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CONTINUOUS VERSION OF FILIPPOV'S THEOREM FOR A STURM-LIOUVILLE TYPE DIFFERENTIAL INCLUSION

AURELIAN CERNEA

Abstract. Using Bressan-Colombo results, concerning the existence of continuous selections of lower semicontinuous multifunctions with decomposable values, we prove a continuous version of Filippov's theorem for a Sturm-Liuoville differential inclusion. This result allows to obtain a continuous selection of the solution set of the problem considered.

1. Introduction

In this paper we study second-order differential inclusions of the form

$$(p(t)x'(t))' \in F(t,x(t)) \quad \text{a. e. in } [0,T]), \quad x(0)=x_0, \quad x'(0)=x_1, \tag{1.1}$$

where $F:[0,T]\times X \to \mathcal{P}(X)$ is a set-valued map, X is a separable Banach space, $x_0,x_1 \in X$ and $p:[0,T]\to(0,\infty)$ is continuous.

In some recent papers [6, 9] several existence results for problem (1.1) are obtained using fixed point techniques. In [5] it is shown that Filippov's ideas [8] can be suitably adapted in order to prove the existence of mild solutions to problem (1.1).

The aim of this paper is to prove the existence of solutions continuously depending on a parameter for the problem (1.1). Our result may be interpreted as a continuous variant of the celebrated Filippov's theorem [8] for problem (1.1). In addition, as usual at a Filippov existence type theorem, our result provides an estimate between the starting "quasi" solution and the solution of the differential inclusion. At the same time we obtain a continuous selection of the solution set of problem (1.1)

The key tool in the proof of our theorem is a result of Bressan and Colombo [2] concerning the existence of continuous selections of lower semicontinuous multifunctions with decomposable values. The proof follows the general ideas as in [1, 3, 4, 7, 10], where similar results are obtained for other classes of differential inclusions.

The paper is organized as follows: in Section 2 we present the notations, definitions and the preliminary results to be used in the sequel and in Section 3 we prove our results.

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2. Preliminaries

Let $T > 0$, $I := [0, T]$ and denote by $\mathcal{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I. Let X be a real separable Banach space with the norm $|\cdot|$. Denote by $\mathcal{P}(X)$ the family of all nonempty subsets of X and by $\mathcal{B}(X)$ the family of all Borel subsets of X. If $A \subset I$ then $\chi_A : I \to \{0, 1\}$ denotes the characteristic function of A. For any subset $A \subset X$ we denote by $\mathrm{cl}(A)$ the closure of A.

As usual, we denote by $C(I, X)$ the Banach space of all continuous functions $x : I \to X$ endowed with the norm $|x(.)|_C = \sup_{t\in I} |x(t)|$ and by $L^1(I, X)$ the Banach space of all (Bochner) integrable functions $x(.) : I \to X$ endowed with the norm $|x(.)|_1 = \int_0^T |x(t)|dt$.

We recall first several preliminary results we shall use in the sequel.

Lemma 2.1 ([11])**.** *Let $u : I \to X$ be measurable and let $G : I \to \mathcal{P}(X)$ be a measurable closed-valued multifunction. Then, for every measurable function $r : I \to (0, \infty)$, there exists a measurable selection $g : I \to X$ of $G(\cdot)$ (i.e. such that $g(t) \in G(t)$ a.e. (I)) such that*

$$|u(t) - g(t)| < d(u(t), G(t)) + r(t) \quad \text{a.e. in } (I),$$

where the distance between a point $x \in X$ and a subset $A \subset X$ is defined as usual by $d(x, A) = \inf\{|x - a| : a \in A\}$.

Definition 2.2. A subset $D \subset L^1(I, X)$ is said to be *decomposable* if for any $u(\cdot), v(\cdot) \in D$ and any subset $A \in \mathcal{L}(I)$ one has $u\chi_A + v\chi_B \in D$, where $B = I\backslash A$. We denote by $\mathcal{D}(I, X)$ the family of all decomposable closed subsets of $L^1(I, X)$.

Next (S, d) is a separable metric space; we recall that a multifunction $G(\cdot) : S \to \mathcal{P}(X)$ is said to be lower semicontinuous (l.s.c.) if for any closed subset $C \subset X$, the subset $\{s \in S; G(s) \subset C\}$ is closed.

Lemma 2.3 ([2])**.** *Let $F^* : I\times S \to \mathcal{P}(X)$ be a closed-valued $\mathcal{L}(I)\otimes\mathcal{B}(S)$-measurable multifunction such that $F^*(t, .)$ is l.s.c. for any $t \in I$. Then the multifunction $G : S \to \mathcal{D}(I, X)$ defined by*

$$G(s) = \{v \in L^1(I, X) : v(t) \in F^*(t, s) \text{ a.e. } (I)\}$$

is l.s.c. with nonempty closed values if and only if there exists a continuous mapping $q : S \to L^1(I, X)$ such that

$$d(0, F^*(t, s)) \le q(s)(t) \quad \text{a.e. in } (I),\ \forall s \in S.$$

Lemma 2.4 ([2])**.** *Let $G(.) : S \to \mathcal{D}(I, X)$ be a l.s.c. multifunction with closed decomposable values and let $\phi(.) : S \to L^1(I, X)$, $\psi : S \to L^1(I, \mathbb{R})$ be continuous such that the multifunction $H : S \to \mathcal{D}(I, X)$ defined by*

$$H(s) = \mathrm{cl}\{v \in G(s) : |v(t) - \phi(s)(t)| < \psi(s)(t) \text{ a. e. } (I)\}$$

has nonempty values. Then $H(.)$ has a continuous selection, i.e. there exists a continuous mapping $h : S \to L^1(I, X)$ such that

$$h(s) \in H(s) \quad \forall s \in S.$$

Consider $F : I \times X \to \mathcal{P}(X)$ a set-valued map, $x_0, x_1 \in X$ and $p : I \to (0, \infty)$ a continuous mapping that defined the Cauchy problem (1.1).

A continuous mapping $x \in C(I, X)$ is called a solution of problem (1.1) if there exists a (Bochner) integrable function $f \in L^1(I, X)$ such that:

$$f(t) \in F(t, x(t)) \quad \text{a.e. } (I), \tag{2.1}$$

$$x(t) = x_0 + p(0)x_1 \int_0^t \frac{1}{p(s)} ds + \int_0^t \frac{1}{p(s)} \int_0^s f(u)du\, ds \quad \forall t \in I. \tag{2.2}$$

Note that, if we denote $G(t) := \int_0^t \frac{1}{p(s)}$, $t \in I$, then (2.2) may be rewrite as

$$x(t) = x_0 + p(0)x_1 G(t) + \int_0^t G(t-u) f(u)du \quad \forall t \in I, \tag{2.3}$$

We shall call $(x(.), f(.))$ a *trajectory-selection pair* of (1.1) if (2.1) and (2.2) are satisfied.

We shall use the following notation for the solution sets of (1.1).

$$\mathcal{S}(x_0, x_1) = \{x : x \text{ is a solution of (1.1)}\}. \tag{2.4}$$

3. The main results

To establish our continuous version of Filippov theorem for problem (1.1) we need the following hypotheses.

Hypothesis 3.1. (i) $F : I \times X \to \mathcal{P}(X)$ has nonempty closed values and is $\mathcal{L}(I) \otimes \mathcal{B}(X)$ measurable.

(ii) There exists $L(.) \in L^1(I, \mathbb{R}_+)$ such that, for almost all $t \in I, F(t, .)$ is $L(t)$-Lipschitz in the sense that

$$d_H(F(t, x), F(t, y)) \le L(t)|x - y| \quad \forall x, y \in X,$$

where $d_H(., .)$ is the Hausdorff distance

$$d(A, B) = \max\{d^*(A, B), d^*(B, A)\}, \quad d^*(A, B) = \sup\{d(a, B); a \in A\}$$

Hypothesis 3.2. (i) *S is a separable metric space and $a, b : S \to X$, $c(.) : S \to (0, \infty)$ are continuous mappings.*

(ii) *There exists the continuous mappings $g(.), q(.) : S \to L^1(I, X)$, $y : S \to C(I, X)$ such that*

$$(p(t)(y(s))'(t))' = g(s)(t) \quad \forall s \in S, t \in I,$$
$$d(g(s)(t), F(t, y(s)(t)) \le q(s)(t) \quad a.e.\ (I),\ \forall\, s \in S.$$

Let $M := \sup_{t \in I} \frac{1}{p(t)}$. Note that $|G(t)| \le Mt$ for all $t \in I$. For the next result, we use the following notation: $m(t) = \int_0^t L(u)du$ and

$$\begin{aligned}\xi(s)(t) = e^{MTm(t)}\Big(&tMTc(s) + |a(s) - y(s)(0)| + MTp(0)|b(s) - (y(s))'(0)|\Big) \\ &+ MT \int_0^t q(s)(u) e^{MT(m(t)-m(u))} du.\end{aligned} \tag{3.1}$$

Theorem 3.3. *Assume that Hypotheses 3.1 and 3.2 are satisfied. Then there exist the continuous mappings $x : S \to C(I, X)$, $f : S \to L^1(I, X)$ such that for any $s \in S$, $(x(s)(.), f(s)(.))$ is a trajectory-selection pair of*

$$(p(t)x'(t))' \in F(t, x(t)), \quad x(0) = a(s), \quad x'(0) = b(s)$$

and

$$|x(s)(t)-y(s)(t)|\le \xi(s)(t)\quad \forall (t,s)\in I\times S, \tag{3.2}$$
$$|f(s)(t)-g(s)(t)|\le L(t)\xi(s)(t)+q(s)(t)+c(s)\quad \text{a.e. } (I),\ \forall s\in S. \tag{3.3}$$

Proof. We make the following notations $\varepsilon_n(s)=c(s)\frac{n+1}{n+2}$, $n=0,1,\dots$, $d(s)=|a(s)-y(s)(0)|+MTp(0)|b(s)-(y(s))'(0)|$,

$$\begin{aligned} q_n(s)(t) &= (MT)^n\int_0^t q(s)(u)\frac{(m(t)-m(u))^{n-1}}{(n-1)!}du\\ &\quad + (MT)^{(n-1)}\frac{(m(t))^{n-1}}{(n-1)!}(MTt\varepsilon_n(s)+d(s)),\quad n\ge 1. \end{aligned}$$

Set also $x_0(s)(t)=y(s)(t)$, $\forall s\in S$.

We consider the multifunctions $G_0(.), H_0(.)$ defined, respectively, by

$$\begin{gathered} G_0(s)=\{v\in L^1(I,X): v(t)\in F(t,y(s)(t))\quad a.e.\,(I)\},\\ H_0(s)=\operatorname{cl}\{v\in G_0(s): |v(t)-g(s)(t)|<q(s)(t)+\varepsilon_0(s)\}. \end{gathered}$$

Since $d(g(s)(t),F(t,y(s)(t))\le q(s)(t)<q(s)(t)+\varepsilon_0(s)$, according with Lemma 2.1, the set $H_0(s)$ is not empty.

Set $F_0^*(t,s)=F(t,y(s)(t))$ and note that

$$d(0,F_0^*(t,s))\le |g(s)(t)|+q(s)(t)=q^*(s)(t)$$

and $q^*(.):S\to L^1(I,X)$ is continuous.

Applying now Lemmas 2.3 and 2.4 we obtain the existence of a continuous selection f_0 of H_0, i.e. such that

$$\begin{gathered} f_0(s)(t)\in F(t,y(s)(t))\quad a.e.\,(I),\ \forall s\in S,\\ |f_0(s)(t)-g(s)(t)|\le q_0(s)(t)=q(s)(t)+\varepsilon_0(s)\quad \forall s\in S,\ t\in I. \end{gathered}$$

We define $x_1(s)(t)=a(s)+p(0)G(t)b(s)+\int_0^t G(t-u)f_0(s)(u)du$ and one has

$$\begin{aligned} &|x_1(s)(t)-x_0(s)(t)|\\ &\le |a(s)-y(s)(0)|+MTp(0)|b(s)-(y(s))'(0)|+MT\int_0^t |f_0(s)(u)-g(s)(u)|du\\ &\le d(s)+MT\int_0^t q_0(s)(u)(u)du+MTt\varepsilon_0(s)\le q_1(s)(t). \end{aligned}$$

We shall construct, using the same idea as in [7], two sequences of approximations $f_n:S\to L^1(I,X)$, $x_n:S\to C(I,X)$ with the following properties

(a) $f_n(.):S\to L^1(I,X)$, $x_n(.):S\to C(I,X)$ are continuous.
(b) $f_n(s)(t)\in F(t,x_n(s)(t))$, a.e. (I), $s\in S$.
(c) $|f_n(s)(t)-f_{n-1}(s)(t)|\le L(t)q_n(s)(t)$, a.e. (I), $s\in S$.
(d) $x_{n+1}(s)(t)=a(s)+p(0)G(t)b(s)+\int_0^t G(t-u)f_n(s)(u)du$, $\forall t\in I, s\in S$.

Suppose we have already constructed $f_i(.), x_i(.)$ satisfying (a)-(c) and define $x_{n+1}(.)$ as in (d). From (c) and (d) one has

$$\begin{aligned}
&|x_{n+1}(s)(t) - x_n(s)(t)| \\
&\le MT \int_0^t |f_n(s)(u) - f_{n-1}(s)(u)| du \\
&\le MT \int_0^t L(u) q_n(s)(u) du \\
&= (MT)^{n+1} \int_0^t q(s)(u) \frac{(m(t)-m(u))^n}{n)!} du + (MT)^n \frac{(m(t))^n}{n!} (MTt\varepsilon_n(s) + d(s)) \\
&< q_{n+1}(s)(t).
\end{aligned} \tag{3.4}$$

On the other hand,

$$d(f_n(s)(t), F(t, x_{n+1}(s)(t)) \le L(t)|x_{n+1}(s)(t) - x_n(s)(t)| < L(t) q_{n+1}(s)(t). \tag{3.5}$$

Consider the following multifunctions for any $s \in S$

$$\begin{gathered}
G_{n+1}(s) = \{v \in L^1(I, X) : v(t) \in F(t, x_{n+1}(s)(t)) \quad a.e.\,(I)\}, \\
H_{n+1}(s) = \operatorname{cl}\{v \in G_{n+1}(s) : |v(t) - f_n(s)(t)| < L(t) q_{n+1}(s)(t) \; a.e.\,(I)\}.
\end{gathered}$$

To prove that $H_{n+1}(s)$ is nonempty we note first that the real function $t \to r_n(s)(t) = c(s)\frac{(MT)^{n+1} t L(t)(m(t))^n}{(n+2)(n+3)n!}$ is measurable and strictly positive for any s. Using (3.5) we get

$$\begin{aligned}
d(f_n(s)(t), F(t, x_{n+1}(s)(t)) &\le L(t)|x_{n+1}(s)(t) - x_n(s)(t)| - r_n(s)(t) \\
&\le L(t) q_{n+1}(s)(t)
\end{aligned}$$

and therefore according to Lemma 2.1 there exists $v \in L^1(I, X)$ such that $v(t) \in F(t, x_n(s)(t))$ a.e. (I) and

$$|v(t) - f_n(s)(t)| < d(f_n(s)(t), F(t, x_n(s)(t)) + r_n(s)(t)$$

and hence $H_{n+1}(s)$ is not empty.

Set $F^*_{n+1}(t, s) = F(t, x_{n+1}(s)(t))$ and note that we may write

$$\begin{aligned}
d(0, F^*_{n+1}(t, s)) &\le L(t)|x_{n+1}(s)(t) - x_n(s)(t)| \\
&\le |f_n(s)(t)| + L(t) q_{n+1}(s)(t) \\
&= q^*_{n+1}(s)(t) \quad \text{a.e. } (I)
\end{aligned}$$

and $p^*_{n+1} : S \to L^1(I, X)$ is continuous.

By Lemmas 2.3 and 2.4 there exists a continuous map $f_{n+1} : S \to L^1(I, X)$ such that

$$\begin{gathered}
f_{n+1}(s)(t) \in F(t, x_{n+1}(s)(t)) \quad \text{a.e. } (I), \forall s \in S, \\
|f_{n+1}(s)(t) - f_n(s)(t)| \le L(t) q_{n+1}(s)(t) \quad a.e.\,(I), \forall s \in S.
\end{gathered}$$

From (3.4) and (d) we obtain

$$\begin{aligned}
|x_{n+1}(s)(.) - x_n(s)(.)|_C &\le MT|f_{n+1}(s)(.) - f_n(s)(.)|_1 \\
&\le \frac{(MTm(T))^n}{n!} (MT|q(s)(.)|_1 + MT^2 c(s) + d(s)).
\end{aligned} \tag{3.6}$$

Therefore $f_n(s)(.)$, $x_n(s)(.)$ are Cauchy sequences in the Banach space $L^1(I,X)$ and $C(I,X)$, respectively. Let $f(.) : S \to L^1(I,X)$, $x(.) : S \to C(I,X)$ be their limits. The function $s \to MT|q(s)(.)|_1 + MT^2c(s) + d(s)$ is continuous, hence locally bounded. Therefore (3.6) implies that for every $s' \in S$ the sequence $f_n(s')(.)$ satisfies the Cauchy condition uniformly with respect to s' on some neighborhood of s. Hence, $s \to f(s)(.)$ is continuous from S into $L^1(I,X)$.

From (3.6), as before, $x_n(s)(.)$ is Cauchy in $C(I,X)$ locally uniformly with respect to s. So, $s \to x(s)(.)$ is continuous from S into $C(I,X)$. On the other hand, since $x_n(s)(.)$ converges uniformly to $x(s)(.)$ and

$$d(f_n(s)(t), F(t, x(s)(t)) \leq L(t)|f_n(s)(t) - x(s)(t)| \quad a.e.\ (I),\ \forall s \in S$$

passing to the limit along a subsequence of $f_n(.)$ converging pointwise to $f(.)$ we obtain

$$f(s)(t) \in F(t, x(s)(t)) \quad a.e.\ (I),\ \forall s \in S.$$

Passing to the limit in d) we obtain

$$x(s)(t) = a(s) + p(0)G(t)b(s) + \int_0^t G(t-u)f(s)(u)du.$$

By adding inequalities (c) for all n and using the fact that $\sum_{i\geq 1} q_i(s)(t) \leq \xi(s)(t)$ we obtain

$$\begin{aligned} |f_{n+1}(s)(t) - g(s)(t)| &\leq \sum_{l=0}^{n} |f_{l+1}(s)(u) - f_l(s)(u)| + |f_0(s)(t) - g(s)(t)| \\ &\leq \sum_{l=0}^{n} L(t)q_{l+1}(s)(t) + q(s)(t) + \varepsilon_0(s) \\ &\leq L(t)\xi(s)(t) + q(s)(t) + c(s). \end{aligned} \tag{3.7}$$

Similarly, by adding (3.4) we get

$$|x_{n+1}(s)(t) - y(s)(t)| \leq \sum_{l=0}^{n} q_l(s)(t) \leq \xi(s)(t). \tag{3.8}$$

By passing to the limit in (3.7) and (3.8) we obtain (3.2) and (3.3), respectively.

Theorem 3.3 allows to obtain the next corollary which is a general result concerning continuous selections of the solution set of problem (1.1).

Hypothesis 3.4. Hypothesis 3.1 is satisfied and there exists $q_0 \in L^1(I, \mathbb{R}_+)$ such that $d(0, F(t,0)) \leq q_0(t)$ a.e. (I).

Theorem 3.5. *Assume that Hypothesis 3.4 are satisfied. Then there exists a function $x : I \times X^2 \to X$ such that*

(a) $x(.,(\xi,\eta)) \in \mathcal{S}(\xi,\eta)$, $\forall(\xi,\eta) \in X^2$.
(b) $(\xi,\eta) \to x(.,(\xi,\eta))$ *is continuous from* X^2 *into* $C(I,X)$.

Proof. We take $S = X \times X$, $a(\xi,\eta) = \xi$, $b(\xi,\eta) = \eta$ for all $(\xi,\eta) \in X \times X$, $c : X \times X \to (0,\infty)$ an arbitrary continuous function, $g(.) = 0$, $y = 0$, $q(\xi,\eta)(t) = q_0(t)$ $\forall(\xi,\eta) \in X \times X$, $t \in I$ and we apply Theorem 3.3 in order to obtain the conclusion of the theorem. $\square$

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Aurelian Cernea

Faculty of Mathematics and Informatics, University of Bucharest, Academiei 14, 010014 Bucharest, Romania

E-mail address: `acernea@math.math.unibuc.ro`