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EXISTENCE OF SOLUTIONS FOR A FOURTH-ORDER BOUNDARY-VALUE PROBLEM

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Abstract. In this paper, we use the lower and upper solution method to obtain an existence theorem for the fourth-order boundary-value problem

$$u^{(4)}(t)=f(t,u(t),u'(t),u''(t),u'''(t)),\quad 0<t<1,$$
$$u(0)=u'(1)=u''(0)=0,\quad u'''(1)=g\Big(\int_0^1 u''(t)d\theta(t)\Big),$$

where $f:[0,1]\times\mathbb{R}^4\to\mathbb{R}$, $g:\mathbb{R}\to\mathbb{R}$ are continuous and may be nonlinear, and $\int_0^1 u''(t)d\theta(t)$ denotes the Riemann-Stieltjes integral.

1. Introduction

It is well known that the bending of an elastic beam can be described with fourth-order boundary-value problems. Recently, many authors have investigated the existence of solutions for fourth-order boundary-value problems subject to a variety of boundary-value conditions, see for example [1, 3, 4, 6, 7].

Very recently, Bai [2] used the lower and upper solution method to obtain the existence of solutions for the problem

$$\begin{gathered}u^{(4)}(t)=f(t,u(t),u'(t),u''(t),u'''(t)),\quad 0<t<1,\\ u(0)=u'(1)=u''(0)=u'''(1)=0,\end{gathered}$$

where $f:[0,1]\times\mathbb{R}^4\to\mathbb{R}$ is increasing.

Motivated by the above-mentioned papers and the main ideas in [5], in this paper, we use the lower and upper solution method to establish the existence of solutions for the fourth-order boundary-value problem

$$\begin{gathered}u^{(4)}(t)=f(t,u(t),u'(t),u''(t),u'''(t)),\quad 0<t<1,\\ u(0)=u'(1)=u''(0)=0,\quad u'''(1)=g\Big(\int_0^1 u''(t)d\theta(t)\Big),\end{gathered}\tag{1.1}$$

where $f:[0,1]\times\mathbb{R}^4\to\mathbb{R}$, $g:\mathbb{R}\to\mathbb{R}$ are continuous and may be nonlinear. $\theta:[0,1]\to\mathbb{R}$ is increasing nonconstant function defined on [0,1] and $\theta(0)=0$.

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If $g(x) = x$, $x \in \mathbb{R}$, then the problem (1.1) educes a three-point boundary-value problem by applying the following property of the Riemann-Stieltjes integral.

Lemma 1.1. *Assume that*

(1) *$u(t)$ is a bounded function value on $C^2[a,b]$, i.e., there exist c, $C \in \mathbb{R}$ such that $c \le u''(t) \le C$, $\forall t \in [a,b]$;*
(2) *$\theta(t)$ is increasing on $[a,b]$;*
(3) *the Riemann-Stieltjes integral $\int_a^b u''(t)d\theta(t)$ exists.*

Then there is a number $v \in \mathbb{R}$ with $c \le v \le C$ such that

$$\int_a^b u''(t)d\theta(t) = v(\theta(b) - \theta(a)).$$

For any continuous solution $u(t)$ of (1.1), by Lemma 1.1, there exists $\eta \in (0,1)$ such that

$$\int_a^b u''(t)d\theta(t) = u''(\eta)(\theta(1) - \theta(0)) = u''(\eta)\theta(1).$$

Let $\sigma = \theta(1)$ and $g(x) = x$, $x \in \mathbb{R}$. Then problem (1.1) can be rewritten as the three-point boundary-value problem

$$\begin{gathered} u^{(4)}(t) = f(t, u(t), u'(t), u''(t), u'''(t)), \quad 0 < t < 1, \\ u(0) = u'(1) = u''(0) = 0, \quad u'''(1) = \sigma u''(\eta), \end{gathered} \tag{1.2}$$

The paper is organized as follows. In the next section, we present some preliminaries and lemmas. Section 3 is devote to our main results.

2. Preliminaries

Definition 2.1. Let $\alpha \in C^3[0,1] \cap C^4(0,1)$. We say α is a lower solution of (1.1) if

$$\begin{gathered} \alpha^{(4)}(t) \le f(t, \alpha(t), \alpha'(t), \alpha''(t), \alpha'''(t)), \quad 0 < t < 1, \\ \alpha(0) \le 0, \quad \alpha'(1) \le 0, \\ \alpha''(0) \ge 0, \quad \alpha'''(1) \ge g(\int_0^1 \alpha''(t)d\theta(t)). \end{gathered}$$

Similarly, $\beta \in C^3[0,1] \cap C^4(0,1)$ is an upper solution of (1.1), if β satisfies similar inequalities in the reverse order.

If we denote by $k(t,s)$ the Green's function of

$$\begin{gathered} -u''(t) = 0, \quad 0 < t < 1, \\ u(0) = u'(1) = 0, \end{gathered} \tag{2.1}$$

then

$$k(t,s) = \begin{cases} t, & 0 \le t \le s \le 1, \\ s, & 0 \le s \le t \le 1. \end{cases}$$

Setting $-u'' = v$, by standard calculation, we get

$$u(t) = \int_0^1 k(t,s)v(s)ds =: (Av)(t),$$

$$u'(t) = \int_t^1 v(s)ds =: (Bv)(t).$$

Obviously, A, B are monotone increasing operators.

We assume that g is an odd function on $\mathbb{R}$. Then problem (1.1) is equivalent to the following integral-differential boundary-value problem

$$\begin{gathered} -v''(t) = f(t, (Av)(t), (Bv)(t), -v(t), -v'(t)), \quad 0 < t < 1, \\ v(0) = 0, \quad v'(1) = g(\int_0^1 v(t) d\theta(t)). \end{gathered} \tag{2.2}$$

For $v \in C[0, 1]$, we define the operator $\hat{f}$ by

$$\hat{f}(v(t), v'(t)) = f(t, (Av)(t), (Bv)(t), -v(t), -v'(t)).$$

Then (2.2) is equivalent to

$$\begin{gathered} -v''(t) = \hat{f}(v(t), v'(t)), \quad 0 < t < 1, \\ v(0) = 0, \quad v'(1) = g(\int_0^1 v(t) d\theta(t)). \end{gathered} \tag{2.3}$$

Suppose α, β are the lower and upper solutions of BVP (1.1) such that $\alpha'' \geq \beta''$ and let $\psi = -\beta'', \phi = -\alpha''$. Then we have

$$\begin{gathered} -\phi''(t) \leq \hat{f}(\phi(t), \phi'(t)), \quad \phi(0) \leq 0, \quad \phi'(1) \leq g(\int_0^1 \phi(t) d\theta(t)), \\ -\psi''(t) \geq \hat{f}(\psi(t), \psi'(t)), \quad \psi(0) \geq 0, \quad \psi'(1) \geq g(\int_0^1 \psi(t) d\theta(t)). \end{gathered}$$

Since A, B are monotone continuous operators, there exists M such that

$$M = \sup_{\phi \leq v \leq \psi} \{\|Av\|_\infty, \|Bv\|_\infty\} > 0.$$

Definition 2.2 ([2])**.** Let $f \in C([0, 1] \times \mathbb{R}^4, \mathbb{R})$, $\phi, \psi \in C([0, 1], \mathbb{R})$ and $\phi(t) \leq \psi(t), t \in [0, 1]$. We say that $f(t, x_1, x_2, x_3, x_4)$ satisfies a Nagumo-type condition with respect to ϕ, ψ if there exists a positive continuous function $h(s)$ on $[0, \infty)$ satisfying

$$|f(t, x_1, x_2, x_3, x_4)| \leq h(|x_4|), \tag{2.4}$$

for all $(t, x_1, x_2, x_3, x_4) \in [0, 1] \times [-M, M]^2 \times [\phi(t), \psi(t)] \times \mathbb{R}$, and

$$\int_\lambda^\infty \frac{s}{h(s)} ds > \max_{0 \leq t \leq 1} \psi(t) - \min_{0 \leq t \leq 1} \phi(t), \tag{2.5}$$

where $\lambda = \max\{|\psi(1) - \phi(0)|, |\psi(0) - \phi(1)|\}$.

Lemma 2.3. *Suppose f satisfies the Nagumo-type condition with respect to $\phi, \psi \in C^2[0, 1]$ and $\phi \leq \psi$. If BVP (2.3) has a solution $v(t)$ such that $\phi(t) \leq v(t) \leq \psi(t)$, then there exists $N > 0$ such that $|v'(t)| \leq N$, for $t \in [0, 1]$.*

The proof of the above lemma is similar to that in [2], therefore, we omit it.

3. Main results

Theorem 3.1. *Suppose α, β are lower and upper solutions to BVP (1.1) such that $\alpha''(t) \geq \beta''(t)$ and f satisfies a Nagumo-type condition with respect to α'', β''. In addition, we assume that g is odd, continuous and increasing on $\mathbb{R}$, θ is increasing on $[0, 1]$ and $\theta(0) = 0$. Then BVP (1.1) has a solution $u(t)$ such that*

$$\alpha(t) \leq u(t) \leq \beta(t), \quad \alpha''(t) \geq u''(t) \geq \beta''(t).$$

Proof. Since f satisfies the Nagumo-type condition with respect to $\phi = -\alpha'', \psi = -\beta''$, there exists a constant $N > 0$ depending on ϕ, ψ, h such that

$$\int_{\lambda}^{N} \frac{s}{h(s)} ds > \max_{0\le t\le 1} \psi(t) - \min_{0\le t\le 1} \phi(t). \tag{3.1}$$

Take $C > \max\{N, \|\phi'\|, \|\psi'\|\}$ and $p(v') = \max\{-C, \min\{v', C\}\}$. By modifying $\hat{f}$ and g with respect to ϕ, ψ, we aim at obtaining a second-order boundary-value problem and reformulating the new problem as an integral equation. We show that solutions of the modified problem lie in the region where $\hat{f}, g$ are unmodified and hence are solutions of problem (2.3). Let $\varepsilon > 0$ be a fixed small number and $F(v(t), v'(t)), G(\int_0^1 v(t)d\theta(t))$ are the modifications of $\hat{f}(v(t), v'(t))$ and $g(\int_0^1 v(t)d\theta(t))$ as follows

$$\begin{aligned}
&F(v(t), v'(t))\\
&= \begin{cases}
\hat{f}(\psi(t), \psi'(t)) + \frac{v(t)-\psi(t)}{1+|v(t)-\psi(t)|}, & \text{if } v(t) \ge \psi(t) + \varepsilon,\\
\hat{f}(\psi(t), p(v')) + [\hat{f}(\psi(t), \psi'(t)) - \hat{f}(\psi(t), p(v'(t))) & \\
+\frac{v(t)-\psi(t)}{1+|v(t)-\psi(t)|}] \times \frac{v(t)-\psi(t)}{\varepsilon}, & \text{if } \psi(t) \le v(t) < \psi(t) + \varepsilon,\\
\hat{f}(v(t), p(v'(t))), & \text{if } \phi(t) \le v(t) \le \psi(t),\\
\hat{f}(\phi(t), p(v'(t))) + [\hat{f}(\phi(t), \phi'(t)) - \hat{f}(\phi(t), p(v'(t))) & \\
+\frac{\phi(t)-v(t)}{1+|\phi(t)-v(t)|}] \times \frac{\phi(t)-v(t)}{\varepsilon}, & \text{if } \phi(t) - \varepsilon < v(t) \le \phi(t),\\
\hat{f}(\phi(t), \phi'(t)) + \frac{\phi(t)-v(t)}{1+|\phi(t)-v(t)|}, & \text{if } v(t) \le \phi(t) - \varepsilon,
\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&G\big(\int_0^1 v(t)d\theta(t)\big)\\
&= \begin{cases}
g(\int_0^1 \psi(t)d\theta(t)) + \frac{\int_0^1 v(t)d\theta(t) - \int_0^1 \psi(t)d\theta(t)}{1+|\int_0^1 v(t)d\theta(t) - \int_0^1 \psi(t)d\theta(t)|}, & \text{if } v(t) > \psi(t),\\
g(\int_0^1 v(t)d\theta(t)), & \text{if } \phi(t) \le v(t) \le \psi(t),\\
g(\int_0^1 \phi(t)d\theta(t)) + \frac{\int_0^1 \phi(t)d\theta(t) - \int_0^1 v(t)d\theta(t)}{1+|\int_0^1 \phi(t)d\theta(t) - \int_0^1 v(t)d\theta(t)|}, & \text{if } v(t) < \phi(t).
\end{cases}
\end{aligned}$$

Obviously, $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R} \to \mathbb{R}$ are continuous and bounded. Consider the modified problem

$$\begin{gathered}
-v''(t) = F(v(t), v'(t)), \quad 0 < t < 1,\\
v(0) = 0, \quad v'(1) = G(\int_0^1 v(t)d\theta(t)).
\end{gathered} \tag{3.2}$$

Then, the BVP (3.2) is equivalent to the integral equation

$$v(t) = G(\int_0^1 v(t)d\theta(t))t + \int_0^1 k(t, s)F(v(s), v'(s))ds. \tag{3.3}$$

Since F and G are continuous and bounded, there exist $M > C$, $m > 0$ such that

$$\begin{gathered}
|F(v(t), v'(t))| < M \quad \text{on} \quad \mathbb{R} \times \mathbb{R},\\
|G(\int_0^1 v(t)d\theta(t))| < m \quad \text{on} \quad \mathbb{R}.
\end{gathered}$$

Choose $\tilde{M} \geq \delta + m + M$ and consider the open bounded and convex set

$$\Omega = \{v \in C^1[0,1] : \|v\| < \tilde{M}, \|v'\| < \tilde{M}\}.$$

Define $\tilde{F} : C^1[0,1] \times \mathbb{R} \to C[0,1]$ and $\tilde{G} : C^1[0,1] \to C[0,1]$ by

$$\tilde{F}(v)(t) = \int_0^1 k(t,s)F(v(s), v'(s))ds,$$
$$\tilde{G}(v)(t) = G(\int_0^1 v(t)d\theta(t))t.$$

It is obvious that $\tilde{F}, \tilde{G}$ are compact. Let $\tilde{T} = \tilde{G} + \tilde{F}$, it is easy to see that (3.2) is equivalent to the fixed point equation

$$\tilde{T}v = v. \tag{3.4}$$

Then, it follows from Schauder fixed point theorem that the integral equation (3.4) has a fixed point v_*. In other words, the BVP (3.2) has a solution v_*. Also, from the definitions of ϕ, ψ, F and G and the choice of C, we have

$$-\phi''(t) \leq \hat{f}(\phi(t), \phi'(t)) = F(\phi(t), \phi'(t)), \quad 0 \leq t \leq 1,$$
$$\phi(0) \leq 0, \quad \phi'(1) \leq g(\int_0^1 \phi(t)d\theta(t)) = G(\int_0^1 \phi(t)d\theta(t))$$

and

$$-\psi''(t) \geq \hat{f}(\psi(t), \psi'(t)) = F(\psi(t), \psi'(t)), \quad 0 \leq t \leq 1,$$
$$\psi(0) \geq 0, \quad \psi'(1) \geq g(\int_0^1 \psi(t)d\theta(t)) = G(\int_0^1 \psi(t)d\theta(t)).$$

That is, ϕ and ψ are the lower and upper solutions of (3.2).

We claim that the solution v_* of (3.2) satisfies $\phi(t) \leq v_*(t) \leq \psi(t)$ for $t \in [0,1]$. We only prove $\phi(t) \leq v_*(t)$, $t \in [0,1]$, the other part is proved in a similar way. Let $w(t) = \phi(t) - v_*(t)$ for $t \in [0,1]$. Assume that $w(t_0) = \max_{0 \leq t \leq 1} w(t) > 0$. We divide the proof into three cases.

Case 1. $t_0 = 0$. Then we have $w(0) = \phi(0) - v_*(0) = \phi(0) > 0$. It contradict the definition of ϕ.

Case 2. $t_0 = 1$. Then $w(1) > 0$ and $w'(1) \geq 0$. The boundary value conditions of (3.2) imply

$$w'(1) = \phi'(1) - {v_*}'(1) \leq g(\int_0^1 \phi(t)d\theta(t)) - G(\int_0^1 v_*(t)d\theta(t)).$$

If $v_*(t) < \phi(t)$, then

$$\begin{aligned} G(\int_0^1 v_*(t)d\theta(t)) &= g(\int_0^1 \phi(t)d\theta(t)) + \frac{\int_0^1 \phi(t)d\theta(t) - \int_0^1 v_*(t)d\theta(t)}{1 + \int_0^1 \phi(t)d\theta(t) - \int_0^1 v_*(t)d\theta(t)} \\ &> g(\int_0^1 \phi(t)d\theta(t)), \end{aligned}$$

which implies $w'(1) < 0$. It is a contradiction. If $v_*(t) > \psi(t)$, then

$$\begin{aligned} G(\int_0^1 v_*(t)d\theta(t)) &= g(\int_0^1 \psi(t)d\theta(t)) + \frac{\int_0^1 v_*(t)d\theta(t) - \int_0^1 \psi(t)d\theta(t)}{1 + \int_0^1 v_*(t)d\theta(t) - \int_0^1 \psi(t)d\theta(t)} \\ &> g(\int_0^1 \psi(t)d\theta(t)) \\ &\geq g(\int_0^1 \phi(t)d\theta(t)), \end{aligned}$$

we can also get $w'(1) < 0$, which is a contradiction. Hence, $\phi(t) \leq v_*(t) \leq \psi(t)$. So

$$G(\int_0^1 v_*(t)d\theta(t)) = g(\int_0^1 v_*(t)d\theta(t)) \geq g(\int_0^1 \phi(t)d\theta(t)),$$

which implies $w'(1) \leq 0$. If $w'(1) < 0$, it is a contradiction. So we have $w'(1) = 0$. Since $t_0 \neq 0$, there exists $t_1 \in [0, 1)$ such that $w(t_1) = 0$ and $w(t) > 0$ on $(t_1, 1]$. Then for each $t \in [t_1, 1]$, we have

$$w''(t) = \phi''(t) - {v_*}''(t) \geq -\hat{f}(\phi(t), \phi'(t)) + \Big[\hat{f}(\phi(t), \phi'(t)) + \frac{w(t)}{1 + w(t)}\Big] > 0.$$

Thus, by $w'(1) = 0$, we get $w'(t) \leq 0 \quad \text{on} \quad [t_1, 1]$, which implies that w is decreasing on $[t_1, 1]$ and hence $w(1) \leq 0$, it is a contradiction.

Case 3. $t_0 \in (0, 1)$. Then, we have $w'(t_0) = 0$ and $w''(t_0) \leq 0$. However, for $0 < w(t_0) < \varepsilon$, we have

$$\begin{aligned} w''(t_0) &= \phi''(t_0) - {v_*}''(t_0) \\ &\geq -\hat{f}(\phi(t_0), \phi'(t_0)) + F(v_*(t_0), {v_*}'(t_0)) \\ &= \frac{w^2(t_0)}{(1 + w(t_0))\varepsilon} > 0, \end{aligned}$$

a contradiction. For $w(t_0) \geq \varepsilon$, we obtain

$$w''(t_0) = \phi''(t_0) - {v_*}''(t_0) \geq \frac{w(t_0)}{1 + w(t_0)} > 0,$$

it is also a contradiction. Thus, $\phi(t) \leq v_*(t), t \in [0, 1]$. By the similar discussion, we can get $v_*(t) \leq \psi(t)$.

According to the Lemma 2.3 and the choice of C, for the solution v_* of (3.2) with $\phi(t) \leq v_*(t) \leq \psi(t)$, $t \in [0, 1]$, we have

$$|{v_*}'(t)| \leq N < C.$$

Thus,

$$\begin{gathered} F(v_*(t), {v_*}'(t)) = \hat{f}(v_*(t), {v_*}'(t)), \\ G(\int_0^1 v_*(t)d\theta(t)) = g(\int_0^1 v_*(t)d\theta(t)). \end{gathered}$$

Hence, the solution v_* of (3.2) with $\phi(t) \leq v_*(t) \leq \psi(t)$, $t \in [0, 1]$, is a solution of (2.3). The proof is complete. □

Using the Theorem 3.1, we can prove the following result.

Corollary 3.2. *Suppose α, β are lower and upper solutions to* (1.2) *such that $\alpha''(t) \geq \beta''(t)$ and f satisfies a Nagumo-type condition with respect to α'', β''. Then* (1.2) *has a solution $u(t)$ such that*

$$\alpha(t) \leq u(t) \leq \beta(t), \quad \alpha''(t) \geq u''(t) \geq \beta''(t).$$

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