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LYAPUNOV-RAZUMIKHIN METHOD FOR ASYMPTOTIC STABILITY OF SETS FOR IMPULSIVE FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In the present paper, we study the global stability of sets of sufficiently general type with respect to impulsive functional differential equations with variable impulsive perturbations. The main results are obtained by means of piecewise continuous Lyapunov functions and the use of the Razumikhin technique.

1. Introduction

Stability of impulsive ordinary differential equations is discussed in [2, 3, 9, 10], and recently the stability of impulsive functional differential equations is investigated in [4, 5, 13, 14, 15]. When the impulses are realized at fixed moments the results are easier to obtain by means of the corresponding results in the continuous case. In the investigation of the impulsive functional differential equations with variable impulsive perturbations there arise a number of difficulties related to the phenomena of "beating" of the solutions, bifurcation, loss of the property of autonomy, etc. The wider application, however, of these type of equations requires the formulation of effective criteria for stability of their solutions.

In the present paper the problem of global stability of sets with respect to systems of impulsive functional differential equations with variable impulsive perturbations is considered by means of Lyapunov's direct method. We use the piecewise continuous Lyapunov's functions. Moreover, the technique of investigation essentially depends on the choice of minimal subsets of a suitable space of piecewise continuous functions, by the elements of which the derivatives of Lyapunov's functions are estimated [10, 12]. It is well known that Lyapunov-Razumikhin function method has been widely used in the treatment of the stability of functional differential equations without impulses [6, 7, 8].

2. Statement of the problem, preliminary notes and definitions

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with norm $|\cdot|$, scalar product $\langle .,. \rangle$ and distance $d(\cdot,\cdot)$; $\mathbb{R}_+ = [0,\infty)$; $\mathbb{R} = (-\infty,\infty)$.

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Let $t_0 \in \mathbb{R}$, $r > 0$. Consider the system of impulsive functional differential equations

$$\begin{aligned} \dot{x}(t) &= f(t, x_t), \quad t \neq \tau_k(x(t)), \\ \Delta x(t) &= I_k(x(t-0)), \quad t = \tau_k(x(t)),\ k = 1, 2, \dots, \end{aligned} \tag{2.1}$$

where $f : (t_0, \infty) \times D \to \mathbb{R}^n$; $D = \{\phi : [-r, 0] \to \mathbb{R}^n, \quad \phi(t)$ is continuous everywhere except at finite number of points $\tilde{t}$ at which $\phi(\tilde{t}-0)$ and $\phi(\tilde{t}+0)$ exist and $\phi(\tilde{t}-0) = \phi(\tilde{t})\}$; $I_k : \mathbb{R}^n \to \mathbb{R}^n, k = 1, 2, \dots$; $\tau_k : \mathbb{R}^n \to (t_0, \infty)$; $\Delta x(t) = x(t+0) - x(t-0)$ and for $t > t_0$, $x_t \in D$ is defined by $x_t = x(t+s)$, $-r \le s \le 0$.

Let $\tau_0(x) \equiv t_0$ for $x \in \mathbb{R}^n$. We shall assume that:

(a) $\tau_k \in C[\mathbb{R}^n, (t_0, \infty)]$, $k = 1, 2, \dots$.
(b) $t_0 < \tau_1(x) < \tau_2(x) < \dots$, $x \in \mathbb{R}^n$.
(c) $\tau_k(x) \to \infty$ as $k \to \infty$ uniformly on $x \in \mathbb{R}^n$.

Assuming that (a), (b) and (c) are fulfilled, we introduce the notation:

$$G_k = \big\{(t, x) \in [t_0, \infty) \times \mathbb{R}^n : \tau_{k-1}(x) < t < \tau_k(x)\big\}, \quad k = 1, 2, \dots,$$
$$\sigma_k = \Big\{(t, x) \in [t_0, \infty) \times \mathbb{R}^n :\ t = \tau_k(x)\Big\};$$

i.e., σ_k, $k = 1, 2, \dots$ are hypersurfaces of the equations $t = \tau_k(x(t))$.

Let $\varphi_0 \in D$. Denote by $x(t) = x(t; t_0, \varphi_0)$ the solution of (2.1) satisfying the initial conditions

$$\begin{aligned} x(t; t_0, \varphi_0) &= \varphi_0(t - t_0), \quad t_0 - r \le t \le t_0, \\ x(t_0 + 0; t_0, \varphi_0) &= \varphi_0(0) \end{aligned} \tag{2.2}$$

and by $J^+(t_0, \varphi_0)$ - the maximal interval of the type (t_0, β), at which the solution $x(t; t_0, \varphi_0)$ is defined. The precise description of the solution $x(t; t_0, \varphi_0)$ of (2.1), (2.2) is given in [4, 14].

Let $M \subset [t_0 - r, \infty) \times \mathbb{R}^n$. Introduce the following notation: $M(t) = \{x \in \mathbb{R}^n : (t, x) \in M, t \in (t_0, \infty)\}$; $M_0(t) = \{x \in \mathbb{R}^n : (t, x) \in M, t \in [t_0 - r, t_0]\}$;
$d(x, M(t)) = \inf_{y \in M(t)} |x - y|$ is the distance between $x \in \mathbb{R}^n$ and $M(t)$;
$M(t, \varepsilon) = \{x \in \mathbb{R}^n : d(x, M(t)) < \varepsilon\} \quad (\varepsilon > 0)$ is an ε- neighbourhood of $M(t)$;
$C_0 = C[[-r, 0], \mathbb{R}^n]$; $d_0(\varphi, M_0(t)) = \max_{t \in [t_0 - r, t_0]} d(\varphi(t - t_0), M_0(t)), \varphi \in C_0$;
$M_0(t, \varepsilon) = \{\varphi \in C_0 : d_0(\varphi, M_0(t)) < \varepsilon\}$;
$S_\alpha = \{x \in \mathbb{R}^n : |x| < \alpha\}$, $\alpha > 0$; $\overline{S_\alpha} = \{x \in \mathbb{R}^n : |x| \le \alpha\}$;
$\overline{S_\alpha}(C_0) = \{\varphi \in C_0 : ||\varphi|| \le \alpha\}$, where $||\varphi|| = \max_{t \in [t_0 - r, t_0]} |\varphi(t - t_0)|$ is the norm of the function $\varphi \in C_0$;
$K = \{a \in C[R_+, R_+] : a(r)$ is strictly increasing and $a(0) = 0\}$;
$CK = \{a \in C[(t_0, \infty) \times R_+, R_+] : a(t, .) \in K$ for any fixed $t \in (t_0, \infty)\}$;
$K^* = \{a \in C[R_+ \times R_+, R_+] : a(., s) \in K$ for any fixed $s \in R_+\}$.

We also introduce the following conditions:

(H1) $M(t) \neq \emptyset$ for $t \in (t_0, \infty)$.
(H2) $M_0(t) \neq \emptyset$ for $t \in [t_0 - r, t_0]$.
(H3) For any compact subset F of $(t_0, \infty) \times \mathbb{R}^n$ there exists a constant $K > 0$ depending on F such that if (t, x), $(t', x) \in F$, then the following inequality is valid
$$|d(x, M(t)) - d(x, M(t'))| \le K|t - t'|.$$
(H4) The integral curves of the (2.1) meet successively each one of the hypersurfaces $\sigma_1, \sigma_2, \dots.$ exactly once.

Condition (H4) guarantees absence of the phenomenon "beating" of the solutions to the (2.1); i.e., a phenomenon when a given integral curve meets more than once or infinitely many times one and the same hypersurface. Efficient sufficient conditions which guarantee the absence of "beating" of the solutions of such systems are given in [1].

Let $t_1, t_2, \dots$ $(t_0 < t_1 < t_2 < \dots)$ be the moments in which the integral curve $(t, x(t; t_0, \varphi_0))$ of the (2.1), (2.2) meets the hypersurfaces σ_k, $k = 1, 2, \dots$.

We shall assume existence of solutions of (2.1) for all $t > t_0$. Note that [1, 2, 9] if $f \in C[(t_0, \infty) \times D, \mathbb{R}^n]$, the function f is Lipschitz continuous with respect to its second argument in $(t_0, \infty) \times D$ uniformly on $t \in (t_0, \infty)$, $|f(t, \tilde{x})| \le L < \infty$ for$(t, \tilde{x}) \in (t_0, \infty) \times D$, $L > 0$, for any $k = 1, 2, \dots$ the following inequality is valid $|I_k(x_1) - I_k(x_2)| \le c|x_1 - x_2|$, $x_1, x_2 \in \mathbb{R}^n$, $\quad c > 0$, and (H4) are met, then $t_k \to \infty$ as $k \to \infty$ and $J^+(t_0, \varphi_0) = (t_0, \infty)$.

Definition 2.1. The solutions of (2.1) are said to be uniformly M-bounded if

$$(\forall \eta > 0)(\exists \beta = \beta(\eta) > 0)(\forall t_0 \in R)(\forall \alpha > 0)$$
$$(\forall \varphi_0 \in \overline{S_\alpha}(C_0) \cap \overline{M_0(t, \eta)})(\forall t > t_0) : x(t; t_0, \varphi_0) \in M(t, \beta);$$

Definition 2.2. The set M is said to be

(a) stable with respect to (2.1) if
$$(\forall t_0 \in R)(\forall \alpha > 0)(\forall \varepsilon > 0)(\exists \delta = \delta(t_0, \alpha, \varepsilon) > 0)$$
$$(\forall \varphi_0 \in \overline{S_\alpha}(C_0) \cap M_0(t, \delta))(\forall t > t_0) : x(t; t_0, \varphi_0) \in M(t, \varepsilon);$$

(b) uniformly stable with respect to (2.1) if the number δ from point (a) depends only on ε;

(c) uniformly globally attractive with respect to (2.1) if
$$(\forall \eta > 0)(\forall \varepsilon > 0)(\exists \sigma = \sigma(\eta, \varepsilon) > 0)$$
$$(\forall t_0 \in R)(\forall \alpha > 0)(\forall \varphi_0 \in \overline{S_\alpha}(C_0) \cap M_0(t, \eta))$$
$$(\forall t \ge t_0 + \sigma) : x(t; t_0, \varphi_0) \in M(t, \varepsilon);$$

(d) uniformly globally asymptotically stable with respect to (2.1) if M is a uniformly stable and uniformly globally attractive set of (2.1) and if the solutions of (2.1) are uniformly M-bounded.

Also we introduce the notations: $I = [t_0 - r, \infty)$; $I_0 = [t_0, \infty)$. In the further considerations we shall use the class V_0 of piecewise continuous auxiliary functions $V : I_0 \times \mathbb{R}^n \to R_+$ which are analogues of Lyapunov's functions.

Definition 2.3. We say that the function $V :\ [t_0, \infty) \times \mathbb{R}^n \to R_+$, belongs to the class V_0 if the following conditions are fulfilled:

(1) The function V is continuous in $\cup_{k=1}^{\infty} G_k$ and locally Lipschitz continuous with respect to its second argument x on each of the sets G_k, $k = 1, 2, \dots$.
(2) $V(t, x) = 0$ for $(t, x) \in M$, $t \ge t_0$ and $V(t, x) > 0$ for $(t, x) \in \{[t_0, \infty) \times \mathbb{R}^n\} \setminus M$.
(3) For each $k = 1, 2, \dots$ and $(t_0^*, x_0^*) \in \sigma_k$ there exist the finite limits
$$V(t_0^* - 0, x_0^*) = \lim_{(t,x) \to (t_0^*, x_0^*),\, (t,x) \in G_k} V(t, x),$$
$$V(t_0^* + 0, x_0^*) = \lim_{(t,x) \to (t_0^*, x_0^*),\, (t,x) \in G_{k+1}} V(t, x)$$

(4) For each $k = 1, 2, \dots$ the following equalities are valid

$$V(t_0^* - 0, x_0^*) = V(t_0^*, x_0^*).$$

(5) For each $k = 1, 2, \dots$ the following inequalities are valid

$$V(t + 0, x(t) + I_k(x(t))) \le V(t, x(t)), \quad t = \tau_k(x(t)), \quad k = 1, 2, \dots \tag{2.3}$$

Let $J \subset R$ be an interval. Define the following classes of functions:
$PC[J, R^n] = \{\sigma : J \to \mathbb{R}^n : \sigma(t)$ is continuous everywhere except some points t_k at which $\sigma(t_k - 0)$ and $\sigma(t_k + 0)$ exist and $\sigma(t_k - 0) = \sigma(t_k),\ k = 1, 2, \dots\}$;

$PC^1[J, R^n] = \{\sigma \in PC[J, R^n] : \sigma(t)$ is continuously differentiable everywhere except some points t_k at which $\dot\sigma(t_k - 0)$ and $\dot\sigma(t_k + 0)$ exist and $\dot\sigma(t_k - 0) = \dot\sigma(t_k)$, $k = 1, 2, \dots\}$;
$\Omega_1 = \{x \in PC[I_0, \mathbb{R}^n] : V(s, x(s)) \le V(t, x(t)),\ t - r \le s \le t,\ t \in I_0,\ V \in V_0\}$.

Let $V \in V_0$. For $x \in PC[I_0, \mathbb{R}^n]$ and $t \in I_0$, $t \ne t_k(x(t))$, $k = 1, 2, \dots$ we define the function

$$D_-V(t, x(t)) = \lim_{h \to 0^-} \inf h^{-1}[V(t + h, x(t) + hf(t, x_t)) - V(t, x(t))].$$

Definition 2.4 ([2])**.** Let $\lambda : (t_0, \infty) \to R_+$ be measurable. Then we say that $\lambda(t)$ is integrally positive if $\int_J \lambda(t)dt = \infty$ whenever $J = \bigcup_{k=1}^\infty [\alpha_k, \beta_k]$, $\alpha_k < \beta_k < \alpha_{k+1}$ and $\beta_k - \alpha_k \ge \theta > 0$, $k = 1, 2, \dots$.

In the proof of the main results we shall use the following lemma.

Lemma 2.5 ([14])**.** *Let* (H4) *and the following conditions hold:*

(1) *The solution $x = x(t; t_0, \varphi_0)$ of the problem* (2.1), (2.2) *is such that $x \in PC[I, S_\rho] \cap PC^1[I_0, S_\rho]$.*
(2) *$g \in PC[[t_0, \infty) \times R_+, R]$ and $g(t, 0) = 0$ for $t \in [t_0, \infty)$.*
(3) *$B_k \in C[R_+, R_+]$, $B_k(0) = 0$ and $\psi_k(u) = u + B_k(u)$ are nondecreasing with respect to u, $k = 1, 2, \dots$.*
(4) *The maximal solution $u^+(t; t_0, u_0)$ of the problem*

$$\dot u(t) = g(t, u(t)), \quad t > t_0,\ t \ne t_k,\ k = 1, 2, \dots,$$
$$u(t_0 + 0) = u_0 \ge 0,$$
$$\Delta u(t_k) = B_k(u(t_k)),\ k = 1, 2, \dots$$

is defined in the interval $[t_0, \infty)$.
(5) *The function $V \in V_0$, $V : I_0 \times S_\rho \to R_+$ is such that $V(t_0 + 0, \varphi_0(0)) \le u_0$ and the inequalities*

$$D_-V(t, x(t)) \le g(t, V(t, x(t))), \quad t \ne \tau_k(x(t)),\ k = 1, 2, \dots$$
$$V(t + 0, x(t) + I_k(x(t))) \le B_k(V(t, x(t))), \quad t = \tau_k(x(t)),\ k = 1, 2, \dots$$

are valid for each $t \in I_0$ and $x \in \Omega_1$.

Then $V(t, x(t; t_0, \varphi_0)) \le u^+(t; t_0, u_0)$, $t \in I_0$.

Corollary 2.6. *Let the condition* (H4) *be satisfies and the function $V \in V_0$ be such that the inequality*

$$D_-V(t, x(t)) \le 0, \quad t \ne \tau_k(x(t)), \quad k = 1, 2, \dots$$

is valid for each $t > t_0$ and $x \in \Omega_1$. Then $V(t, x(t; t_0, \varphi_0)) \le V(t_0, \varphi_0(t_0))$, $t \in [t_0, \infty)$.

3. Main results

Theorem 3.1. *Assume that* (H1)-(H4) *and the following conditions are met:*

(1) *The functions $V \in V_0$ and $a, b \in K$ are such that*
$$a(d(x, M(t))) \le V(t, x) \le b(d(x, M(t))),$$
for $(t, x) \in [t_0, \infty) \times \mathbb{R}^n$ and $a(r) \to \infty$ as $r \to \infty$.

(2) *The inequality*
$$D_- V(t, x(t)) \le -p(t)c(d(x(t), M(t))), \quad t \ne \tau_k(x(t)), \quad k = 1, 2 \dots$$
is valid for any $t > t_0$, $x \in \Omega_1, V \in V_0$, $p : [t_0, \infty) \to (0, \infty)$, $c \in K$.

(3) *$\int_0^\infty p(s)c[b^{-1}(\eta)]ds = \infty$ for each sufficiently small value of $\eta > 0$.*

Then the set M is uniformly globally asymptotically stable with respect to (2.1).

Proof. Let $\varepsilon > 0$. Choose $\delta = \delta(\varepsilon) > 0$, $\delta < \varepsilon$ so that $b(\delta) < a(\epsilon)$. Let $\alpha > 0$ be arbitrary, $\varphi_0 \in \overline{S_\alpha}(C_0) \cap M_0(t, \delta)$ and $x(t) = x(t; t_0, \varphi_0)$. From conditions 1 and 2, and (2.3) it follows that for $t \in J^+(t_0, \varphi_0)$,
$$\begin{aligned} a(d(x(t; t_0, \varphi_0), M(t))) &\le V(t, x(t)) \\ &\le V(t_0, \varphi_0(t_0)) \\ &\le b(d(\varphi_0(t_0), M_0(t_0))) \\ &\le b(d_0(\varphi_0, M_0(t))) \\ &< b(\delta) < a(\varepsilon). \end{aligned}$$
Since $J^+(t_0, \varphi_0) = (t_0, \infty)$, then $x(t) \in M(t, \varepsilon)$ for all $t > t_0$. This proves that the set M is uniformly stable.

Now let $\eta > 0$ and $\varepsilon > 0$ be given and let the number $\sigma = \sigma(\eta, \varepsilon) > 0$ be chosen so that
$$\int_{t_0}^{t_0+\sigma} p(s)c[b^{-1}(\frac{a(\varepsilon)}{2})]ds > b(\eta). \tag{3.1}$$
(This is possible in view of condition 3).

Let $\alpha > 0$ be arbitrary, $\varphi_0 \in \overline{S_\alpha}(C_0) \cap M_0(t, \eta)$ and $x(t) = x(t; t_0, \varphi_0)$. Assume that for any $t \in [t_0, t_0 + \sigma]$,
$$d(x(t), M(t)) \ge b^{-1}(\frac{a(\varepsilon)}{2}).$$
Then by condition 2 and (3.1), it follows that
$$\int_{t_0}^{t_0+\sigma} D_- V(s, x(s))ds \le -\int_{t_0}^{t_0+\sigma} p(s)c[b^{-1}(\frac{a(\varepsilon)}{2})]ds < -b(\eta). \tag{3.2}$$
On the other hand, if $t_0 + \sigma \in (\tau_r, \tau_{r+1}]$, then from (2.3) we obtain
$$\begin{aligned} &\int_{t_0}^{t_0+\sigma} D_- V(s, x(s))ds \\ &= \sum_{k=1}^{r} \int_{\tau_{k-1}}^{\tau_k} D_- V(s, x(s))ds + \int_{\tau_r}^{t_0+\sigma} D_- V(s, x(s))ds \\ &= \sum_{k=1}^{r} [V(\tau_k, x(\tau_k)) - V(\tau_{k-1} + 0, x(\tau_{k-1} + 0))] + V(t_0 + \sigma, x(t_0 + \sigma)) \\ &\quad - V(\tau_r + 0, x(\tau_r + 0)) \ge V(t_0 + \sigma, x(t_0 + \sigma)) - V(t_0, \varphi_0(t_0)), \end{aligned}$$

whence, in view of (3.2) and condition 2, it follows that $V(t_0+\sigma, x(t_0+\sigma)) < 0$, which contradicts condition 1.

The contradiction obtained shows that there exists $t^* \in [t_0, t_0+\sigma]$, such that

$$d(x(t^*), M(t^*)) < b^{-1}(\frac{a(\epsilon)}{2}).$$

Then for $t \geq t^*$ (hence for any $t \geq t_0+\sigma$ as well) the following inequalities are valid

$$\begin{aligned} a(d(x(t), M(t))) &\leq V(t, x(t)) \\ &\leq V(t^*+0, x(t^*+0)) \\ &\leq b(d(x(t^*), M(t^*))) \\ &< \frac{a(\varepsilon)}{2} < a(\epsilon). \end{aligned}$$

Hence $x(t) \in M(t, \epsilon)$ for $t \geq t_0+\sigma$; i.e., the set M is uniformly globally attractive with respect to (2.1).

Finally we shall prove that the solutions of (2.1) are uniformly M-bounded. Let $\eta > 0$ and let $\beta = \beta(\eta) > 0$ be such that $a(\beta) > \gamma b(\eta)$. Choose arbitrary $\alpha > 0$, $\varphi_0 \in \overline{S_\alpha}(C_0) \cap \overline{M_0(t,\eta)}$ and let $x(t) = x(t; t_0, \varphi_0)$. Then for $t > t_0$,

$$\begin{aligned} a(d(x(t), M(t))) &\leq V(t, x(t)) \\ &\leq V(t_0, \varphi_0(t_0)) \\ &\leq \gamma b(d(\varphi_0(t_0), M_0(t_0))) \\ &\leq \gamma b(d_0(\varphi_0, M_0(t))) \\ &\leq \gamma b(\eta) < a(\beta). \end{aligned}$$

Hence $x(t) \in M(t, \beta)$ for $t > t_0$. □

Theorem 3.2. *Assume that* (H1)–(H4) *and Condition 1 of Theorem 3.1 are met. Also assume that there exists an integrally positive function* $\lambda(t)$ *such that*

$$D_-V(t, x(t)) \leq -\lambda(t)c(d(x(t), M(t))), \quad t \neq \tau_k(x(t)),\ k = 1, 2 \dots$$

holds for any $t > t_0$, $x \in \Omega_1, V \in V_0$ *and* $c \in K$. *Then the set* M *is uniformly globally asymptotically stable with respect to* (2.1).

Proof. The fact that the set M is uniformly stable with respect to the (2.1) and the uniform M-boundedness of the solutions of (2.1) are proved as in the proof of Theorem 3.1.

Now we shall prove that the set M is uniformly globally attractive with respect to the (2.1). Let again $\varepsilon > 0$ and $\eta > 0$ be given. Choose the number $\delta = \delta(\varepsilon) > 0$ so that $b(\delta) < a(\varepsilon)$.

We shall prove that there exists $\sigma = \sigma(\varepsilon, \eta) > 0$ such that for any solution $x(t) = x(t; t_0, \varphi_0)$ of (2.1) for which $t_0 \in R$, $\varphi_0 \in \overline{S_\alpha}(C_0) \cap M_0(t, \eta)$ ($\alpha > 0$ - arbitrary) and for any $t^* \in [t_0, t_0+\sigma]$,

$$d(x(t^*), M(t^*)) < \delta(\varepsilon). \tag{3.3}$$

Suppose that this is not true. Then for any $\sigma > 0$ there exists a solution $x(t) = x(t; t_0, \varphi_0)$ of (2.1) for which $t_0 \in R$, $\varphi_0 \in \overline{S_\alpha}(C_0) \cap M_0(t, \eta)$, $\alpha > 0$, such that

$$d(x(t), M(t)) \geq \delta(\varepsilon), \tag{3.4}$$

for $t \in [t_0, t_o + \sigma]$. From the third condition in this theorem and (2.3) it follows that

$$
\begin{aligned}
V(t, x(t)) - V(t_0, \varphi_0(t_0)) &\le \int_{t_0}^{t} D_- V(s, x(s)) ds \\
&\le - \int_{t_0}^{t} \lambda(s) c(d(x(s), M(s))) ds, \quad t > t_0.
\end{aligned}
$$

From the properties of the function $V(t, x(t))$ in the interval (t_0, ∞) it follows that there exists the finite limit

$$
\lim_{t \to \infty} V(t, x(t)) = v_0 \ge 0. \tag{3.5}
$$

Then from condition 1 of Theorem 3.1, (3.4)-(3.5) it follows that

$$
\int_{t_0}^{\infty} \lambda(t) c(d(x(t), M(t))) dt \le b(\eta) - v_0.
$$

From the integral positivity of the function $\lambda(t)$ it follows that the number σ can be chosen so that

$$
\int_{t_0}^{t_0+\sigma} \lambda(t) dt > \frac{b(\eta) - v_0 + 1}{c(\delta(\varepsilon))}.
$$

Then

$$
\begin{aligned}
b(\eta) - v_0 &\ge \int_{t_0}^{\infty} \lambda(t) c(d(x(t), M(t))) dt \\
&\ge \int_{t_0}^{t_0+\sigma} \lambda(t) c(d(x(t), M(t))) dt \\
&\ge c(\delta(\varepsilon)) \int_{t_0}^{t_0+\sigma} \lambda(t) dt \\
&> b(\eta) - v_0 + 1.
\end{aligned}
$$

The contradiction obtained shows that there exists a positive constant $\sigma = \sigma(\epsilon, \eta)$ such that for any solution $x(t) = x(t; t_0, \varphi_0)$ of (2.1) for which $t_0 \in R$, $\varphi_0 \in \overline{S_\alpha}(C_0) \cap M_0(t, \eta)$, $\alpha > 0$, there exists $t^* \in [t_0, t_0 + \sigma]$ such that (3.3) holds. Then for $t \ge t^*$ (hence for any $t \ge t_0 + \sigma$ as well) the following inequalities are valid

$$
\begin{aligned}
a(d(x(t), M(t))) &\le V(t, x(t)) \\
&\le V(t^* + 0, x(t^* + 0)) \\
&\le b(d(x(t^*), M(t^*))) \\
&< b(\delta) < a(\epsilon),
\end{aligned}
$$

which proves that the set M is uniformly globally attractive with respect to (2.1). $\square$

4. An example

We shall use Theorem 3.2 to prove the global uniform asymptotic stability of a set with respect to the system

$$
\begin{gathered}
\dot{x}(t) = \begin{cases} A(t)x(t) + B(t)x(t-h(t)), & x(t) > 0,\ t \neq \tau_k(x(t)),\\ 0, & x(t) \leq 0,\ t \neq \tau_k(x(t)); \end{cases} \\
\Delta x(t) = \begin{cases} C_k x(t), & x(t) > 0,\ t = \tau_k(x(t)),\\ 0, & x(t) \leq 0,\ t = \tau_k(x(t)), \end{cases}
\end{gathered} \tag{4.1}
$$

where $t > t_0$; $x \in PC[(t_0,\infty),\mathbb{R}^n]$; $A(t)$ and $B(t)$ are $(n \times n)$ matrix-valued functions, C_k, $k = 1,2,\dots$ are $(n \times n)$ matrices; $h \in C[(t_0,\infty),R_+]$.

Such systems seem to have application, among other things, in the study of active suspension height control. In the interest of improving the overall performance of automotive vehicles, in recent years, suspension incorporating active components have been developed. The designs may cover a spectrum of of performance capabilities, but the active components alter only the vertical force reactions of the suspensions, not the kinematics. The conventional passive suspensions consist of usual components with spring and damping properties, which are time-invariant. The interest in active or semi-active suspensions derives from the potential for improvements to vehicle ride performance with no compromise or enhancement in handling. The full active suspensions incorporate actuators to generate the desired forces in the suspension. They actuators are normally hydraulic cylinders.

Let $\tau = \inf_{t\geq t_0}(t - h(t))$ and $\varphi_1 \in C[[\tau,t_0],\mathbb{R}^n]$. Denote by $x(t) = x(t;t_0,\varphi_1)$ the solution of system (4.1) satisfying the initial condition

$$
x(t;t_0,\varphi_1) = \varphi_1(t), \quad \tau \leq t \leq \quad t_0, \tag{4.2}
$$

and by $J^+(t_0,\varphi_1)$ - the maximal interval of the type (t_0,β), at which the solution $x(t;t_0,\varphi_1)$ is defined.

Theorem 4.1. *Let* (H4) *and the following conditions hold:*

1. *The matrix functions $A(t)$ and $B(t)$ are continuous for $t \in (t_0,\infty)$.*
2. *$t - h(t) \to \infty$ as $t \to \infty$.*
3. *For each $k = 1,2,\dots$ the elements of the matrix C_k are nonnegative.*
4. *There exists a continuous real $(n \times n)$ matrix $D(t)$, $t \in (t_0,\infty)$, which is symmetric, positive definite, differentiable for $t \neq \tau_k(x(t))$, $k = 1,2,\dots$ and such that for each $k = 1,2\dots,$*
$$
x^T[A^T(t)D(t) + D(t)A(t) + \dot{D}(t)]x \leq -c(t)|x|^2, \quad x \in \mathbb{R}^n,\ t \neq \tau_k(x(t)), \tag{4.3}
$$
$$
x^T[C_k^T D(t) + D(t)C_k + C_k^T D(t)C_k]x \leq 0, \quad t = \tau_k(x(t)), \tag{4.4}
$$
where $c(t) > 0$ is a continuous function.
5. *There exists an integrally positive function $\lambda(t)$ such that*
$$
d(t) = c(t) - \max\{\alpha(t)\lambda(t),\beta(t)\lambda(t)\} \geq 0, \tag{4.5}
$$
$$
\frac{2\beta^{1/2}(t)}{\alpha^{1/2}(t-h(t))}|D(t)B(t)| \leq d(t), \tag{4.6}
$$
where $\alpha(t)$ and $\beta(t)$ are respectively the smallest and the greatest eigenvalues of matrix $D(t)$.

Then the set $M = [\tau - t_0, \infty) \times \{x \in \mathbb{R}^n : x \leq 0\}$ is uniformly globally asymptotically stable with respect to system (4.1).

Proof. Consider the function

$$V(t,x) = \begin{cases} x^T D(t)x, & \text{for } x > 0, \\ 0, & \text{for } x \leq 0. \end{cases}$$

From the condition that $D(t)$ is real symmetric matrix it follows that for $x \in \mathbb{R}^n$ and $x \neq 0$ it holds

$$\alpha(t)|x|^2 \leq x^T D(t)x \leq \beta(t)|x|^2. \tag{4.7}$$

From thins inequalities it follows that condition 1 of Theorem 3.1 is satisfied.

For the chosen function $V(t,x)$ the set Ω_1 is

$$\Omega_1 = \{x \in PC[I_0, \mathbb{R}^n] : x^T(s)D(s)x(s) \leq x^T(t)D(t)x(t),\ \tau \leq s \leq t,\ t \in I_0\}.$$

For $t > t_0$ and $x \in \Omega_1$ the following inequalities are valid:

$$\begin{aligned} \alpha(t-h(t))|x(t-h(t))|^2 &\leq x^T(t-h(t))D(t-h(t))x(t-h(t)) \\ &\leq x^T(t)D(t)x(t) \leq \beta(t)|x(t)|^2, \end{aligned}$$

from which we obtain the estimate

$$|x(t-h(t))| \leq \frac{\beta^{1/2}(t)}{\alpha^{1/2}(t-h(t))}|x(t)|. \tag{4.8}$$

Let $t \neq \tau_k(x(t))$ and $x \in \Omega_1$. ¿From (4.3), (4.5), (4.6) and (4.8), we have

$$\begin{aligned} D_-V(t,x(t)) &= \begin{cases} -c(t)|x(t)|^2 + 2|D(t)B(t)||x(t)||x(t-h(t))|, & x(t) > 0, \\ 0, & x(t) \leq 0 \end{cases} \\ &\leq \begin{cases} -[c(t)-d(t)]|x(t)|^2, & x(t) > 0, \\ 0, & x(t) \leq 0 \end{cases} \\ &\leq -\lambda(t)V(t,x(t)). \end{aligned}$$

Let $t = \tau_k(x(t))$. Then from (4.4) we have

$$\begin{aligned} &V(t+0, x(t) + C_k x(t)) \\ &= \begin{cases} (x^T(t) + x^T(t)C_k^T)D(t)(x(t) + C_k x(t)), & x > 0, \\ 0, & x(t) \leq 0 \end{cases} \\ &= \begin{cases} x^T(t)D(t)x(t) + x^T(t)[C_k^T D(t) + D(t)C_k + C_k^T D(t)C_k]x(t), & x(t) > 0, \\ 0, & x(t) \leq 0 \end{cases} \\ &\leq V(t, x(t)). \end{aligned}$$

Thus we have checked that all the conditions of Theorem 3.2 are satisfied. Hence the set $M = [\tau - t_0, \infty) \times \{x \in \mathbb{R}^n : x \leq 0\}$ is uniformly globally asymptotically stable with respect to system (4.1). □

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