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ANOTHER UNDERSTANDING OF FOURTH-ORDER FOUR-POINT BOUNDARY-VALUE PROBLEMS

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Abstract. In this article we investigate the existence of positive and/or negative solutions of a classes of four-point boundary-value problems for fourth-order ordinary differential equations. The assumptions in this article are more relaxed than the known assumptions. Our technique relies on the continuum property (connectedness and compactness) of the solutions funnel (Knesser's Theorem), combined with the corresponding vector field's ones. This approach permits the extension of results (getting positive solutions) to nonlinear boundary conditions, whenever the corresponding Green's kernel is not of definite sign or there does not exist (see the last Corollary).

1. Introduction

In recent years, boundary-value problems for second and higher order differential equations have been extensively studied. Due to their important role in both theory and applications, BVPs have generated a great deal of interest over the years. They are often used to model various phenomena in physics, biology, chemistry and engineering (see [14] and the references there in).

Erbe and Wang [8], by using a Green's function and the Krasnoselskii's fixed point theorem in a cones proved the existence of a positive solution of the boundary value problem

$$x''(t) = f(t, x(t)), \quad 0 \le t \le 1,$$
$$ax(0) - bx'(0) = 0, \quad cx(1) + dx'(1) = 0,$$

under the following assumptions:

(A1) f is continuous and positive; i.e. $f \in C([0,1] \times [0,\infty), [0,\infty))$, $\delta = ad + bc + ac > 0$;

(A2)
$$\lim_{x\to 0+} \frac{\max_{0\le t\le 1} f(t,x)}{x} = 0 \quad \text{and} \quad \lim_{x\to +\infty} \frac{\min_{0\le t\le 1} f(t,x)}{x} = +\infty$$
or
$$\lim_{x\to 0+} \frac{\min_{0\le t\le 1} f(t,x)}{x} = +\infty \quad \text{and} \quad \lim_{x\to +\infty} \frac{\max_{0\le t\le 1} f(t,x)}{x} = 0.$$

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The literature for last BVP is voluminous. Suggestively we refer to [3, 9, 17, 18, 19] and the references therein. The monographs of Agarwal [1] and Agarwal, O'Regan and Wong [2] contain excellent surveys of known results.

Recently an increasing interest in studying the existence of solutions and positive solutions to boundary-value problems for higher order differential equations is observed; see for example [4, 5, 11, 12, 13]. Especially, Graef and Yang [11] and Hao et all [15] proved existence results on nonlinear boundary-value problem for fourth order equations.

Recently, Ge and Bai [10] investigated the fourth-order nonlinear boundary-value problem

$$\begin{gathered} u^{(4)}(t) = -f(t, u(t), u''(t)), \quad 0 \le t \le 1, \\ u(0) = u(1) = 0 \\ au''(\xi_1) - bu'''(\xi_1) = 0, \quad cu''(\xi_2) + du'''(\xi_2) = 0\,, \end{gathered} \tag{1.1}$$

where $0 \le \xi_1 < \xi_1 \le 0$. Precisely, by using a fixed point theorem due to Krasnoselskii and Zabreiko in [16], they proved the following result.

Theorem 1.1. *Assume that*

(H1) a, b, c, d *are nonnegative constants such that* $\rho = ad + bc + ac(\xi_2 - \xi_1) \neq 0$, $b - a\xi_1 \ge 0$ *and* $0 \le \xi_1 < \xi_2 \le 1$*;*

(H2) *The nonlinearity can separated as* $f(t, u, v) = p(t)g(u) + q(t)h(v)$*, where* $g, h : \mathbb{R} \to \mathbb{R}$ *are continuous,*

$$\lim_{u\to\infty} \frac{g(u)}{u} = \lambda, \quad \lim_{v\to\infty} \frac{h(v)}{v} = \mu, \tag{1.2}$$

and $p, q \in C[0, 1]$*. Moreover, there exists* $t_0 \in [0, 1]$ *such that* $p(t_0)g(0) + q(t_0)h(0) \neq 0$*, and there exists a continuous nonnegative function* $w : [0, 1] \to \mathbb{R}^+$ *such that* $|p(s)| + |q(s)| \le w(s)$ *for each* $s \in [0, 1]$*.*

(H3) $\max\{|\lambda|, |\mu|\} < \min\{\frac{1}{L_1}, \frac{1}{L_2}\}$*, where*

$$\begin{aligned} L_1 = \frac{1}{12}\Big[&\int_0^{\xi_1} \tau^3(2-\tau)w(\tau)d\tau + \int_{\xi_1}^1 (1-\tau)^3(1+\tau)w(\tau)d\tau \\ &+ \frac{2(b - a\xi_1) + a}{d} \int_{\xi_1}^{\xi_2} (c(\xi_2 - \tau) + d)w(\tau)d\tau\Big], \end{aligned}$$

and

$$L_2 = \int_{\xi_1}^1 (1-\tau)w(\tau)d\tau + \frac{1}{d}\int_{\xi_1}^{\xi_2} (b + a(1-\xi_1))(c(\xi_2 - \tau) + d)w(\tau)d\tau.$$

Then the BVP (1.1) *admits at least one nontrivial solution* $u \in C^2[0, 1]$*.*

Graef et al. [14] obtained existence and multiplicity results for the BVP

$$\begin{gathered} u^{(4)}(t) = g(t)f(u(t)), \\ u(0) = u(1) = u''(1) = u''(0) - u''(p) = 0, \quad p \in (0, 1)\,. \end{gathered}$$

Cui and Zou [7], proved existence results for the same differential equation with the boundary conditions

$$u(0) = u(1) = u'(0) = u'(1) = 0,$$

under sub or super-linearity conditions on the nonlinearity.

Remark 1.2. We note that in Theorem 1.1, it is not assumed any positivity of the nonlinearity and this fact does not guarantee positivity of the obtained solution. Furthermore, whenever the nonlinearity is nonnegative, the assumption $p(t_0)g(0) + q(t_0)h(0) \neq 0$ in (H2) yields

$$\lim_{|u|+|v|\to 0} \frac{\sup_{0\le t\le 1} f(t,u,v)}{|u|+|v|} = +\infty$$

that is, the nonlinearity has an asymptotic behavior of sublinearity type, at least at the origin. Indeed, if $p(t_0)g(0) + q(t_0)h(0) \neq 0$ and $t_0 \in [\xi_1, \xi_2]$, then the above condition can be written as

$$\lim_{|u|+|v|\to 0} \frac{\sup_{\xi_1\le t\le \xi_2} f(t,u,v)}{|u|+|v|} = \infty.$$

The case $f(t,u,v) \equiv 0$, $\xi_1 \le t \le \xi_2$ is impossible, since then the BVP (1.1) accepts only the trivial solution, contrary to the assumption $p(t_0)g(0) + q(t_0)h(0) \neq 0$.

Restricting our consideration on the linear case, notice as far as the author is aware, that only the conditions in (1.1) have been studied, where the constants $a, b, c, d \ge 0$. If $f(t, 0, 0) = 0$, then the boundary value problem (1.1) always has the trivial solution, but here we are only interested in a positive (negative) solution; i.e., whenever $u(t) > 0$ on $(0, 1)$.

The aim of this work is to prove the existence of a positive and/or a negative solution for the boundary value problem (1.1), where still $a, b, c, d \ge 0$, but without the assumptions $\rho = ad + bc + ac(\xi_2 - \xi_1) \neq 0$ and/or $b - a\xi_1 \ge 0$. Moreover the nonlinearity is not necessarily separated and the obtained solutions are of definite sign.

Furthermore we study BVPs of the form

$$\begin{gathered} u^{(4)}(t) = \pm f(t, u(t), u''(t)), \quad 0 \le t \le 1, \\ u(0) = u(1) = 0 \\ au''(\xi_1) \pm bu'''(\xi_1) = 0, \quad cu''(\xi_2) \mp du''(\xi_2) = 0, \end{gathered}$$

where the constants $a, b, c, d \ge 0$ are chosen positive and suitable.

In some of these cases, seems that the Green's functions does not exists or fails to be nonnegative. This makes Erbe and Wang's method not applicable in those cases.

Remark 1.3. Assume the nonlinearity is negative. The differential equation (2.3) defines a vector field, the properties of which will be crucial for our study. More specifically, let us look at the (v, v') face semi-plane $(v > 0)$. By the sign condition on f, we obtain that $v'' < 0$. Thus any trajectory $(v(t), v'(t))$, $t \ge 0$, emanating from the semi-line

$$E_0 := \{(v, v') : av - bv' = 0,\ v > 0\}$$

"evolutes" naturally, initially (when $v'(t) > 0$) toward the positive v-semi-axis and then (when $v'(t) < 0$) turns toward the semi-line

$$E_1 := \{(v, v') : cv + dv' = 0,\ v > 0\}.$$

Setting a certain growth rate on f (say superlinearity), we can control the vector field, so that some trajectory reaches on E_1 at the time $t = 1$. These properties will be referred as *the nature of the vector field* throughout the rest of paper.

The technique presented here is different from those in the above mentioned papers. Actually, we rely on the above "nature of the vector field" and the Knesser's property (continuum) of the cross-sections of the solutions funnel. For completeness we restate the well-known Knesser's theorem.

Theorem 1.4 ([6]). *Consider a system*

$$x' = f(t,x), \quad (t,x) \in \Omega := [a,b] \times \mathbb{R}^n, \tag{1.3}$$

with f continuous. Let $\hat{E}_0$ be a continuum (compact and connected) in $\Omega_0 := \{(t,x) \in \Omega : t = a\}$ and let $\mathcal{X}(\hat{E}_0)$ be the family of solutions of (1.3) *emanating from $\hat{E}_0$. If any solution $x \in \mathcal{X}(\hat{E}_0)$ is defined on the interval $[a,\tau]$, then the set (*cross-section)

$$\mathcal{X}(\tau;\hat{E}_0) := \{x(\tau) : x \in \mathcal{X}(\hat{E}_0)\}$$

is a continuum in $\mathbb{R}^n$.

2. Main Results

Consider the differential equation

$$u^{(4)}(t) = -f(t,u(t),u''(t)), \quad 0 \le t \le 1, \tag{2.1}$$

with boundary conditions

$$\begin{gathered} u(0) = u(1) = 0 \\ au''(\xi_1) - bu'''(\xi_1) = 0, \quad cu''(\xi_2) + du'''(\xi_2) = 0. \end{gathered} \tag{2.2}$$

Remark 2.1. The change of variable $v(t) = u''(t)$ reduces the above boundary value problem to

$$\begin{gathered} v''(t) = -f(t,u(t),v(t)), \quad t \in [0,1], \\ au''(\xi_1) - bu'''(\xi_1) = 0, \quad cu''(\xi_2) + du'''(\xi_2) = 0, \end{gathered} \tag{2.3}$$

where

$$u(t) = \int_0^t s(t-1)v(s)ds + \int_t^1 t(s-1)v(s)ds, \quad 0 \le t \le 1.$$

We note that

$$\begin{gathered} 0 \le v(t) \le M,\ t \in [0,1] \ \Rightarrow\ -\frac{M}{8} \le u(t) \le 0,\ t \in [0,1]; \\ v(t) \ge M,\ t \in [0,1] \ \Rightarrow\ u(t) \le -\frac{M}{8},\ t \in [0,1] \end{gathered} \tag{2.4}$$

Moreover, whenever we are interesting in negative and convex solutions, without loss of generality, we may extend the nonlinearity as

$$f(t,u,v) = f(t,0,0), \quad u \ge 0 \text{ and } v \le 0$$

and if we are asking for positive and concave solutions, we may set

$$f(t,u,v) = f(t,0,0), \quad u \le 0 \text{ and } v \ge 0.$$

We will use the following assumptions: The nonlinearity f is a continuous and nonnegative function; that is,

$$f(t,u,v) \in C([0,1] \times (-\infty,0] \times [0,+\infty), [0,+\infty)). \tag{2.5}$$

It is asymptotically linear at infinity; that is,

$$f_\infty = \lim_{u\to-\infty, v\to+\infty} \frac{\min_{\xi_1\le t\le\xi_2} f(t,u,v)}{v} = \mu \le +\infty. \tag{2.6}$$

It is superlinear at the origin; that is,

$$f_0 = \lim_{u\to 0-, v\to 0+} \frac{\max_{\xi_1\le t\le\xi_2} f(t,u,v)}{v} = 0. \tag{2.7}$$

Similarly, we assume that:

$$f(t,u,v) \in C([0,1]\times[0,+\infty)\times(-\infty,0],[0,+\infty)); \tag{2.8}$$

$$f_\infty = \lim_{u\to+\infty, v\to-\infty} \frac{\min_{\xi_1\le t\le\xi_2} f(t,u,v)}{-v} = \mu; \tag{2.9}$$

$$f_0 = \lim_{u\to 0+, v\to 0-} \frac{\min_{\xi_1\le t\le\xi_2} f(t,u,v)}{v} = 0. \tag{2.10}$$

For technical reasons and since readers are more familiar with the boundary conditions at (2.3), we prefer to establish first an existence result for the boundary value problem (2.1)–(2.2) and then (see Theorems 2.5, 2.6 and 2.7) we exhibit results for other BVPs.

Theorem 2.2. *Assume* (2.5)–(2.7) *hold. Then the boundary value problem* (2.1)–(2.2) *admits a negative and convex solution* $u(t)$, $0\le t\le 1$, *provided that*

$$\mu > \frac{48}{(\xi_2-\xi_1)^2}.$$

Proof. Consider the BVP (2.3). From assumptions (2.6) and (2.4), it follows that for every $K\in\big(\frac{48}{(\xi_2-\xi_1)^2},\mu\big)$, there is an $H>0$, such that

$$f(t,u,v)\ge Kv,\quad 0\le t\le 1,\quad u\le -\frac{H}{8},\quad v\ge H.$$

We assert that there is v_1 sufficiently large and a solution $v\in\mathcal{X}(P_1)$, $P_1 = (v_1, \frac{a}{b}v_1)\in E_0$ such that

$$v(\xi_2)\le 0. \tag{2.11}$$

Considering then the function

$$G(P_1) := cv(\xi_2) + dv'(\xi_2).$$

We note that

$$G(P_1) < 0 \tag{2.12}$$

since, in view of Remark 1.3, $v'(\xi_2)<0$.

We assume on the contrary, that for every $P_1=(v_1,\frac{a}{b}v_1)\in E_0$, where $v_1\ge H$ and any solution $v\in\mathcal{X}(P_1)$,

$$v(t)>0,\quad \xi_1\le t\le\xi_2.$$

Note first that for $v_1=H$, the sign property of f yields

$$v(t) = v_1 + (t-\xi_1)\frac{a}{b}v_1 - \frac{(t-\xi_1)^2}{2!} f(\bar t, u(\bar t), v(\bar t)) \le H\big(1+\frac{a}{b}(\xi_2-\xi_1)\big), \tag{2.13}$$

for $\xi_1\le t\le\xi_2$. We assert that there exists $v_1\ge H$, for which

$$\begin{gathered} v(t) < 2H(1+\frac{a}{b}(\xi_2-\xi_1)),\quad v'(t)>0,\quad \xi_1\le t\le \xi_1+(\xi_2-\xi_1)/4,\\ v(\xi_1+(\xi_2-\xi_1)/4\ ) = 2H(1+\frac{a}{b}(\xi_2-\xi_1)). \end{gathered} \tag{2.14}$$

Let us assume that is not true. By the Knesser's property, the set

$$C = \{(t, v(t)) : \xi_1 \le t \le \xi_1 + (\xi_2 - \xi_1)/4,\ v \in \mathcal{X}(\Omega)\},$$

where $\Omega = ([H, +\infty) \times [\frac{a}{b}H, +\infty)) \cap E_0$ is connected. Hence by (2.13) we obtain that

$$v(t) < 2H\big(1 + \frac{a}{b}(\xi_2 - \xi_1)\big), \quad \xi_1 \le t \le \xi_1 + (\xi_2 - \xi_1)/4,\ v \in \mathcal{X}(\Omega).$$

This conclusion is impossible. Indeed, for that (fixed) H, we chose

$$M = \max\Big\{f(t, u, v) : \xi_1 \le t \le \xi_1 + (\xi_2 - \xi_1)/4,\ 0 \le v \le 2H(1 + \frac{a}{b}(\xi_2 - \xi_1)),\\ -\frac{2H}{8}(1 + \frac{a}{b}(\xi_2 - \xi_1)) \le u \le 0\Big\}$$

Then by the Taylor formula, we get for some $\bar{t} \in [\xi_1, \xi_1 + (\xi_2 - \xi_1)/4]$,

$$\begin{aligned} v(\xi_1 + (\xi_2 - \xi_1)/4\ ) &= v_1 + \frac{a}{b}v_1\frac{(\xi_2 - \xi_1)}{4} - \frac{(\xi_2 - \xi_1)^2}{4^2 2!}f(\bar{t}, u(\bar{t}), v(\bar{t}))\\ &\ge v_1\big(1 + \frac{a}{b}\frac{(\xi_2 - \xi_1)}{4}\big) - M\frac{(\xi_2 - \xi_1)^2}{32}. \end{aligned}$$

Hence

$$\lim_{v_1 \to +\infty} v(\xi_1 + (\xi_2 - \xi_1)/4\ ) = \lim_{v_1 \to +\infty} v_1\big(1 + \frac{a}{b}\frac{(\xi_2 - \xi_1)}{4}\big) - M\frac{(\xi_2 - \xi_1)^2}{32} = +\infty,$$

a contradiction. This and the positivity of the nonlinearity prove the assertion (2.14). Since

$$v(\xi_1 + (\xi_2 - \xi_1)/4\ ) = v(\xi_1\ ) + \int_{\xi_1}^{\xi_1 + (\xi_2 - \xi_1)/4} v'(t)dt \ge \frac{(\xi_2 - \xi_1)}{4}v'(\xi_1 + (\xi_2 - \xi_1)/4\ ),$$

we get

$$v'(\xi_1 + (\xi_2 - \xi_1)/4\ ) \le 8H\frac{1 + \frac{a}{b}(\xi_2 - \xi_1)}{(\xi_2 - \xi_1)}. \tag{2.15}$$

If

$$v(t) \ge 2H\big(1 + \frac{a}{b}(\xi_2 - \xi_1)\big), \quad \xi_1 + (\xi_2 - \xi_1)/4 \le t \le \xi_2,$$

we obtain the contradiction

$$\begin{aligned} v(\xi_2) &= v(\xi_1 + (\xi_2 - \xi_1)/4) + \frac{3(\xi_2 - \xi_1)}{4}v'(\xi_1 + (\xi_2 - \xi_1)/4)\\ &\quad - \frac{1}{2!}\frac{9(\xi_2 - \xi_1)^2}{16}f(t, u(t), v(t))\\ &\le 2H(1 + \frac{a}{b}(\xi_2 - \xi_1)) + \frac{3(\xi_2 - \xi_1)}{4}8H\frac{1 + \frac{a}{b}(\xi_2 - \xi_1)}{(\xi_2 - \xi_1)}\\ &\quad - \frac{9}{2!}\frac{(\xi_2 - \xi_1)^2}{16}K2H(1 + \frac{a}{b}(\xi_2 - \xi_1)) < 0, \end{aligned}$$

due to the choice of $K > 48/(\xi_2 - \xi_1)^2 > 128/9(\xi_2 - \xi_1)^2$. Thus we may consider $t_0 = (\xi_1, \xi_1 + (\xi_2 - \xi_1)/4)$ and $t_1 \in (\xi_1 + (\xi_2 - \xi_1)/4, \xi_2)$ such that

$$\begin{gathered} v(t_0) = H(1 + \frac{a}{b}(\xi_2 - \xi_1)) = v(t_1)\\ v'(t_0) > 0, \quad v'(t_1) < 0,\\ v(t) \ge H(1 + \frac{a}{b}(\xi_2 - \xi_1)), \quad t_0 \le t \le t_1. \end{gathered} \tag{2.16}$$

Suppose that $t_1 \ge \xi_1 + (3/4)(\xi_2 - \xi_1)$. Then from (2.15) and the choice of $K \ge 48/(\xi_2-\xi_1)^2 \ge 16/(\xi_2-\xi_1)$, we get the contradiction

$$\begin{aligned} v(t_1) &= v(\xi_1 + (\xi_2-\xi_1)/4) + [t_1 - (\xi_1 + (\xi_2-\xi_1)/4)]v'(\xi_1 + (\xi_2-\xi_1)/4) \\ &\quad - \frac{(t_1 - (\xi_1 + (\xi_2-\xi_1)/4))^2}{2!} f(t, u(t), v(t)) \\ &\le H(1 + \frac{a}{b}(\xi_2-\xi_1)) + [t_1 - (\xi_1 + (\xi_2-\xi_1)/4)] 8H \frac{1+\frac{a}{b}(\xi_2-\xi_1)}{(\xi_2-\xi_1)} \\ &\quad - \frac{(t_1 - (\xi_1 + (\xi_2-\xi_1)/4))^2}{2!} KH(1 + \frac{a}{b}(\xi_2-\xi_1)) \\ &< H(1 + \frac{a}{b}(\xi_2-\xi_1)) \end{aligned}$$

Hence

$$t_1 \in (\xi_1 + (\xi_2-\xi_1)/4,\ \xi_1 + (3/4)(\xi_2-\xi_1)) \tag{2.17}$$

Assume that

$$v'(t_1) > -4H \frac{1+\frac{a}{b}(\xi_2-\xi_1)}{(\xi_2-\xi_1)}. \tag{2.18}$$

Then

$$\begin{aligned} v(t_1) &= v(\xi_1 + (\xi_2-\xi_1)/4) + \int_{\xi_1 + (\xi_2-\xi_1)/4}^{t_1} v'(t)dt \\ &\ge 2H(1 + \frac{a}{b}(\xi_2-\xi_1)) - 4H \frac{1+\frac{a}{b}(\xi_2-\xi_1)}{(\xi_2-\xi_1)} (t_1 - (\xi_1 + (\xi_2-\xi_1)/4)). \end{aligned}$$

Consequently, in view of (2.16)-(2.18), we obtain

$$(2H - H)\big(1 + \frac{a}{b}(\xi_2-\xi_1)\big) \le 4H \frac{1+\frac{a}{b}(\xi_2-\xi_1)}{(\xi_2-\xi_1)} (t_1 - (\xi_1 + (\xi_2-\xi_1)/4))$$

and then

$$t_1 \ge \xi_1 + \frac{\xi_2-\xi_1}{2} = \frac{\xi_2+\xi_1}{2}. \tag{2.19}$$

Thus, noting (2.16) and the choice $K > 48/(\xi_2-\xi_1)^2$, we get the contradiction

$$\begin{aligned} v'(t_1) &= v'\big(\xi_1 + \frac{\xi_2-\xi_1}{4}\big) - [t_1 - (\xi_1 + \frac{\xi_2-\xi_1}{4})] f(t, u(t), v(t)) \\ &\le 8H \frac{1+\frac{a}{b}(\xi_2-\xi_1)}{(\xi_2-\xi_1)} - (t_1 - \xi_1 - \frac{\xi_2-\xi_1}{4}) K v(t) \\ &\le 8H \frac{1+\frac{a}{b}(\xi_2-\xi_1)}{(\xi_2-\xi_1)} - (t_1 - \xi_1 - \frac{\xi_2-\xi_1}{4}) KH(1 + \frac{a}{b}(\xi_2-\xi_1)) \\ &\le -4H \frac{1+\frac{a}{b}(\xi_2-\xi_1)}{(\xi_2-\xi_1)}. \end{aligned}$$

Hence

$$v'(t_1) < -4H \frac{1+\frac{a}{b}(\xi_2-\xi_1)}{(\xi_2-\xi_1)}.$$

Recalling (2.17), we obtain the final contradiction

$$\begin{aligned} v(\xi_2) = v(t_1) + \int_{t_1}^{\xi_2} v'(t)dt &\le H(1+\frac{a}{b}(\xi_2-\xi_1)) + (\xi_2-t_1)v'(t_1) \\ &\le H(1+\frac{a}{b}(\xi_2-\xi_1)) - (\xi_2-t_1)4H\frac{1+\frac{a}{b}(\xi_2-\xi_1)}{(\xi_2-\xi_1)} \le 0. \end{aligned}$$

Consequently, the assertion (2.11) and then (2.12) are proved.

On the other hand, by the superlinearity of $f(t,u,v)$ at $v=0$ (see the assumption (2.7), for any $\lambda>0$ there is an $\eta>0$ such that

$$0<v\le\eta,\quad -\frac{\eta}{8}\le u<0 \quad \text{implies} \quad \max_{\xi_1\le t\le\xi_2} f(t,u,v)<\lambda v. \tag{2.20}$$

Consider any positive number $\varepsilon < b/[b+a(\xi_2-\xi_1)]<1$ and choose

$$\lambda<\min\big\{\frac{\varepsilon a}{b(\xi_2-\xi_1)}, \frac{2}{b(\xi_2-\xi_1)^2}[b-\varepsilon(b+a(\xi_2-\xi_1))]\big\}. \tag{2.21}$$

We assert that for $P_0=(v_0,\frac{a}{b}v_0)\in E_0$, where $v_0=\varepsilon\eta$, and any solution $v\in\mathcal{X}(P_0)$, it follows that

$$\varepsilon\eta\le v(t)\le\eta,\quad t\in[\xi_1,\xi_2]. \tag{2.22}$$

Indeed in view of Remark 1.3, let's assume that there exists $t^*\in(\xi_1,\xi_2]$ such that

$$\varepsilon\eta\le v(t)\le\eta,\quad v'(t)>0,\quad \xi_1\le t<t^*,\quad v(t^*)=\eta.$$

Then by the Taylor's formula and (2.20), we get $\bar t\in(0,t^*)$, such that

$$\begin{aligned} \eta = v(t^*) &= v_0[1+\frac{a}{b}(t^*-\xi_1)] - \frac{(t^*-\xi_1)^2}{2}f(\bar t,u(\bar t),v(\bar t)). \\ &\le \varepsilon\eta[1+\frac{a(t^*-\xi_1)}{b}] + \frac{(t^*-\xi_1)^2}{2}\lambda v(\bar t) \\ &\le \varepsilon\eta[1+\frac{a}{b}(\xi_2-\xi_1)] + \frac{(\xi_2-\xi_1)^2}{2}\lambda\eta. \end{aligned}$$

Consequently,

$$\lambda\ge\frac{2}{b(\xi_2-\xi_1)^2}[b-\varepsilon(b+a(\xi_2-\xi_1))],$$

contrary to the choice of λ in (2.21). Hence, the assertion $v(t)\le\eta$, $t\in[\xi_1,\xi_2]$ in (2.22) is proved. Moreover, if there is $t^*\in(\xi_1,\xi_2)$ such that

$$0\le v'(t)\le\frac{a\varepsilon\eta}{b},\ \xi_1\le t<t^*,\quad v'^*)=0,$$

then by (2.20) and (2.22),

$$\begin{aligned} 0 = v'^*) &= v_0\frac{a}{b} - (t^*-\xi_1)f(\bar t,u(\bar t),v(\bar t)). \\ &\ge \varepsilon\eta\frac{a}{b} - (t^*-\xi_1)\lambda v(\bar t) \\ &\ge \varepsilon\eta\frac{a}{b} - (\xi_2-\xi_1)\lambda v(\bar t) \ge \varepsilon\eta\frac{a}{b} - (\xi_2-\xi_1)\lambda\eta, \end{aligned}$$

a contradiction to (2.21). Thus $v'(t)\ge0$, $\xi_1\le t<\xi_2$ and (2.22) is proved. Consequently

$$G(P_0)=cv(\xi_2)+dv'(\xi_2)>0. \tag{2.23}$$

Finally consider the segment

$$[P_0,P_1]:=\{(v,v')\in E_0 : v_0\le v\le v_1\}$$

and furthermore the cross-section

$$\mathcal{X}(1;[P_0,P_1]) := \{(v(1),v'(1)) : v \in \mathcal{X}(P) \text{ and } P \in [P_0,P_1]\}$$

of the solutions funnel emanating from the segment $[P_0,P_1]$. By the definition of the function $G(P_1) := cv(\xi_2) + dv'(\xi_2)$, (2.12) and (2.23), it is clear (recall that $E_1 := \{(v,v') : cv - dv' = 0,\ v > 0\}$) that

$$E_1 \cap \mathcal{X}(1;[P_0,P_1]) \neq \emptyset.$$

This means that there is a point $P \in [P_0,P_1]$ such that $G(P) = 0$ and thus a solution $v_0(t) \in \mathcal{X}(P)$ satisfying the boundary value problem (2.3).

Furthermore, by the above analysis, the obtained solution $v_0(t)$, $\xi_1 \le t \le \xi_2$, is positive. We extend $v_0(t)$ on the entire interval, as follows:

$$v(t) = \begin{cases} v_0(\xi_1), & 0 \le t \le \xi_1 \\ v_0(t), & \xi_1 \le t \le \xi_2 \\ v_0(\xi_2), & \xi_2 \le t \le 1. \end{cases}$$

Then, the function $v(t)$, $0 \le t \le 1$, is positive and continuous. In view of the transformation $v_0(t) = u''(t)$, we consider the boundary-value problem

$$\begin{gathered} u'' = v_0(t) \\ u(0) = 0 = u(1). \end{gathered} \tag{2.24}$$

It is well known that its Green function is

$$G(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1 \\ t(1-s), & 0 \le t \le s \le 1. \end{cases}$$

Consequently (see (2.4)) the desired negative and convex solution of the boundary value problem (2.1)-(2.2) is given by the formula

$$u(t) = -\int_0^1 G(t,s)v(s)ds = \int_0^t s(t-1)v(s)ds + \int_t^1 t(s-1)v(s)ds,$$

for $0 \le t \le 1$. □

Example 2.3. Consider the fourth-order four-point boundary-value problem

$$\begin{gathered} u^{(4)}(t) = -\frac{t}{t^2+1}u^2(t) - e^t(u''(t))^3 \\ u(0) = u(1) = 0 = u''(1/3) - (1/4)u'''(1/3) = u''(2/3) + u'''(2/3) = 0. \end{gathered}$$

Since the nonlinearity is continuous, positive and superlinear, the conditions (2.5)-(2.7) are obviously satisfied, where $\mu = +\infty$. To show that BVP has at least one nontrivial solution we apply Theorem 2.2 with $\xi_1 = 1/3$, $\xi_2 = 2/3$, $a = b = c = d = 1$. On the other hand, the conditions (H1) and (H3) of Theorem 1.1 clearly do not hold. Hence the result in [10], does not guarantee existence of a solution to the above BVP.

Remark 2.4. The condition (2.7) can be replaced by the more general condition

$$\begin{aligned} f_0 &= \lim_{u\to 0+, v\to 0-} \frac{\min_{\xi_1 \le t \le \xi_2} f(t,u,v)}{-v} \\ &= \mu^* < \min\Big\{\frac{a}{(\xi_2-\xi_1)[b+a(\xi_2-\xi_1)]}, \frac{2}{(\xi_2-\xi_1)^2}\Big\}. \end{aligned}$$

Indeed, at (2.21) we may choose

$$\frac{\mu^* b(\xi_2-\xi_1)}{a} \le \varepsilon \le \frac{b(2-\mu^*(\xi_2-\xi_1))}{b+a(\xi_2-\xi_1)}$$

and then

$$\mu^* < \lambda < \min\Big\{\frac{\varepsilon a}{b(\xi_2-\xi_1)}, \frac{2}{b(\xi_2-\xi_1)^2}[b-\varepsilon(b+a(\xi_2-\xi_1))]\Big\}.$$

Consider the boundary-value problem

$$u^{(4)}(t) = f(t,u(t),u''(t)), \quad 0 \le t \le 1, \tag{2.25}$$

with

$$\begin{gathered} u(0)=u(1)=0 \\ au''(\xi_1)-bu'''(\xi_1)=0, \quad cu''(\xi_2)+du'''(\xi_2)=0. \end{gathered} \tag{2.26}$$

Theorem 2.5. *Under assumptions* (2.8)–(2.10)*, the boundary value problem* (2.25) - (2.26) *has a positive and concave solution* $u(t)$*,* $0 \le t \le 1$*, provided that*

$$\mu > \frac{48}{(\xi_2-\xi_1)^2}.$$

Proof. We set

$$F(t,u,v) = f(t,-u,-v), \quad t\in[0,1],\ u\le 0,\ v\ge 0.$$

Since f satisfies conditions (2.8)-(2.10), we easily check that F suits the conditions (2.5)-(2.7). In view of Theorem 2.2, considering a solution $u(t)$, $0 \le t \le 1$, of the BVP (2.1)– (2.2) (where f is replaced by F), we set

$$y(t) \equiv -u(t), \quad 0 \le t \le 1.$$

Then we obtain

$$\begin{gathered} (-y(t))^{(4)} = (u(t))^{(4)} = -F(t,u(t), \\ u''(t)) = -f(t,-u(t), \quad -u''(t)) = -f(t,y(t),y''(t)). \end{gathered}$$

Hence, the function $y(t)$, $0 \le t \le 1$, is a solution of the differential equation (2.25). Moreover, the boundary conditions (2.26) are satisfied by the function $y(t)$, since the solution $u(t)$ fulfils the boundary conditions (2.2). Consequently, $y(t)$ is the required solution of (2.25)-(2.26). □

Consider (2.1) with the boundary condition

$$\begin{gathered} u(0)=u(1)=0 \\ au''(\xi_1)+bu'''(\xi_1)=0, \quad cu''(\xi_2)-du'''(\xi_2)=0. \end{gathered} \tag{2.27}$$

Theorem 2.6. *Assume* (2.8)–(2.10) *hold. Then the boundary value problem* (2.1)–(2.27) *has a negative and convex solution* $v(t)$*,* $0 \le t \le 1$*, provided that*

$$\mu > \frac{48}{(\xi_2-\xi_1)^2}.$$

Proof. Consider any function $u(t)$, $0 \le t \le 1$. We define a map F by the formula

$$F(\xi_1+\xi_2-t,u(t),v(t)) = f(t,-u(t),-v(t)), \quad t\in[0,1],\ u\ge 0,\ v\le 0.$$

Since f satisfies the conditions (2.8)–(2.10), we easily check that F suits the conditions (2.5)–(2.7) on the interval $[\xi_1+\xi_2-1, \xi_1+\xi_2]$. In view of Theorem 2.2 , consider a solution $u^*(t)$, $[\xi_1+\xi_2-1 \le t \le \xi_1+\xi_2]$, of the BVP:

$$u^{(4)}(t) = -F(t, u(t), v(t))$$

with boundary conditions (2.2). and set

$$u(t) \equiv -u^*(\xi_1+\xi_2-t), \quad 0 \le t \le 1.$$

Then we obtain

$$\begin{aligned}
(-u(t))^{(4)} &= (u^*(\xi_1+\xi_2-t))^{(4)} \\
&= -F(\xi_1+\xi_2-t, u^*(\xi_1+\xi_2-t), v^*(\xi_1+\xi_2-t)) \\
&= -F(\xi_1+\xi_2-t, -u(t), -v(t)) \\
&= -f(t, u(t), v(t));
\end{aligned}$$

that is, the function $u(t)$, $0 \le t \le 1$ is a solution of the differential equation (2.1). Moreover, since the solution $u^*(t)$ fulfils the boundary conditions (2.2), we get via the above transformation

$$cu^{*\prime\prime}(\xi_1) - du^{*\prime\prime\prime}(\xi_1) = 0 \Rightarrow -cu''(\xi_2) + du'''(\xi_2) = 0 \Rightarrow cu''(\xi_2) - du'''(\xi_2),$$
$$au^{*\prime\prime}(\xi_2) + bu^{*\prime\prime\prime}(\xi_2) = 0 \Rightarrow -au''(\xi_1) - bu'''(\xi_1) = 0 \Rightarrow au''(\xi_1) + bu'''(\xi_1) = 0$$

that is, the boundary conditions (2.27) are satisfied by the function $u(t)$. Consequently, $u(t)$ is the requited solution of (2.1)– (2.27). □

Consider now the equation

$$u^{(4)}(t) = f(t, u(t), u''(t)), \quad 0 \le t \le 1,$$

with boundary condition (2.27).

Theorem 2.7. *Assume* (2.5)-(2.7) *hold. Then the boundary value problem* (2.25)-(2.27) *has a negative and convex solution* $u(t)$, $0 \le t \le 1$, *provided that*

$$\mu > \frac{48}{(\xi_2-\xi_1)^2}.$$

Proof. We set

$$F(t, u, v) = f(t, -u, -v), \quad t \in [0, 1], \quad u \ge 0, \quad v \le 0.$$

Since f satisfies the conditions (2.5)-(2.7), we easily check that F fulfills the conditions (2.8)-(2.10). Thus, in view of Theorem 2.6, the BVP

$$u^{(4)}(t) = -F(t, u(t), u''(t)), \quad 0 \le t \le 1,$$

with boundary condition (2.27) admits a positive and concave solution $u(t)$, $0 \le t \le 1$. We set

$$u^*(t) \equiv -u(t), \quad 0 \le t \le 1.$$

Then we obtain

$$u^{*(4)}(t) = -u^{(4)}(t) = F(t, u(t),$$
$$u''(t)) = f(t, -u(t), \quad -u''^*(t), u^{*\prime\prime}(t)).$$

Consequently, the function $u^*(t)$, $0 \le t \le 1$ is a solution of the differential equation (2.25). Moreover, the boundary conditions (2.27) are satisfied by the function

$u(t)$, since the solution $u^*(t)$ fulfils the same conditions. Consequently, $u(t)$ is the requited solution. □

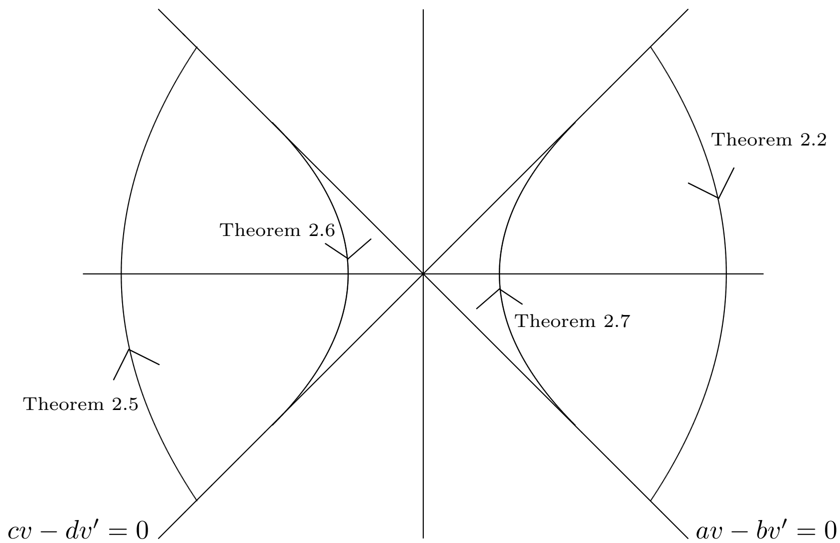

Figure 1. Summary of results

3. Additional Results

In this section, we consider the equation

$$u^{(4)}(t) = -f(t, u(t), u''(t)), \quad 0 \le t \le 1,$$

with boundary condition (2.2), under the following assumptions: The nonlinearity f is a continuous and positive function; that is,

$$f(t, u, v) \in C([0, 1] \times (-\infty, 0] \times [0, +\infty), [0, +\infty)); \tag{3.1}$$

It is asymptotically linear at infinity; that is,

$$f_\infty = \lim_{u\to-\infty, v\to+\infty} \frac{\max_{\xi_1\le t\le\xi_2} \ f(t, u, v)}{v} = \mu \tag{3.2}$$

It is sublinear at the origin; that is,

$$f_0 = \lim_{u\to0-, v\to0+} \frac{\min_{\xi_1\le t\le\xi_2} \ f(t, u, v)}{v} = +\infty. \tag{3.3}$$

Similarly, assume that

$$f(t, u, v) \in C([0, 1] \times [0, +\infty) \times (-\infty, 0], [0, +\infty)); \tag{3.4}$$

$$f_\infty = \lim_{u\to+\infty, v\to-\infty} \frac{\min_{\xi_1\le t\le\xi_2} f(t, u, v)}{-v} = \mu; \tag{3.5}$$

$$f_0 = \lim_{u\to0+, v\to0-} \frac{\max_{\xi_1\le t\le\xi_2} \ f(t, u, v)}{-v} = +\infty. \tag{3.6}$$

Theorem 3.1. *Assume that conditions* (3.1)–(3.3) *hold. Then the boundary-value problem* (2.1)–(2.2) *admits a negative and convex solution* $u(t)$, $0 \le t \le 1$, *provided that*
$$\mu < \frac{a}{b + a(\xi_2 - \xi_1)}$$

Proof. Consider the BVP (2.3). By the assumption (3.3) and (2.4), it follows that for every $K \ge 32[1 + \frac{a}{b}(\xi_2 - \xi_1)]/(\xi_2 - \xi_1)^2$ there is an $\eta > 0$, such that
$$\begin{gathered} f(t,u,v) \ge Kv, \quad 0 \le t \le 1, \\ -\frac{\eta[1 + \frac{a}{b}(\xi_2 - \xi_1)]}{8} \le u \le -\eta, \quad 0 < v \le \eta[1 + \frac{a}{b}(\xi_2 - \xi_1)]. \end{gathered}$$
For $P = (\eta, \frac{a}{b}\eta) \in E_0$ and any solution $v \in \mathcal{X}(P)$ and $t \in [\xi_1, \xi_2]$, we get
$$v(t) = [1 + (t - \xi_1)\frac{a}{b}]\eta - \frac{(t-\xi_1)^2}{2!} f(\bar{t}, u(\bar{t}), v(\bar{t})) \le \eta(1 + \frac{a}{b}(\xi_2 - \xi_1)).$$
Moreover, by the Knesser's property and the connectedness of the set
$$\{P_1 = (v_1, \frac{a}{v} v_1) \in E_0 : \eta \le v_1 \le \eta(1 + \frac{a}{b}(\xi_2 - \xi_1))\}$$
(see also the proof of (2.14)), it follows that
$$\begin{gathered} v_1 \le v(t) < \eta(1 + \frac{a}{b}(\xi_2 - \xi_1)), \quad v'(t) > 0, \quad \xi_1 \le t \le \xi_1 + (\xi_2 - \xi_1)/4, \\ v(\xi_1 + (\xi_2 - \xi_1)/4\ ) = \eta(1 + \frac{a}{b}(\xi_2 - \xi_1)), \end{gathered} \tag{3.7}$$
for some $v_1 \in [\eta, \eta(1 + \frac{a}{b}(\xi_2 - \xi_1))]$ and $v \in \mathcal{X}(P_1)$. If $u(\xi_2) \le 0$ and since then $u'(\xi_2) \le 0$, we immediately obtain
$$G(P) := cv(\xi_2) + dv'(\xi_2) < 0. \tag{3.8}$$
Assume that $u(t) > 0$, $\xi_1 \le t \le \xi_2$. By (3.7) and the choice of K, we obtain
$$\begin{aligned} v(\xi_1 + (\xi_2 - \xi_1)/4\ ) &= [1 + \frac{(\xi_2 - \xi_1)}{4}\frac{a}{b}]v_1 - \frac{(\xi_2 - \xi_1)^2}{4^2 2!} f(\bar{t}, u(\bar{t}), v(\bar{t})) \\ &\le [1 + \frac{(\xi_2 - \xi_1)}{4}\frac{a}{b}]v_1 - \frac{(\xi_2 - \xi_1)^2}{32} Kv(\bar{t}) \\ &\le [1 + (\xi_2 - \xi_1)\frac{a}{b}]v_1 - \frac{(\xi_2 - \xi_1)^2}{32} Kv_1 < 0. \end{aligned}$$
Moreover, by Remark 1.3 (the nature of the vector field), we know that $v'(\xi_2) \le 0$ and hence (3.8) holds.

On the other hand, by assumption (3.2), for every $\lambda \in (\mu, \frac{a}{b + a(\xi_2 - \xi_1)})$, there is $H > 0$, such that
$$f(t,u,v) \le \lambda v, \quad 0 \le t \le 1,\ u \le -\frac{H}{8}, \quad v \ge H. \tag{3.9}$$
Hence, setting $P = (H, \frac{a}{b}H)$ and $v \in \mathcal{X}(P)$, (2.13) still holds; i.e.,
$$v(t) \le H(1 + \frac{a}{b}(\xi_2 - \xi_1)), \quad \xi_1 \le t \le \xi_2. \tag{3.10}$$
We assert that
$$v'(t) \ge 0,\ \xi_1 \le t \le \xi_2. \tag{3.11}$$
Assume on the contrary, that there exists a $t^* \in (\xi_1, \xi_2)$ such that
$$0 < v'(t) \le \frac{a}{b}H, \quad \xi_1 \le t \le t^*, \quad v'^*) = 0.$$

Then, by (3.10),

$$H \le v(t) \le H(1 + \frac{a}{b}(\xi_2 - \xi_1)), \quad \xi_1 \le t \le t^*.$$

Hence in view of the choice of λ and (3.9), we obtain the contradiction

$$\begin{aligned} v'^*) &= \frac{a}{b}H - f(\bar{t}, u(\bar{t}), v(\bar{t})) \\ &\ge \frac{a}{b}H - \lambda u(\bar{t}) \\ &\ge H[\frac{a}{b} - \lambda(1 + \frac{a}{b}(\xi_2 - \xi_1))] > 0. \end{aligned}$$

Consequently (3.11) yields $G(\xi_2; P) = cv(\xi_2) + dv'(\xi_2) > 0$. □

Theorem 3.2. *Assume (3.4)–(3.6) hold. Then the boundary value problem* (2.25)–(2.26) *admits a positive and concave solution* $u(t)$, $0 \le t \le 1$, *provided that*

$$\mu < \frac{a}{b + a(\xi_2 - \xi_1)}.$$

The proof of the above theorem is similar to that of Theorem 2.5, and thus omitted.

Remark 3.3. We may easily obtain more results similar to the above given, in Theorems 2.6-2.7.

The final Corollary-example illustrates the power of our approach. Indeed, it is unknown, at least to the authors of this paper, if we are able to construct a Green's function for (2.1) with the the boundary conditions

$$\begin{gathered} u(0) = u(1) = 0 \\ a[u''(\xi_1)]^{1/2} - bu'''(\xi_1) = 0, \quad c[u''(\xi_2)]^{5/3} + du'''(\xi_2) = 0. \end{gathered} \tag{3.12}$$

Corollary 3.4. *Under the assumptions of Theorem 2.6, the BVP* (2.1) *and* (3.12) *admits a negative and convex solution.*

Proof. We must only replace the semi-lines E_0 and E_1, by the relating semi-parabolas defined by the boundary conditions (3.12), instead of those in (2.2). The rest of the proof follows readily by that of Theorem 2.6. □

Remark 3.5. More general, we could replace the semi-lines E_0 and E_1, by suitable continua.

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