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ALMOST AUTOMORPHY OF SEMILINEAR PARABOLIC EVOLUTION EQUATIONS

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Abstract. This paper studies the existence and uniqueness of almost automorphic mild solutions to the semilinear parabolic evolution equation

$$u'(t) = A(t)u(t) + f(t, u(t)),$$

assuming that the linear operators $A(\cdot)$ satisfy the 'Acquistapace–Terreni' conditions, the evolution family generated by $A(\cdot)$ has an exponential dichotomy, and the resolvent $R(\omega, A(\cdot))$, and f are almost automorphic.

1. Introduction

In this work we investigate the almost automorphy of the solutions to the parabolic evolution equations

$$u'(t) = A(t)u(t) + g(t), \quad t \in \mathbb{R}, \tag{1.1}$$

$$u'(t) = A(t)u(t) + f(t, u(t)), \quad t \in \mathbb{R}, \tag{1.2}$$

in a Banach space X, where the linear operators $A(t)$ satisfy the 'Acquistapace–Terreni' conditions and that the evolution family U generated by $A(\cdot)$ has an exponential dichotomy. The asymptotic behavior of these equations was studied by several authors. The most extensively studied cases are the autonomous case $A(t) = A$ and the periodic case $A(t + T) = A(t)$, see [3, 4, 7, 13, 14, 22, 26] for almost periodicity and [6, 10, 12, 16, 20, 21] for almost automorphy. Maniar and Schnaubelt [19] studied the general case, where some resolvent $R(\omega, A(\cdot))$ of $A(\cdot)$ is only almost periodic.

In this paper, we follow the idea of [19] and assume that the function $t \mapsto R(\omega, A(t)) \in \mathcal{L}(X)$, for $\omega \geq 0$, is almost automorphic. We show first the almost automorphy of the Green's function corresponding to U, following the strategy of [19] which consists in using Yosida-approximations of $A(\cdot)$. This result will yield the existence of a unique almost automorphic mild solution $u : \mathbb{R} \to X$ of (1.1) given by

$$u(t) = \int_{\mathbb{R}} \Gamma(t, \tau) g(\tau)\, d\tau, \quad t \in \mathbb{R}, \tag{1.3}$$

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for every almost automorphic function g. Using an interpolation argument, as in [5], we show that the solution u of (1.1) given by (1.3) is also almost automorphic in every time invariant interpolation space $X_\alpha,\ 0 \le \alpha < 1$.

Finally, by a fixed point technique, if the semilinear term $f : \mathbb{R} \times X_\alpha \to X$ is almost automorphic and globally small Lipschitzian; i.e., the Lipshitz constant is small, we show that there is a unique almost automorphic mild solution on X_α to the semilinear parabolic evolution problem (1.2). This is an extension of [20, Theorem 3.1].

To illustrate our results, we also study an example of a reaction diffusion equation with time-varying coefficients. If the coefficients and the semilinear term f are almost automorphic, we show that the solutions are almost automorphic.

2. Prerequisites

A set $U = \{U(t,s) : t \ge s,\ t,s \in \mathbb{R}\}$ of bounded linear operators on a Banach space X is called an *evolution family* if

(E1) $U(t,s) = U(t,r)U(r,s)$ and $U(s,s) = I$ for $t \ge r \ge s$ and
(E2) $(t,s) \mapsto U(t,s)$ is strongly continuous for $t > s$.

We say that an evolution family U has an *exponential dichotomy* if there are projections $P(t)$, $t \in \mathbb{R}$, being uniformly bounded and strongly continuous in t and constants $\delta > 0$ and $N \ge 1$ such that

1. $U(t,s)P(s) = P(t)U(t,s)$,
2. the restriction $U_Q(t,s) : Q(s)X \to Q(t)X$ of $U(t,s)$ is invertible (and we set $U_Q(s,t) := U_Q(t,s)^{-1}$),
3. $\|U(t,s)P(s)\| \le Ne^{-\delta(t-s)}$ and $\|U_Q(s,t)Q(t)\| \le Ne^{-\delta(t-s)}$

for $t \ge s$ and $t,s \in \mathbb{R}$. Here and below we let $Q(\cdot) = I - P(\cdot)$. Exponential dichotomy is a classical concept in the study of the long–term behaviour of evolution equations; see e.g., [8, 9, 11, 15, 17, 23, 25]. If U has an exponential dichotomy, then the operator family

$$\Gamma(t,s) := \begin{cases} U(t,s)P(s), & t \ge s,\ t,s \in \mathbb{R}, \\ -U_Q(t,s)Q(s), & t < s,\ t,s \in \mathbb{R}, \end{cases}$$

is called the *Green's function* corresponding to U and $P(\cdot)$. If $P(t) = I$ for $t \in \mathbb{R}$, then U is *exponentially stable*. The evolution family is called *exponentially bounded* if there are constants $M > 0$ and $\gamma \in \mathbb{R}$ such that $\|U(t,s)\| \le Me^{\gamma(t-s)}$ for $t \ge s$.

In the present work, we study operators $A(t)$, $t \in \mathbb{R}$, on X subject to the following hypothesis introduced by P. Acquistapace and B. Terreni in [2].

(H1) There is an $\omega \ge 0$ such that the operators $A(t)$, $t \in \mathbb{R}$, satisfy $\Sigma_\phi \cup \{0\} \subseteq \rho(A(t) - \omega)$, $\|R(\lambda, A(t) - \omega)\| \le \frac{K}{1+|\lambda|}$, and

$$\|(A(t)-\omega)R(\lambda, A(t)-\omega)\,[R(\omega, A(t)) - R(\omega, A(s))]\| \le L\,|t-s|^\mu |\lambda|^{-\nu}$$

for $t,s \in \mathbb{R}$, $\lambda \in \Sigma_\phi := \{\lambda \in \mathbb{C}\backslash\{0\} : |\arg\lambda| \le \phi\}$, and constants $\phi \in (\frac{\pi}{2}, \pi)$, $L, K \ge 0$, and $\mu, \nu \in (0,1]$ with $\mu + \nu > 1$.

This assumption implies that there exists a unique evolution family U on X such that $(t,s) \mapsto U(t,s) \in \mathcal{L}(X)$ is continuous for $t > s$, $U(\cdot,s) \in C^1((s,\infty), \mathcal{L}(X))$, $\partial_t U(t,s) = A(t)U(t,s)$, and

$$\|A(t)^k U(t,s)\| \le C\,(t-s)^{-k} \tag{2.1}$$

for $0 < t - s \le 1$, $k = 0, 1$, $0 \le \alpha < \mu$, $x \in D((\omega - A(s))^\alpha)$, and a constant C depending only on the constants in (H1). Moreover, $\partial_s^+ U(t,s)x = -U(t,s)A(s)x$ for $t > s$ and $x \in D(A(s))$ with $A(s)x \in \overline{D(A(s))}$. We say that $A(\cdot)$ *generates* U. Note that U is exponentially bounded by (2.1) with $k = 0$.

We further suppose that

(H2) the evolution family U generated by $A(\cdot)$ has an exponential dichotomy with constants $N, \delta > 0$, dichotomy projections $P(t)$, $t \in \mathbb{R}$, and Green's function Γ.

For the sequel, we need the following estimates, see [5] for the proof.

Proposition 2.1. *For every* $0 \le \alpha \le 1$, *we have the following assertions:*

(i) *There is a constant* $c(\alpha)$, *such that*

$$\|U(t,s)P(s)x\|_\alpha^t \le c(\alpha)e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\alpha}\|x\|; \tag{2.2}$$

(ii) *there is a constant* $m(\alpha)$, *such that*

$$\|\widetilde{U}_Q(s,t)Q(t)x\|_\alpha^s \le m(\alpha)e^{-\delta(t-s)}\|x\| \tag{2.3}$$

for every $x \in X$ *and* $t > s$.

We need to introduce the following definition, and we refer to [21] for more information.

Definition 2.2 (S. Bochner)**.** (i) A continuous function $f : \mathbb{R} \to X$ is called almost automorphic if for every sequence $(\sigma_n)_{n\in N}$ there exists a subsequence $(s_n)_{n\in N} \subset (\sigma_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n,m\to+\infty} f(t + s_n - s_m) = f(t) \quad \text{for each } t \in \mathbb{R}.$$

This is equivalent to

$$g(t) := \lim_{n\to+\infty} f(t + s_n) \quad \text{and} \quad f(t) = \lim_{n\to+\infty} g(t - s_n)$$

are well defined for each $t \in \mathbb{R}$. We note that $f \in AA(\mathbb{R}, X)$.

(ii) A function $f : \mathbb{R} \times Y \to X$ is said to be almost automorphic if it satisfies the following conditions: $f(\cdot, y)$ is almost automorphic for every $y \in Y$ and f is continuous jointly in (t, x). We note $f \in AA(\mathbb{R} \times Y, X)$.

The function g in the definition above is measurable, but not necessarily continuous. It is well-known that $AA(\mathbb{R}, X)$ is a Banach space under the sup-norm $\|f\|_{AA(\mathbb{R},X)} = \sup_{t\in\mathbb{R}} \|f(t)\|$.

3. Main results

In this section, we study the existence of almost automorphic solutions to the semilinear evolution equations

$$u'(t) = A(t)u(t) + f(t, u(t)), \quad t \in \mathbb{R}, \tag{3.1}$$

where $A(t), t \in \mathbb{R}$, satisfy (H1) and (H2), and the following assumptions hold:

(H3) $R(\omega, A(\cdot)) \in AA(\mathbb{R}, \mathcal{L}(X))$;

(H4) there are $0 \le \alpha < \beta < 1$ such $X_\alpha^t = X_\alpha$, $t \in \mathbb{R}$, $X_\beta^t = X_\beta$, $t \in \mathbb{R}$, with uniform equivalent norms;

(H5) the function $f : \mathbb{R} \times X_\alpha \to X$ belongs to $AA(\mathbb{R} \times X_\alpha, X)$ and is globally small Lipschitzian; i.e., there is a small $K_f > 0$ such that

$$\|f(t,u) - f(t,v)\| \leq K_f \|u - v\|_\alpha \quad \text{for all } t \in \mathbb{R} \text{ and } u, v \in X_\alpha.$$

By a mild solution of (3.1) we understand a continuous function $u : \mathbb{R} \to X_\alpha$, which satisfies the following variation of constants formula

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\sigma)f(\sigma,u(\sigma))d\sigma \quad \text{for all } t \geq s,\ t,s \in \mathbb{R}. \tag{3.2}$$

To achieve the goal of this section, we show some intermediate results. Let us define the Yosida approximations $A_n(t) = nA(t)R(n, A(t))$ of $A(t)$ for $n > \omega$ and $t \in \mathbb{R}$. These operators generate an evolution family U_n on X. It has been shown in [19, Lemma 3.1, Proposition 3.3, Corollary 3.4] that assumptions (H1) and (H2) are satisfied by $A_n(\cdot)$ with the same constants for every $n \geq n_0$.

In the following lemma, we show that the Yosida approximations $A_n(\cdot)$ satisfy also assumption (H3) for large n. The formulas on the resolvent used in the proof are taken from [19].

Lemma 3.1. *If* (H1) *and* (H3) *hold, then there is a number* $n_1 \geq n_0$ *such that* $R(\omega, A_n(\cdot)) \in AA(\mathbb{R}, \mathcal{L}(X))$ *for* $n \geq n_1$.

Proof. Let $(s'_l)_{l\in\mathbb{N}}$ be a sequence of real numbers, as $R(\omega, A(\cdot))$ is almost automorphic, there is a subsequence $(s_l)_{l\in\mathbb{N}}$ such that

$$\lim_{l,\ k\to+\infty} \|R(\omega, A(t + s_l - s_k)) - R(\omega, A(t))\| = 0, \tag{3.3}$$

for each $t \in \mathbb{R}$ If $n \geq n_0$ and $|\arg(\lambda - \omega)| \leq \phi$, we have

$$\begin{aligned} &R(\omega, A_n(t + s_l - s_k)) - R(\omega, A_n(t)) \\ &= \frac{n^2}{(\omega+n)^2}\Big(R\Big(\frac{\omega n}{\omega+n}, A(t+s_l-s_k)\Big) - R\Big(\frac{\omega n}{\omega+n}, A(t)\Big)\Big) \\ &= \frac{n^2}{(\omega+n)^2} R(\omega, A(t+s_l-s_k))\big[1 - \frac{\omega^2}{\omega+n}R(\omega, A(t+s_l-s_k))\big]^{-1} \\ &\quad - \frac{n^2}{(\omega+n)^2} R(\omega, A(t))\big[1 - \frac{\omega^2}{\omega+n}R(\omega, A(t))\big]^{-1}. \end{aligned} \tag{3.4}$$

We can also see that

$$\|\frac{\omega^2}{\omega+n}R(\omega, A(s))\| \leq \frac{\omega^2}{\omega+n}\frac{K}{1+\omega} \leq \frac{\omega K}{n} \leq \frac{1}{2}$$

for $n \geq n_1 := \max\{n_0, 2\omega K\}$ and $s \in \mathbb{R}$. In particular,

$$\|\big[1 - \frac{\omega^2}{\omega+n}R(\omega, A(s))\big]^{-1}\| \leq 2. \tag{3.5}$$

Hence, (3.4) implies

$$\begin{aligned} &\|R(\omega, A_n(t+s_l-s_k)) - R(\omega, A_n(t))\| \\ &\leq 2\|R(\omega, A(t+s_l-s_k)) - R(\omega, A(t))\| \\ &\quad + \frac{K}{1+\omega}\|\big[1 - \frac{\omega^2}{\omega+n}R(\omega, A(t+s_l-s_k))\big]^{-1} - \big[1 - \frac{\omega^2}{(\omega+n)^2}R(\omega, A(t))\big]^{-1}\|. \end{aligned}$$

Employing (3.5) again, we obtain

$$\begin{aligned}
&\big\| \big[1-\frac{\omega^2}{\omega+n}R(\omega,A(t+s_l-s_k))\big]^{-1}-\big[1-\frac{\omega^2}{\omega+n}R(\omega,A(t))\big]^{-1}\big\| \\
&\le 4\big\| \big[1-\frac{\omega^2}{\omega+n}R(\omega,A(t+s_l-s_k))\big]-\big[1-\frac{\omega^2}{\omega+n}R(\omega,A(t))\big]\big\| \\
&\le 4\omega\, \|R(\omega,A(t+s_l-s_k))-R(\omega,A(t))\|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|R(\omega,A_n(t+s_l-s_k))-R(\omega,A_n(t))\| \\
&\le (2+4K)\|R(\omega,A(t+s_l-s_k))-R(\omega,A(t))\|
\end{aligned} \tag{3.6}$$

for $n \ge n_1$ and $t \in \mathbb{R}$. The assertion thus follows from (3.3). □

The following technical lemma is also needed.

Lemma 3.2. *Assume that* (H1)– (H3) *hold. For every sequence $(s'_l)_{l\in\mathbb{N}} \in \mathbb{R}$, there is a subsequence $(s_l)_{l\in\mathbb{N}}$ such that for every $\eta > 0$, and $t,\ s \in \mathbb{R}$ there is $l(\eta,t,s) > 0$ such that*

$$\|\Gamma_n(t+s_l-s_k,s+s_l-s_k)-\Gamma_n(t,s)\| \le cn^2\eta \tag{3.7}$$

for a large n and $l,\ k \ge l(\eta,t,s)$.

Proof. Let a sequence $(s'_l)_{l\in\mathbb{N}} \in \mathbb{R}$. Since $R(\omega,A(\cdot)) \in AA(\mathbb{R},X)$, then we can extract a subsequence (s_l) such that

$$\|R(\omega,A(\sigma+s_l-s_k))-R(\omega,A(\sigma))\| \to 0, \quad k,l \to \infty, \tag{3.8}$$

for all $\sigma \in \mathbb{R}$. As in [19], we have

$$\begin{aligned}
&\Gamma_n(t+s_l-s_k,s+s_l-s_k)-\Gamma_n(t,s) \\
&= \int_{\mathbb{R}} \Gamma_n(t,\sigma)(A_n(\sigma)-\omega)[R(\omega,A_n(\sigma+s_l-s_k))-R(\omega,A_n(\sigma))] \\
&\quad \times (A_n(\sigma+s_l-s_k)-\omega)\Gamma_n(\sigma+s_l-s_k,s+s_l-s_k)\,d\sigma
\end{aligned}$$

for $s,t \in \mathbb{R}$ and $l,k, \in \mathbb{N}$ and large n. This formula, the estimate (3.6) and [19, Corollary 3.4] imply

$$\begin{aligned}
&\|\Gamma_n(t+s_l-s_k,s+s_l-s_k)-\Gamma_n(t,s)\| \\
&\le cn^2 \int_{\mathbb{R}} e^{-\frac{3\delta}{4}|t-\sigma|}e^{-\frac{3\delta}{4}|\sigma-s|}\|R(\omega,A_n(\sigma+s_l-s_k))-R(\omega,A_n(\sigma))\|\,d\sigma \\
&\le cn^2(2+4K)\int_{\mathbb{R}} e^{-\frac{3\delta}{4}|t-\sigma|}e^{-\frac{3\delta}{4}|\sigma-s|}\|R(\omega,A(\sigma+s_l-s_k))-R(\omega,A(\sigma))\|\,d\sigma \to 0,
\end{aligned}$$

as $k,l \to \infty$, by (3.8) and the Lebesgue's Dominated Convergence Theorem. Hence, for $\eta > 0$ there is $l(\eta,t,s) > 0$ such that

$$\|\Gamma_n(t+s_l-s_k,s+s_l-s_k)-\Gamma_n(t,s)\| < cn^2\eta$$

for large n and $l,\ k \ge l(\eta,t,s)$. □

The almost automorphy of the Green function Γ is proved in the next proposition. An analogous result for the almost periodicity is shown in [19].

Proposition 3.3. *Assume that* (H1)– (H2) *hold. Let a sequence* $(s'_l)_{l\in\mathbb{N}} \in \mathbb{R}$ *there is a subsequence* $(s_l)_{l\in\mathbb{N}}$ *such that for every* $h > 0$

$$\|\Gamma(t + s_l - s_k, s + s_l - s_k) - \Gamma(t, s)\| \to 0, \quad k, l \to \infty$$

for $|t - s| \geq h$.

Proof. Let $(s'_l)_{l\in\mathbb{N}}$ be a sequence in $\mathbb{R}$, and consider the subsequence (s_l) given by Lemma 3.2. Let $\varepsilon > 0$ and $h > 0$. There is $t_\varepsilon > h$ such that

$$\|\Gamma(t + s_l - s_k, s + s_l - s_k) - \Gamma(t, s)\| \leq \varepsilon$$

for $|t - s| \geq t_\varepsilon$ and $l,\ k \in \mathbb{N}$. For $h \leq |t - s| \leq t_\varepsilon$, by [19, Lemma 4.2] we have

$$\|\Gamma(t + s_l - s_k, s + s_l - s_k) - \Gamma_n(t + s_l - s_k, s + s_l - s_k)\| \leq c(t_\varepsilon)n^{-\theta}, \tag{3.9}$$

$$\|\Gamma(t, s) - \Gamma_n(t, s)\| \leq c(t_\varepsilon)n^{-\theta} \tag{3.10}$$

for all k, l and large n. Let $n_\varepsilon > 0$ large enough such that $n^{-\theta} < \frac{\varepsilon}{4c(t_\varepsilon)}$ for $n \geq n_\varepsilon$. Take $0 < \eta < \frac{\varepsilon}{2cn_\varepsilon^2}$. Hence, by (3.9), (3.10) and Lemma 3.2, one has

$$\|\Gamma(t + s_l - s_k, s + s_l - s_k) - \Gamma(t, s)\| \leq 2c(t_\varepsilon)n_\varepsilon^{-\theta} + cn_\varepsilon^2\eta \leq \varepsilon$$

for all $k, l \geq l(\varepsilon, t, s)$. Consequently, $\|\Gamma(t + s_l - s_k, s + s_l - s_k) - \Gamma(t, s)\| \to 0$ as $l,\ k \to +\infty$ for $|t - s| > h > 0$. ☐

Using Proposition 3.3, we show the existence of a unique almost automorphic solution to the inhomogeneous evolution equation

$$u'(t) = A(t)u(t) + g(t), \quad t \in \mathbb{R}. \tag{3.11}$$

More precisely, we state the following main result.

Theorem 3.4. *Assume* (H1)–(H4). *Then, for every* $g \in AA(\mathbb{R}, X)$*, the unique bounded mild solution* $u(\cdot) = \int_{\mathbb{R}} \Gamma(\cdot, s)g(s)\, ds$ *of* (3.11) *belongs to* $AA(\mathbb{R}, X_\alpha)$.

Proof. First we prove that the mild solution u is almost automorphic in X. Let a sequence $(s'_l)_{l\in\mathbb{N}}$ and $h > 0$. As $g \in AA(\mathbb{R}, X)$ there exists a subsequence $(s_l)_{l\in\mathbb{N}}$ such that $\lim_{l,\ k\to+\infty} \|g(t + s_l - s_k) - g(t)\| \to 0$. Now, we write

$$\begin{aligned}
&u(t + s_l - s_k) - u(t)\\
&= \int_{\mathbb{R}} \Gamma(t + s_l - s_k, s + s_l - s_k)g(s + s_l - s_k)\, ds - \int_{\mathbb{R}} \Gamma(t, s)g(s)\, ds\\
&= \int_{\mathbb{R}} \Gamma(t + s_l - s_k, s + s_l - s_k)(g(s + s_l - s_k) - g(s))\, ds\\
&\quad + \int_{|t-s|\geq h} (\Gamma(t + s_l - s_k, s + s_l - s_k) - \Gamma(t, s))g(s)\, ds\\
&\quad + \int_{|t-s|\leq h} (\Gamma(t + s_l - s_k, s + s_l - s_k) - \Gamma(t, s))g(s)\, ds.
\end{aligned}$$

For $\varepsilon' > 0$, we deduce from Proposition 3.3 and (H2) that

$$\|u(t+s_l-s_k)-u(t)\| \leq 2N \int_{\mathbb{R}} e^{-\delta|t-s|}\|g(s+s_l-s_k)-g(s)\|\, ds + (\tfrac{4}{\delta}\, \varepsilon' + 4Nh)\|g\|_\infty$$

for $t \in \mathbb{R}$ and $l,\ k > l(\varepsilon,\ h) > 0$. Now, for $\varepsilon > 0$, take h small and then $\varepsilon' > 0$ small such that

$$\|u(t + s_l - s_k) - u(t)\| \leq 2N \int_{\mathbb{R}} e^{-\delta|t-s|}\|g(s + s_l - s_k) - g(s)\|\, ds + \tfrac{\varepsilon}{2}$$

for $t \in \mathbb{R}$ and $l,\ k > l(\varepsilon) > 0$. Finally, by the Lebesgue's Dominated Convergence Theorem, u is almost automorphic in X.

Using the reiteration theorem, we obtain $X_\alpha = (X, X_\beta)_\theta$, with $\theta = \alpha/\beta$. By the property of interpolation, we have

$$\begin{aligned} &\|u(t+s_l-s_k)-u(t)\|_\alpha \\ &\le c(\alpha,\beta)\|u(t+s_l-s_k)-u(t)\|^{\frac{\beta-\alpha}{\beta}}\|u(t+s_l-s_k)-u(t)\|_\beta^{\frac{\alpha}{\beta}}. \end{aligned}$$

Using estimates in Proposition 2.1 we can show that u is bounded in X_β. Hence,

$$\begin{aligned} \|u(t+s_l-s_k)-u(t)\|_\alpha &\le c(\alpha,\beta)c^{\frac{\beta}{\alpha}}\|u(t+s_l-s_k)-u(t)\|^{\frac{\beta-\alpha}{\beta}} \\ &\le c'\|u(t+s_l-s_k)-u(t)\|^{\frac{\beta-\alpha}{\beta}}. \end{aligned} \tag{3.12}$$

Since u is almost automorphic in X, $u(t+s_l-s_k) \to u(t)$, as $l,k \to \infty$, for $t \in \mathbb{R}$, and thus $x \in AA(\mathbb{R}, X_\alpha)$. □

As a consequence of Theorem 3.4 and a fixed point technique, we achieve the aim of the paper.

Theorem 3.5. *Assume that* (H1)–(H5) *hold. Then* (3.1) *admits a unique mild solution u in $AA(\mathbb{R}, X_\alpha)$.*

Proof. Consider $v \in AA(\mathbb{R}, X_\alpha)$ and $f \in AA(\mathbb{R} \times X_\alpha, X)$. Then, by [21, Theorem 2.2.4, p. 21], the function $g(\cdot) := f(\cdot, v(\cdot)) \in AA(\mathbb{R}, X)$, and from Theorem 3.4, the inhomogeneous evolution equation

$$u'(t) = A(t)u(t) + g(t), \quad t \in \mathbb{R},$$

admits a unique mild solution $u \in AA(\mathbb{R}, X)$ given by

$$u(t) = \int_{\mathbb{R}} \Gamma(t,s) f(s, v(s))ds, \quad t \in \mathbb{R}.$$

Let the operator $F : AA(\mathbb{R}, X_\alpha) \to AA(\mathbb{R}, X_\alpha)$ be defined by

$$(Fv)(t) := \int_{\mathbb{R}} \Gamma(t,s) f(s, v(s))ds \quad \text{for all } t \in \mathbb{R}.$$

Now we prove that F has a unique fixed point. The estimates (2.2) and (2.3) yield

$$\begin{aligned} \|Fx(t) - Fy(t)\|_\alpha &\le c(\alpha)\int_{-\infty}^{t} e^{-\delta(t-s)}(t-s)^{-\alpha}\|f(s,y(s)) - f(s,x(s))\|ds \\ &\quad + c(\alpha)\int_{t}^{+\infty} e^{-\delta(t-s)}\|f(s,y(s)) - f(s,x(s))\|ds. \\ &\le K_f c'(\alpha)\|x-y\|_\infty \end{aligned}$$

for all $t \in \mathbb{R}$ and $x,\ y \in AA(\mathbb{R}, X_\alpha)$. If we assume that $K_f c'(\alpha) < 1$, then F has a unique fixed poind $u \in AA(\mathbb{R}, X_\alpha)$. Thus u is the unique almost automorphic solution to the equation (3.1). □

Example 3.6. Consider the parabolic problem

$$\begin{aligned} \partial_t u(t,x) = A(t,x,D)u(t,x) + h(t, \nabla u(t,x)), \quad t \in \mathbb{R},\ x \in \Omega, \\ B(x,D)u(t,x) = 0, \quad t \in \mathbb{R},\ x \in \partial\Omega, \end{aligned} \tag{3.13}$$

on a bounded domain $\Omega \subseteq \mathbb{R}^n$ with boundary $\partial\Omega$ of class C^2 and outer unit normal vector $\nu(x)$, employing the differential expressions

$$A(t,x,D) = \sum_{k,l} a_{kl}(t,x)\partial_k\partial_l + \sum_k a_k(t,x)\,\partial_k + a_0(t,x),$$

$$B(x,D) = \sum_k b_k(x)\,\partial_k + b_0(x).$$

We require that $a_{kl} = a_{lk}$ and b_k are real–valued, $a_{kl}, a_k, a_0 \in C_b^\mu(\mathbb{R}, C(\overline{\Omega}))$, $b_k, b_0 \in C^1(\partial\Omega)$,

$$\sum_{k,l=1}^n a_{kl}(t,x)\,\xi_k\,\xi_l \geq \eta|\xi|^2\,, \quad \text{and} \quad \sum_{k=1}^n b_k(x)\nu_k(x) \geq \beta$$

for constants $\mu \in (1/2,1)$, $\beta, \eta > 0$ and all $\xi \in \mathbb{R}^n$, $k,l = 1,\cdots,n$, $t \in \mathbb{R}$, $x \in \overline{\Omega}$ resp. $x \in \partial\Omega$. (C_b^μ is the space of bounded, globally Hölder continuous functions.) We set $X = C(\overline{\Omega})$,

$$D(A(t)) = \{u \in \bigcap_{p>1} W_p^2(\Omega) : A(t,\cdot,D)u \in C(\overline{\Omega}),\ B(\cdot,D)u = 0 \text{ on } \partial\Omega\}$$

for $t \in \mathbb{R}$. It is known that the operators $A(t)$, $t \in \mathbb{R}$, satisfy (H1), see [1, 18], or [24, Exa.2.9]. Thus $A(\cdot)$ generates an evolution family $U(\cdot,\cdot)$ on X. Let $\alpha \in (1/2,1)$ and $p > \frac{n}{2(1-\alpha)}$. Then $X_\alpha^t = X_\alpha = \{f \in C^{2\alpha}(\overline{\Omega}) : B(\cdot,D)u = 0\}$ with uniformly equivalent constants due to [18, Theorem 3.1.30], and $X_\alpha \hookrightarrow W_p^2(\Omega)$. It is clear that the function $f(t,u)(x) := h(t,\nabla u(x))$, $x \in \Omega$, is continuous from $\mathbb{R} \times X_\alpha$ to X, and if h is small Lipschitzian and almost automorphic then f is. Under the exponential dichotomy of $U(\cdot,\cdot)$ and almost automorphy of $R(\omega, A(\cdot))$, the parabolic equation (3.13) has a unique almost automorphic solution.

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