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EXISTENCE OF LEAST ENERGY SOLUTIONS TO COUPLED ELLIPTIC SYSTEMS WITH CRITICAL NONLINEARITIES

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Abstract. In this paper we study the existence of nontrivial solutions of elliptic systems with critical nonlinearities and subcritical nonlinear coupling interactions, under Dirichlet or Neumann boundary conditions. These equations are motivated from solitary waves of nonlinear Schrödinger systems in physics. Using minimax theorem and by estimates on the least energy, we prove the existence of nonstandard least energy solutions, i.e. solutions with least energy and each component is nontrivial.

1. Introduction

In this paper, we consider the existence of least energy solutions to the Dirichlet problem

$$\begin{gathered} -\Delta u+\lambda_1 u=\mu_1 u^3+\beta u^{p-1}v^p \quad \text{in } \Omega \\ -\Delta v+\lambda_2 v=\mu_2 v^3+\beta u^{p}v^{p-1} \quad \text{in } \Omega \\ u>0, \quad v>0 \quad \text{in } \Omega \\ u=0, \quad v=0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.1}$$

and to the Neumann prolem

$$\begin{gathered} -\Delta u+\lambda_1 u=\mu_1 u^3+\beta u^{p-1}v^p \quad \text{in } \Omega \\ -\Delta v+\lambda_2 v=\mu_2 v^3+\beta u^{p}v^{p-1} \quad \text{in } \Omega \\ u>0, \quad v>0 \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu}=0, \quad \frac{\partial v}{\partial \nu}=0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^4$ is a smooth bounded domain, λ_i, μ_i, β are constants, $\mu_i > 0$, $i = 1, 2$, and $1 < p < 2$. Since the dimension of Ω is $N = 4$, $4 = \frac{2N}{N-2}$ is the critical Sobolev exponent. Therefore there are critical nonlinearities and coupling interaction terms in the elliptic systems. In this paper, we are interested in positive solutions. The

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solutions of problems (1.1) and (1.2) are equivalent to positive solutions of

$$\begin{aligned} -\Delta u + \lambda_1 u &= \mu_1 |u|^2 u + \beta |u|^{p-2}|v|^p u \quad \text{in } \Omega \\ -\Delta v + \lambda_2 v &= \mu_2 |v|^2 v + \beta |u|^{p}|v|^{p-2} v \quad \text{in } \Omega \end{aligned}$$

By a least energy solution we mean a nontrivial solution with the least energy

$$E(u,v) = \int_\Omega \frac{1}{2}(|\nabla u|^2 + \lambda_1 u^2 + |\nabla v|^2 + \lambda_2 v^2) - \frac{1}{4}(\mu_1 u^4 + \mu_2 v^4) - \frac{\beta}{p}|uv|^p \tag{1.3}$$

among all nontrivial solutions of problem (1.1) or (1.2). By nontrivial solution of system we mean that at least one of its components is nontrivial(nonzero function). Of course, the least energy solution we are interested in is *nonstandard*(a definition in [15]); i.e., each of its components is nontrivial.

Recently, the existence and multiplicity of solutions of classical coupled nonlinear Schrödinger equations (CNLS)

$$\begin{aligned} \Delta u - \lambda_1 u + \mu_1 u^3 + \beta u v^2 &= 0 \\ \Delta v - \lambda_2 v + \mu_2 v^3 + \beta u^2 v &= 0 \end{aligned} \tag{1.4}$$

has been investigated by several authors in the case of subcritical nonlinearities, we recall, among many others, Ambrosetti & Cororado [1], Lin & Wei [7, 8], Maia, Montefusco & Pellacci [9], Sirakov [15], and the author [17]. These CNLS are motivated by nonlinear optics and Bose-Einstein double condensates and have attracted a considerable attention in the last years. On the other hand, systems of one-dimensional NLS

$$\begin{aligned} i\phi_t + \phi_{xx} + \alpha_1 |\phi|^{p_1-2}\phi + \alpha_0 |\psi|^{p_0}|\phi|^{p_0-2}\phi &= \delta\psi_{xx} \\ i\psi_t + \psi_{xx} + \alpha_2 |\psi|^{p_2-2}\psi + \alpha_0 |\phi|^{p_0}|\psi|^{p_0-2}\psi &= \delta\phi_{xx} \end{aligned}$$

where $\alpha_j \geq 0 (j = 1, 2), \alpha_0 \in \mathbb{R}$ and $|\delta| < 1$, appear in several branches of physics, such as in the study of interactions of waves with different polarizations or in the description of nonlinear modulations of two monochromatic waves. These systems have been studied in many physical literatures. See [3] for more references. Standing waves of the form

$$\phi(t,x) = e^{i\lambda_1 t} u(x), \quad \psi(t,x) = e^{i\lambda_2 t} v(x)$$

satisfy

$$\begin{aligned} -u_{xx} + \delta v_{xx} + \lambda_1 u &= \alpha_1 |u|^{p_1-2} u + \alpha_0 |v|^{p_0}|u|^{p_0-2} u \\ -v_{xx} + \delta u_{xx} + \lambda_2 v &= \alpha_2 |v|^{p_2-2} v + \alpha_0 |u|^{p_0}|v|^{p_0-2} v \end{aligned}. \tag{1.5}$$

A natural question is to study the multidimensional accompanist of (1.5). Therefore, the system we consider in this paper can be viewed as a generalization of (1.4) and a high dimensional case of (1.5).

Single elliptic equations with critical nonlinearities have been extensively studied by many authors, including the classical results of Brezis-Nirenberg [2], singular perturbation problem [5, 13, 14, 16] for Neumann problems, multi-peak solutions [4, 12], concentration phenomena [10, 13], and so on. In [11], the authors constructed concentrated solutions for elliptic systems with critical nonlinearities and weakly coupling interactions. To the author's knowledge, there are few results on Schrödinger type systems with critical nonlinearities and strong coupling interactions. This is another motivation of this paper.

Let $\lambda_1(\Omega)$ be the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. The main results of this paper are as follows.

Theorem 1.1. *Assume $-\lambda_1(\Omega) < \lambda_1$, $\lambda_2 < 0$. Then problem* (1.1) *has a nonstandard least energy solution for sufficiently large β.*

Theorem 1.2. *Assume λ_1, λ_2 are sufficiently large (but independent of β). Then problem* (1.2) *has a nonstandard least energy solution for sufficiently large β.*

The proof relies on a variational approach based on the well-known Mountain-Pass Theorem. It can be viewed as adaptation of an an approach which is now classical for CNLS. The compactness is recovered by imposing that β is sufficiently large so that the mountain-pass min-max value c satisfies a suitable inequality involving the best constant S in the Sobolev embedding $H^1(\mathbb{R}^4) \hookrightarrow L^4(\mathbb{R}^4)$. This ensures that the Palais-Smale condition holds at the level c. To prove that the least energy is nonstandard, we use the semitrivial solutions $(U_1, 0)$ and $(0, U_2)$ as comparison functions, where U_i's are the positive least energy solutions of the equation

$$-\Delta U_i + \lambda_i U_i = \mu_i U_i^3 \quad \text{in} \quad \Omega.$$

In the sequel we use the following notation.

$$\|u\|_{\lambda_1}^2 := \int_\Omega |\nabla u|^2 + \lambda_1 u^2, \quad \|v\|_{\lambda_2}^2 := \int_\Omega |\nabla v|^2 + \lambda_2 v^2, \quad |u|_q^q := \int_\Omega |u|^q.$$

2. Dirichlet problem

Let $X = H_0^1(\Omega) \times H_0^1(\Omega)$ and

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} E(\gamma(t))$$

where $\Gamma = \{\gamma \in C([0,1], X) | \gamma(0) = 0, E(\gamma(1)) < 0\}$. Then (e.g. [18]) $c > 0$ and

$$c = \inf_{(u,v)\in X,(u,v)\neq(0,0)} \max_{t>0} E(tu, tv) = \inf_{(u,v)\in\mathcal{N}} E(u, v) \tag{2.1}$$

where

$$\mathcal{N} = \{(u, v) \in X \setminus \{(0, 0)\} | \|u\|_{\lambda_1}^2 + \|v\|_{\lambda_2}^2 = \mu_1 |u|_4^4 + \mu_2 |v|_4^4 + 2\beta |uv|_p^p\}.$$

Existence of nontrivial solution. Using the mountain pass theorem, we first prove the existence of nontrivial solution. In this section, we always assume that

$$-\lambda_1(\Omega) < \lambda_1 < 0, \quad -\lambda_1(\Omega) < \lambda_2 < 0 \tag{2.2}$$

where $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$.

Theorem 2.1. *Assume that condition* (2.2) *holds, there exists a nontrivial least energy solution for problem* (1.1) *for sufficiently large β.*

Proof. By the mountain pass lemma (e.g. [18]), there exists a minimizing sequence $(u_n, v_n) \in X$ such that as $n \to \infty$

$$E(u_n, v_n) \to c, \quad E'(u_n, v_n) \to 0 \quad \text{in} \quad X'. \tag{2.3}$$

We assume that (u_n, v_n) is nonnegative; otherwise we consider $(|u_n|, |v_n|)$. It is routine to prove that $\|u_n\|_{\lambda_1}^2 + \|v_n\|_{\lambda_2}^2$ is bounded and

$$\begin{gathered}\frac{1}{2}(1-\frac{1}{p})(\|u_n\|_{\lambda_1}^2 + \|v_n\|_{\lambda_2}^2) + \frac{1}{2}(\frac{1}{p}-\frac{1}{2})(\mu_1|u_n|_4^4 + \mu_2|v_n|_4^4) = c + o(1), \\ \frac{1}{4}(\mu_1|u_n|_4^4 + \mu_2|v_n|_4^4) + (1-\frac{1}{p})\beta|u_nv_n|_p^p = c + o(1).\end{gathered} \tag{2.4}$$

Going to a subsequence, if necessary, there exists $(u, v) \in X$ such that

$$\begin{gathered}u_j \rightharpoonup u, \quad v_j \rightharpoonup v, \quad \text{in } H_0^1(\Omega), \\ u_j \to u, \quad v_j \to v, \quad \text{in } L^2(\Omega), \\ u_j \to u, \quad v_j \to v, \quad \text{a.e. in } \Omega, \\ u_j^3 \rightharpoonup u^3, \quad v_j^3 \rightharpoonup v^3, \quad \text{in } L^{4/3}(\Omega) \\ u_j^{p-1}v_j^p \rightharpoonup u^{p-1}v^p, \quad u_j^pv_j^{p-1} \rightharpoonup u^pv^{p-1}, \quad \text{in } L^{2/p}(\Omega).\end{gathered} \tag{2.5}$$

It is easy to see that (u, v) is a nonnegative solution of equations (1.1) and has nonnegative energy; i.e.,

$$E'(u, v) = 0, \quad E(u, v) \geq 0. \tag{2.6}$$

Set $\sigma_n = u_n - u$, $\tau_n = v_n - v$, $\gamma_n = u_nv_n - uv$. By Brézis-Lieb theorem and (2.5),

$$\begin{gathered}|u_n|_4^4 = |u|_4^4 + |\sigma_n|_4^4 + o(1), \\ |v_n|_4^4 = |v|_4^4 + |\tau_n|_4^4 + o(1), \\ |u_nv_n|_p^p = |uv|_p^p + o(1).\end{gathered} \tag{2.7}$$

By a direct computation and (2.3), (2.7),

$$\begin{aligned}E(u_n, v_n) &= E(u, v) + \frac{1}{2}(|\nabla\sigma_n|_2^2 + |\nabla\tau_n|_2^2) - \frac{1}{4}(\mu_1|\sigma_n|_4^4 + \mu_2|\tau_n|_4^4) + o(1) \\ &= c + o(1),\end{aligned} \tag{2.8}$$

with

$$\begin{aligned}o(1) &= (E'(u_n, v_n), (u_n, v_n)) \\ &= (E'(u, v), (u, v)) + |\nabla\sigma_n|_2^2 + |\nabla\tau_n|_2^2 - (\mu_1|\sigma_n|_4^4 + \mu_2|\tau_n|_4^4) \\ &= |\nabla\sigma_n|_2^2 + |\nabla\tau_n|_2^2 - (\mu_1|\sigma_n|_4^4 + \mu_2|\tau_n|_4^4).\end{aligned} \tag{2.9}$$

Assuming that $|\nabla\sigma_n|_2^2 + |\nabla\tau_n|_2^2 \to b$, by (2.9),

$$\mu_1|\sigma_n|_4^4 + \mu_2|\tau_n|_4^4 \to b. \tag{2.10}$$

If $b = 0$, the proof is done. Now we assume that $b > 0$. By the Sobolev imbedding theorem,

$$|\nabla\sigma_n|_2^2 > S|\sigma_n|_4^2, \quad |\nabla\tau_n|_2^2 > S|\tau_n|_4^2.$$

Hence

$$\begin{aligned}(|\nabla\sigma_n|_2^2 + |\nabla\tau_n|_2^2)^2 &\geq S^2(|\sigma_n|_4^2 + |\tau_n|_4^2)^2 \\ &\geq \frac{S^2}{\max\{\mu_1, \mu_2\}}(\mu_1|\sigma_n|_4^4 + \mu_2|\tau_n|_4^4)\end{aligned} \tag{2.11}$$

Let $n \to \infty$ on both side of (2.11), we have

$$b^2 > \frac{S^2}{\max\{\mu_1, \mu_2\}}b, \quad \text{i.e. } b > \frac{S^2}{\max\{\mu_1, \mu_2\}}. \tag{2.12}$$

From (2.6), (2.8) and (2.10), we have

$$b \le 4c. \tag{2.13}$$

Therefore,

$$4c > \frac{S^2}{\max\{\mu_1, \mu_2\}}. \tag{2.14}$$

When β is sufficiently large, this is a contradiction with the following lemma. This completes the proof. □

Lemma 2.2. *As $\beta \to \infty$, $c \to 0$.*

Proof. Fix a nontrivial $W \in H_0^1(\Omega)$. There exists $t_0 > 0$ such that $(t_0 W, t_0 W) \in \mathcal{N}$. Indeed,

$$t_0 = \Big(\frac{\|W\|_{\lambda_1}^2 + \|W\|_{\lambda_2}^2}{t_0^{4-2p}(\mu_1+\mu_2)|W|_4^4 + 2\beta|W|_{2p}^{2p}}\Big)^{\frac{1}{2(p-1)}} \le O\Big(\frac{1}{\beta^{1/2(p-1)}}\Big) \tag{2.15}$$

as $\beta \to \infty$. Hence

$$\begin{aligned} c &\le E(t_0W, t_0W) \\ &= \frac{1}{2}\Big(1-\frac{1}{p}\Big)t_0^2(\|W\|_{\lambda_1}^2 + \|W\|_{\lambda_2}^2) + \frac{1}{2}\Big(\frac{1}{p}-\frac{1}{2}\Big)t_0^4(\mu_1+\mu_2)|W|_4^4 \\ &\le O\Big(\frac{1}{\beta^{1/(p-1)}}\Big). \end{aligned} \tag{2.16}$$

□

Nontrivial solution is nonstandard. In this subsection, we will show that the nontrivial least energy solution in subsection 2.1 is nonstandard.

Theorem 2.3. *The solution obtained in Theorem 2.1 is nonstandard.*

Proof. From Brézis-Nirenberg's theorem([2], see also [18, Theorem 1.45]), there exists nontrivial solution $W_i \in H_0^1(\Omega)$ for

$$-\Delta W + \lambda_i W = \mu_i W^3, \quad i = 1, 2. \tag{2.17}$$

In fact, the W_i's are mountain pass solutions and hence they are least energy solutions with respective energies

$$I_i = \frac{1}{4}\|W_i\|_{\lambda_i}^2 = \frac{\mu_i}{4}|W_i|_4^4, \quad i = 1, 2. \tag{2.18}$$

From the proof of Lemma 2.2, (2.15) and (2.16), for sufficiently large β, we have

$$c < \min\{I_1, I_2\}. \tag{2.19}$$

This implies that, for sufficiently large β, any nontrivial solution with the least energy must be nonstandard. □

Proof of Theorem 1.1. By the maximum principle, any nonstandard nonnegative solution of equations (1.1) is positive. Combining this with Theorem 2.1 and Theorem 2.3, we complete the proof. □

3. Neumann problem

In this section, we assume that λ_1, λ_2 are sufficiently large as in [16], but not independent of β. Using the same procedure as in section 2, we come to prove existence of nonstandard solution of problem (1.2).

In this section, except we set $X = H^1(\Omega) \times H^1(\Omega)$ and let W_i be the positive least energy solution of problem (3.1), we use the same notations and definitions $c, \Gamma, \mathcal{N}$ as in section 2.

Proof of Theorem 1.2. We follow the same procedure as in section 2 and we only give a sketch of the proof.
Claim 1. The least energy $c \to 0$ as $\beta \to \infty$. The proof is the same as Lemma 2.2.
Claim 2. Any least energy solution is nonstandard for sufficiently large β. From [16, Theorem 3.1], for $i = 1, 2$, problem

$$-\Delta W + \lambda_i W = \mu_i W^3 \quad \text{in } \Omega, \quad \frac{\partial W}{\partial \nu} = 0 \quad \text{on } \partial\Omega \tag{3.1}$$

possesses a positive solution W_i for λ_i suitably large. In fact, the nonconstant solution in [16] is a mountain pass and hence a least energy solution. Assume $I_i, i = 1, 2$ are their corresponding least energies. By the proof of Theorem 2.3, we have $c < \min\{I_1, I_2\}$. So nontrivial least energy solutions are nonstandard.
Claim 3. Existence of nontrivial solution. Assume that $\{(u_j, v_j\}_{j=1}^{\infty}$ is a nonnegative minimizing sequence for the mountain pass energy c, i.e.

$$E(u_j, v_j) \to c, \quad E'(u_j, v_j) \to 0 \quad \text{in } X'. \tag{3.2}$$

The same procedure as in section 2 (2.4) implies that as $j \to \infty$

$$\frac{1}{2}(1 - \frac{1}{p})(\|u_j\|_{\lambda_1}^2 + \|v_j\|_{\lambda_2}^2) + \frac{1}{2}(\frac{1}{p} - \frac{1}{2})(\mu_1 |u_j|_4^4 + \mu_2 |v_j|_4^4) = c + o(1), \tag{3.3}$$

Hence $\{\|u_j\|_{\lambda_1}\}$ and $\{\|v_j\|_{\lambda_2}\}$ are bounded sequences. Going if necessary to a subsequence, there exists $(u, v) \in X$ such that

$$\begin{gathered}
u_j \rightharpoonup u, \quad v_j \rightharpoonup v, \quad \text{in } H^1(\Omega), \\
u_j \to u, \quad v_j \to v, \quad \text{in } L^2(\Omega), \\
u_j \to u, \quad v_j \to v, \quad \text{a.e. in } \Omega, \\
u_j^3 \rightharpoonup u^3, \quad v_j^3 \rightharpoonup v^3, \quad \text{in } L^{4/3(\Omega)} \\
u_j^{p-1} v_j^p \rightharpoonup u^{p-1} v^p, \quad u_j^p v_j^{p-1} \rightharpoonup u^p v^{p-1}, \quad \text{in } L^{2/p}(\Omega).
\end{gathered}$$

Hence (u, v) is nonnegative and satisfies the equations in (1.2).
Claim: $(u, v) \neq (0, 0)$. Otherwise, $(u_j, v_j) \rightharpoonup (0, 0)$ in $H^1(\Omega) \times H^1(\Omega)$, $(u_j, v_j) \to (0, 0)$ in $L^2(\Omega) \times L^2(\Omega)$, and $u_j v_j \to 0$ in $L^p(\Omega)$. From [16, Lemma 2.1, page 289], for any $\varepsilon > 0$, as $j \to \infty$

$$S_\varepsilon |u_j|_4^2 \leq |\nabla u_j|_2^2 + o(1), \tag{3.4}$$

$$S_\varepsilon |v_j|_4^2 \leq |\nabla v_j|_2^2 + o(1) \tag{3.5}$$

where $S_\varepsilon = (2^{-1/2} S - \varepsilon)(1 + \varepsilon)^{-1}$, S is the Sobolev constant.

Assume that $|\nabla u_j|_2^2 + |\nabla v_j|_2^2 \to b$. Since $(E'(u_j, v_j), (u_j, v_j)) \to 0$ and $|u_j v_j|_p^p \to 0$,

$$|\nabla u_j|_2^2 + |\nabla v_j|_2^2 = \mu_1 |u_j|_4^4 + \mu_2 |v_j|_4^4 + o(1). \tag{3.6}$$

It follows that

$$E(u_j, v_j) = \frac{1}{4}(|\nabla u_j|_2^2 + |\nabla v_j|_2^2) + o(1) = \frac{1}{4}(\mu_1 |u_j|_4^4 + \mu_2 |v_j|_4^4) + o(1) \to c > 0. \quad (3.7)$$

If $b = 0$, this is a contradiction with (3.7). If $b > 0$, by (3.4),(3.5), we have

$$(|\nabla u_j|_2^2 + |\nabla v_j|_2^2)^2 \geq S_\varepsilon^2(|u_j|_4^2 + |v_j|_4^2)^2 + o(1) \geq C_\varepsilon(\mu_1 |u_j|_4^4 + \mu_2 |v_j|_4^4) + o(1) \quad (3.8)$$

where $C_\varepsilon = \frac{S_\varepsilon^2}{\max\{\mu_1, \mu_2\}}$. From (3.6),

$$b^2 \geq C_\varepsilon b, \quad \text{i.e., } b \geq C_\varepsilon. \quad (3.9)$$

This is a contradiction with (3.7) and Claim 1. Hence (u, v) is nontrivial. The same procedure as in [6, page 9] (also [16]) implies that the solution is positive. This completes the proof. □

Remarks. (1) The arguments developed in this paper also work in dimension $N > 4$ with critical nonlinearities. For simplification in writing, we consider only the case $N = 4$.

(2) According to the proof developed in the paper we cannot exclude the possibility that the solutions of Neumann problem are nonconstant for some suitably chosen large β.

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