adjoint matrix equations

$$LX(t) = (PX^{\Delta})^{\Delta}(t) + Q(t)X^{\sigma}(t).$$

In [15], the oscillatory and nonoscillatory behaviour of solutions of second-order linear difference equations is discussed. In literature of time scales, substantial amount of work has been done on oscillation behaviour of nonlinear dynamic equations [2, 3, 4, 5, 6, 16].

In the literature we find a new class of interface problems, termed as mixed pair of equations, discussed in the papers [7, 10, 11, 12, 13, 14, 17, 18, 19, 20, 21, 22, 23] where two different differential equations are defined on adjacent intervals with a common point of interface and the solutions satisfy a matching condition at the point of interface. We observe that the above problem for the regular case has been discussed in [17, 18, 19, 20, 21, 23]. In [7] the authors discuss an application of the classical Weyl limit criterion to define the coefficients with well-known Wronskian boundary conditions to tackle the singularity at the boundary for this class of

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problems. Though this work is specifically for Sturm-Liouville problems, it paves a way to study the problem of singularity at the end boundary points. But the problem of having a singularity at the point of interface remained unexplored. This problem of having singularity at the point of interface is discussed in [8, 9]. In [8], the Green's matrix is obtained for a boundary value problem involving a pair of dynamic equations with a singular interface. In [9], the existence of matching solutions for an initial value problem involving a pair of dynamic equations with a singular interface is discussed.

In this paper we study the oscillation theory for dynamic equations and also deal with the problem of having singularity at the point of interface. In this direction, we intend to study the oscillation behaviour for a pair of dynamic equations having a singularity at the point of interface. In this paper, we prove a theorem that gives the relationship between the number of eigenvalues and zeros of the corresponding eigenfunctions.

In Section 2, we give few mathematical definitions, which we use through the rest of the paper and in Section 3, we define the pair of second order dynamic equations with matching interface conditions. We also define the angle functions Θ and $\hat{\Theta}$. In Section 4, we prove a theorem involving the angle functions Θ and $\hat{\Theta}$. Finally, in Section 5, we prove a theorem that gives the relationship between number of eigenvalues and zeros of the corresponding eigenfunctions.

2. Mathematical Preliminaries

Definition 2.1. Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$ we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\},$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

If $\sigma(t) > t$, we say that t is right-scattered, while $\rho(t) < t$ we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. Points that are right-dense and left-dense at the same time are called dense. Finally, the graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$

Definition 2.2. A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by

$$\mathcal{C}_{rd} = \mathcal{C}_{rd}(\mathbb{T}).$$

The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by

$$\mathcal{C}^1_{rd} = \mathcal{C}^1_{rd}(\mathbb{T}) = \mathcal{C}^1_{rd}(\mathbb{T}, \mathbb{R}).$$

Definition 2.3. The function $\Theta(x, y) = \tan^{-1}\left(\frac{x}{y}\right)$ that gives the angle that the point (x, y) makes with the positive x-axis is called the angle function.

3. Pair of Dynamic Equations and the Angle Functions

Let $I_1 = [a, c]$, $I_2 = [\sigma(c), b]$ and $I = I_1 \cup I_2$ for $-\infty < a, b, c < \infty$. Let

$$L_1 X_1 = \frac{1}{r_1}(-(p_1 X_1^{\Delta})^{\Delta} + q_1 X_1) \tag{3.1}$$

be defined on I_1, and let

$$L_2 X_2 = \frac{1}{r_2}(-(p_2 X_2^{\Delta})^{\Delta} + q_2 X_2) \tag{3.2}$$

be defined on I_2, where $p_i \in \mathcal{C}^1_{rd}(I_i), q_i$ and $r_i \in \mathcal{C}^1_{rd}(I_i)$ are real valued functions, $p_i(t) > 0$ and $r_i(t) > 0$ for all $t \in I_i, i = 1, 2$. We assume that the functions X_1, X_2 satisfy the matching conditions

$$X_1(c) = X_2(\sigma(c)), \tag{3.3}$$

$$p_1(c) X_1^{\Delta}(c) = p_2(\sigma(c)) X_2^{\Delta}(\sigma(c)). \tag{3.4}$$

For a real number λ, let us consider the pair of dynamic equations

$$L_1 X_1 = \lambda X_1 \quad \text{on } I_1, \tag{3.5}$$

$$L_2 X_2 = \lambda X_2 \quad \text{on } I_2, \tag{3.6}$$

together with the matching interface conditions (3.3), (3.4). It follows from [1, Corollary 5.90 and Theorem 5.119] that problem (3.5), (3.6) along with conditions (3.3), (3.4) has two linearly independent real valued solutions. For a real valued nontrivial solution $X = (X_1, X_2)$ of (3.5), (3.6) along with conditions (3.3), (3.4), we define the new dependent variables ρ_1, ρ_2 and Θ_1, Θ_2 by

$$X_i(t) = \rho_i(t) \sin \Theta_i(t), \tag{3.7}$$

$$p_i(t) X_i^{\Delta}(t) = \rho_i(t) \cos \Theta_i(t), \tag{3.8}$$

where the angle function $\Theta_i(t)$ satisfies

$$\Theta_i(t) = tan^{-1} \frac{X_i(t)}{p_i(t) X_i^{\Delta}(t)} \quad \text{for } t \in I_i,\ i = 1, 2. \tag{3.9}$$

From conditions $X_1(c) = X_2(\sigma(c))$, $p_1(c) X_1^{\Delta}(c) = p_2(\sigma(c)) X_2^{\Delta}(\sigma(c))$, and

$$\Theta_i(t) = \tan^{-1} \frac{X_i(t)}{p_i(t) X_i^{\Delta}(t)} \quad \text{for } t \in I_i,\ i = 1, 2,$$

we get that

$$\Theta_1(c) = \Theta_2(\sigma(c)). \tag{3.10}$$

We define

$$\Theta(t) = \begin{cases} \Theta_1(t), & t \in I_1 \\ \Theta_2(t), & t \in I_2. \end{cases}$$

We notice that Θ is a continuous and almost everywhere continuously differentiable real valued function defined on $I = I_1 \cup I_2$. Also, from relations (3.7), (3.8), for any nontrivial solutions of (3.5), (3.6) along with conditions (3.3), (3.4) it follows that $\rho_i(t) \neq 0$ for all $t \in I_i, i = 1, 2$, as $\rho_i(t) = 0$ implies that $X_i(t) = 0$ contradicting our assumption of nontrivial solutions. Also, since $X_1(c) = X_2(\sigma(c))$ and $\Theta_1(c) = \Theta_2(\sigma(c))$, from (3.7) we have $\rho_1(c) = \rho_2(\sigma(c))$, and hence, for $t \in I$. $X(t) = 0$ if and only if $\Theta(t) = n\pi$, for some integer n. If we let $g_i(t) = \lambda r_i(t) - q_i(t)$, $t \in I_i, i = 1, 2$, equations (3.5), (3.6) can be rewritten in the form

$$(p_1 X_1^{\Delta})^{\Delta} + g_1 X_1 = 0 \quad \text{on } I_1 \tag{3.11}$$

$$(p_2 X_2^{\Delta})^{\Delta} + g_2 X_2 = 0 \quad \text{on } I_2 \,. \tag{3.12}$$

Let $\hat{p_i}, \hat{q_i}, \hat{r_i}, \hat{g_i}, \hat{\rho_i}, \hat{\Theta}_i$, $i = 1, 2$, be another set of functions as defined preceding discussions. We define

$$p(t) = \begin{cases} p_1(t), & t \in I_1 \\ p_2(t), & t \in I_2, \end{cases} \quad \hat{p}(t) = \begin{cases} \hat{p}_1(t), & t \in I_1 \\ \hat{p}_2(t), & t \in I_2, \end{cases}$$

$$g(t) = \begin{cases} g_1(t), & t \in I_1 \\ g_2(t), & t \in I_2, \end{cases} \quad \hat{g}(t) = \begin{cases} \hat{g}_1(t), & t \in I_1 \\ \hat{g}_2(t), & t \in I_2, \end{cases}$$

$$\hat{\Theta}(t) = \begin{cases} \hat{\Theta}_1(t), & t \in I_1 \\ \hat{\Theta}_2(t), & t \in I_2. \end{cases}$$

Upon delta differentiation, with respect to t, we have from (3.9) for $\Theta(t)$, $\Theta^{\Delta}(t) = \Theta'(t)$ (as I_1 and I_2 are continuous intervals and (3.4)), which is equal to

$$\begin{aligned} \frac{1}{1 + \big(\frac{X(t)}{p(t)X'(t)}\big)^2} \frac{d}{dx} \frac{X(t)}{p(t)X'(t)} &= \frac{p(t)(X'(t))^2 - X(t)(p(t)X'(t))'}{(p(t)X'(t))^2 + X^2(t)} \\ &= \frac{p(t)(X'(t))^2 + g(t)X^2(t)}{(p(t)X'(t))^2 + X^2(t)}. \end{aligned}$$

Equations (3.7), (3.8) imply

$$\Theta^{\Delta}(t) = \frac{1}{\rho^2(t)} \Big(\frac{1}{p(t)} \rho^2(t) \cos^2(\Theta(t)) + g(t)\rho^2(t) \sin^2 \Theta(t) \Big).$$

Hence,

$$\Theta^{\Delta}(t) = \frac{1}{p(t)} \sin^2 \Theta(t) + g(t) \sin^2 \Theta(t), \quad \text{for } t \in I. \tag{3.13}$$

Similarly, for $\hat{\Theta}(t)$, we have

$$\hat{\Theta}^{\Delta}(t) = \frac{1}{\hat{p}(t)} \sin^2 \hat{\Theta}(t) + g(t) \sin^2 \hat{\Theta}(t), \quad \text{for } t \in I. \tag{3.14}$$

4. A Theorem Involving Angle Functions

Theorem 4.1. *Let $\hat{p}(t) \le p(t)$ and $g(t) \le \hat{g}(t)$ for all $t \in I_1 \cup I_2$. Let $X(t)$ be a solution of* (3.11), (3.12), (3.3), (3.4), *and let $\hat{X}$ be a solution of*

$$(\hat{p}_1 \hat{X}_1^{\Delta})^{\Delta} + \hat{g}_1 \hat{X}_1 = 0 \quad \text{on } I_1, \tag{4.1}$$

$$(\hat{p}_2 \hat{X}_2^{\Delta})^{\Delta} + \hat{g}_2 \hat{X}_2 = 0 \quad \text{on } I_2, \tag{4.2}$$

satisfying the matching conditions

$$\hat{X}_1(c) = \hat{X}_2(\sigma(c)), \tag{4.3}$$

$$\hat{p}_1(c) \hat{X}_1^{\Delta}(c) = \hat{p}_2(\sigma(c)) \hat{X}_2^{\Delta}(\sigma(c)) \tag{4.4}$$

Then, if $\hat{\Theta}(d) \ge \Theta(d)$ for some $d \in (a, c] \cup [\sigma(c), b)$, then $\hat{\Theta}(t) \ge \Theta(t)$ for all $t \in (d, c] \cup [\sigma(c), b)$.

Proof. **Case I** Let us suppose that $d \in [\sigma(c), b)$. Then $\hat{\Theta}(d) \geq \Theta(d)$ implies that $\hat{\Theta_2}(d) \geq \Theta_2(d)$, and from hypothesis of the theorem we have that $\hat{p_2}(t) \leq p_2(t)$ and $\hat{g_2}(t) \geq g_2(t)$ for all $t \in [\sigma(c), b)$. For $t \in I_2$, we have, from relations (3.13) and (3.14)

$$\Theta_2^\Delta(t) = \frac{1}{p_2(t)} \sin^2 \Theta_2(t) + g_2(t) \sin^2 \Theta_2(t),$$
$$\hat{\Theta}_2^\Delta(t) = \frac{1}{\hat{p}_2(t)} \sin^2 \hat{\Theta}_2(t) + \hat{g}_2(t) \sin^2 \hat{\Theta}_2(t).$$

Let $\delta_2 = \hat{\Theta}_2 - \Theta_2$. Then, we have

$$\begin{aligned}
\delta_2^\Delta &= \frac{1}{\hat{p}_2(t)} \sin^2 \hat{\Theta}_2(t) + \hat{g}_2(t) \sin^2 \hat{\Theta}_2(t) - \frac{1}{p_2(t)} \sin^2 \Theta_2(t) - g_2(t) \sin^2 \Theta_2(t) \\
&= \frac{1}{\hat{p}_2(t)} \sin^2 \hat{\Theta}_2(t) - \frac{1}{p_2(t)} \cos^2 \Theta_2(t) + \hat{g}_2(t) \sin^2 \hat{\Theta}_2(t) - g_2(t) \sin^2 \Theta_2(t) \\
&= \Big(\frac{1}{\hat{p}_2} - \frac{1}{p_2}\Big) \cos^2 \hat{\Theta}_2 + (\hat{g}_2 - g_2) \sin^2 \Theta_2 + \frac{1}{p_2}(\cos^2 \hat{\Theta}_2 - \cos^2 \Theta_2) \\
&\quad - \hat{g}_2(\sin^2 \Theta_2 - \sin^2 \hat{\Theta}_2).
\end{aligned}$$

As we know that $\cos^2 \Theta = 1 - \sin^2 \Theta$, we have

$$\begin{aligned}
\delta_2^\Delta &= \Big(\frac{1}{\hat{p}_2} - \frac{1}{p_2}\Big) \cos^2 \hat{\Theta}_2 + (\hat{g}_2 - g_2) \sin^2 \Theta_2 \\
&\quad + \Big(\hat{g}_2 - \frac{1}{p_2}\Big)(\sin \hat{\Theta}_2 + \sin \Theta_2)(\sin \hat{\Theta}_2 - \sin \Theta_2) \\
&\geq h_2(t) + f_2(t)\delta_2(t),
\end{aligned}$$

where $0 \leq h_2(t) \leq \frac{1}{\hat{p}_2} - \frac{1}{p_2} + \hat{g}_2 - g_2$, $|f_2(t)| \leq 2\Big(|\hat{g}_2(t)| + \frac{1}{p_2(t)}\Big)$; as we have $\sin \Theta_2, \sin \hat{\Theta}_2, \cos \Theta_2, \cos \hat{\Theta}_2$ bounded above by one and for small values of $\hat{\Theta}_2$ and Θ_2, we have $\sin \hat{\Theta}_2 \approx \hat{\Theta}_2$ and $\sin \Theta_2 \approx \Theta_2$. Clearly, the functions f_2 and h_2 both are locally integrable on I_2. Let $k_2(t) = exp\Big(-\int_d^t f_2(t)dt\Big) > 0$. Then, we have, $(k_2\delta_2)^\Delta = (k_2\delta_2' + k_2'\delta_2)$, since $\Theta^\Delta = \Theta'$. So, we have $(k_2\delta_2)^\Delta = k_2\delta_2' - k_2 f_2 \delta_2 = k_2(\delta_2' - f_2\delta_2) \geq k_2 h_2 \geq 0$ since, $\delta_2' \geq h_2(t) + f_2(t)\delta_2(t))$, and therefore $k_2\delta_2$ is an increasing function; i.e., $k_2(t_1)\delta_2(t_1) \leq k_2(t_2)\delta_2(t_2)$, for $t_1 \leq t_2$. We have $\delta_2(d) = (\hat{\Theta}_2(d) - \Theta_2(d)) \geq 0$, hence, it follows that $\delta_2(t) \geq 0$ for all $t \in (d, b)$, as $k_2(d)\delta_2(d) \leq k_2(t)\delta_2(t)$, for $d \leq t$ and $k_2(d)\delta_2(d) \geq 0, k_2(t) > 0$ for $t \in I_2$. Hence, $\hat{\Theta}(t) \geq \Theta(t)$ for all $t \in (d, b)$.

Case II Let us suppose that $d \in (a, c]$. Then, $\hat{\Theta}(d) \geq \Theta(d)$ implies $\hat{\Theta}_1(d) \geq \Theta_1(d)$, from the hypothesis of the theorem, we have from relations (3.13) and (3.14)

$$\Theta_1^\Delta(t) = \frac{1}{p(t)} \cos^2 \Theta_1(t) + g_1(t) \sin^2 \Theta_1(t),$$
$$\hat{\Theta}_1^\Delta(t) = \frac{1}{\hat{p}_1(t)} \cos^2 \hat{\Theta}_1(t) + g_1(t) \sin^2 \hat{\Theta}_1(t).$$

Let $\delta_1 = \hat{\Theta}_1 - \Theta_1$. Then as proceeding in Case I, we can show that $\hat{\Theta}(t) \geq \Theta(t)$, for all $t \in (d, c]$. In fact, from the proof of Case I, it follows that $\hat{\Theta}_1(c) \geq \Theta_1(c)$. Now, by continuity of Θ function (see relation (3.10)), we have that $\hat{\Theta}_2(\sigma(c)) = \hat{\Theta}_1(c) \geq \Theta_1(c) = \Theta_2(\sigma(c))$, and hence by Case I we get that $\hat{\Theta}_2(t) \geq \Theta(t)$ for all

$t \in [\sigma(c), b)$. Therefore, $\hat{\Theta}(t) \geq \Theta(t)$ for all $t \in (d, c] \cup [\sigma(c), b)$. Hence, the proof is complete. □

5. Relationship Between Eigenvalues and Zeros of Eigenfunctions

Theorem 5.1. *Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be real valued solutions of* (3.5), (3.6) *along with conditions* (3.3), (3.4). *If u and v are linearly independent, then between any two consecutive zeros of u, there lies exactly one zero of v.*

Proof. Let t_1, t_2 be two consecutive zeros of u. For u, let $\Theta(t) = \tan^{-1} \frac{u(t)}{p(t)u^{\Delta}(t)}$ and for v, let $\tilde{\Theta}(t) = \tan^{-1} \frac{v(t)}{p(t)v^{\Delta}(t)}$, where

$$u_i(t) = \rho_i(t) \sin \Theta_i(t), p_i(t)u_i^{\Delta}(t) = \rho_i(t) \cos \Theta_i(t),$$
$$v_i(t) = \rho_i(t) \sin \tilde{\Theta}_i(t), p_i(t)v_i^{\Delta}(t) = \rho_i(t) \cos \tilde{\Theta}_i(t).$$

Let Θ and $\tilde{\Theta}$, the angel functions defined above, be such that $\Theta(t_1) = 0, 0 \leq \tilde{\Theta}(t_1) < \pi$. This may be accomplished by taking $-u(-v)$ instead of $u(v)$, if necessary.

Case I Suppose that $t_1 \in I_1$. Then, $\Theta_1(t_1) = 0$ and since u_1 and v_1 are linearly independent $\tilde{\Theta}_1(t_1) \neq 0$. Since, $\rho_i(t) \neq 0$ and $\tilde{\Theta}_1(t_1) = 0$ violates the definition of u_1 and v_1 being linearly independent on I_1. Hence, $\Theta_1(t_1) = 0 \leq \tilde{\Theta}_1(t_1) < \pi$; i.e., $\Theta(t_1) = 0 \leq \tilde{\Theta}(t_1) < \pi$. Now, from Theorem 4.1, it follows that $\pi = \Theta(t_2) < \tilde{\Theta}(t_2)$. Since, t_1 and t_2 are two consecutive zeros of $u, \rho_i(t) \neq 0$ and $u_i(t) = \rho_i(t) \sin \Theta_i(t)$. Hence (by continuity) $\tilde{\Theta}$ must take the value π at some point $y_1 \in (t_1, t_2)$. Since $v_i(t) = \rho_i(t) \sin \tilde{\Theta}_i(t), v$ has at least one zero in (t_1, t_2).

Case II Suppose that $t_1 \in I_2$. Then, $\Theta_2(t_1) = 0$, since u_2 and v_2 are linearly independent, $\tilde{\Theta}_2 \neq 0$. Hence, $\Theta_2(t_1) = 0 \leq \tilde{\Theta}_2(t_1) < \pi$; i.e., $\Theta(t_1) = 0 \leq \tilde{\Theta}(t_1) < \pi$, and therefore as in Case I, it follows that v has at least one zero in (t_1, t_2). Now, let us suppose that v has two(consecutive) zeros $y_1, y_2 \in (t_1, t_2)$. Then as shown above, there exists a point $t_3 \in (y_1, y_2)$ such that $u(t_3) = 0$, a contradiction. Hence, v has exactly one zero in (t_1, t_2). This completes the proof. □

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Pallav Kumar Baruah
Department of Mathematics and Computer Science, Sri Sathya Sai University, Prasanthinilayam, India
E-mail address: baruahpk@sssu.edu.in

Dasu Krishna Kiran Vamsi
Department of Mathematics and Computer Science, Sri Sathya Sai University, Prasanthinilayam, India
E-mail address: dkkvamsi@gmail.com