Electronic Journal of Differential Equations, Vol. 2008(2008), No. 63, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

GLOBAL SMOOTH SOLUTIONS OF THE SPIN POLARIZED TRANSPORT EQUATION

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Abstract. In this paper, we prove the existence of global smooth solutions of the spin-polarized transport equation using energy estimates method in dimension two.

1. Introduction

It is well known that the Landau-Lifshitz equation has a fundamental importance in understanding the ferromagnetism in materials, see [11]. It reads as

$$m_t = m \times \Delta m - \mu m \times (m \times \Delta m),$$

where $m \in \mathbb{S}^2$ is the magnetization field and $\mu > 0$ is the Gilbert damping coefficients. There is a long list of work contributed to this equation regarding its existence, uniqueness, self-similar solutions, blowup and partially regularities, the interested reader can refer to [1, 3, 8, 7, 11, 12] for more details. Mathematically, it is closely related to harmonic maps to the sphere, see [10, 8]. There are also several equations closely related to Landau-Lifshitz equation, such as Landau-Lifshitz-Maxwell equation [4], and the equations in ferrimagnetic materials [9] and so on. Many authors also take the spin polarize into account, to consider the spin polarized current-driven magnetization in materials, see [13, 14]. Recently, C.J. Garcia-Cervera and X.P. Wang[6] considered the weak solutions of the following spin-polarized transport equations in materials

$$\begin{gathered}\frac{\partial s}{\partial t} = -\operatorname{div} J_s - D_0(x)s - D_0(x)s \times m \\ \frac{\partial m}{\partial t} = -m \times (h+s) + \alpha m \times \frac{\partial m}{\partial t} \\ s(x,0) = s_0(x), \quad m(x,0) = m_0(x).\end{gathered} \tag{1.1}$$

In this equation, (s, m) is the unknown, $s = (s_1, s_2, s_3) : \Omega \to \mathbb{R}^3$ denotes the spin accumulation and $m = (m_1, m_2, m_3) : \Omega \to \mathbb{S}^2$ is the magnetization field, where $\mathbb{S}^2$ is the unit sphere in $\mathbb{R}^3$. J_s is the spin current

$$J_s = m \otimes J_e - D_0(x)[\nabla s - \beta m \otimes (\nabla s \cdot m)], \tag{1.2}$$

2000 *Mathematics Subject Classification.* 35K15, 35K20.
Key words and phrases. Spin-polarized transport equations; global smooth solutions; Landau-Lifshitz equations.

Submitted July 4, 2007. Published April 27, 2008.

where J_e is the applied electronic current, $0 < \beta < 1$ is the spin polarized parameter and $D_0(x)$ is the diffusion parameter depending on the material. In the second equation of (1.1), $h = -\nabla_m \Phi + \Delta m + h_d$ denotes the anisotropy, exchange and self-induced energy respectively.

As can be seen, this equation is closely related to the Landau-Lifshitz equation in the magnetization field m and to quasilinear parabolic equations in the spin accumulation s. It has great importance physically and interesting mathematical structures as well. In [6], the authors considered the global weak solutions of this equation and obtained interesting results by Galerkin's approximating method. A natural question is whether or not this system has global smooth solutions. Taking into account the results in [8], in particular Theorem 2.6, it is not right in general by technical reasons. However, we can show that under some additional conditions on the initial data, this system admits global smooth solutions. For simplicity, we consider the periodic boundary conditions and assume that the diffusion material $D_0(x)$ is constant, which is set to be $D_0(x) \equiv 1$ in this paper. Furthermore, we neglect to consider the anisotropic energy $\nabla_m \Phi$ and self-induced energy h_d. Under these assumptions and simplifications, we show that there exists a global smooth solution for this system under smallness condition (1.3) in space dimension two. Our main result is the following theorem.

Theorem 1.1. *Let $k \geq 2$, $J_e \in H^k(R^+ \times \Omega)$, $0 < \beta < 1$ and $s_0 \in H^k(\Omega)$, $m_0 \in H^k(\Omega)$. Then there exists a positive constant $\lambda_0 > 0$, such that if the smallness condition*

$$\|s_0\|_{L^2}^2 + \|J_e\|_{L^2(R^+\times\Omega)}^2 + \|\nabla m_0\|_{L^2}^2 < \lambda_0 \tag{1.3}$$

holds, then there exists a global solution (s, m) of (1.1) in $H^k(\Omega)$ satisfying

$$\partial_t^j \partial_x^\alpha \in L^\infty([0,T]; L^2\Omega); \quad \partial_t^k \partial_x^\beta \in L^2([0,T]; L^2\Omega),$$

with $2j + |\alpha| \leq k$ and $2k + |\beta| \leq k+1$. In particular, if the initial data are smooth, the solution is globally smooth.

In the sequel, we denote by $\|\cdot\|_X$ the norm of functions on space X. In particular, if $X = L^2(\Omega)$, we simply denote $\|\cdot\|$ instead of $\|\cdot\|_{L^2(\Omega)}$.

This paper is organized as follows. In the next Section, we show that there exists a local solution in Sobolev spaces $W^{2,p}$ for $p > 2$. By standard bootstrap method, this solution is indeed smooth. In Section 3, we give some a priori global estimates of the solution in H^2. Together with the local theory, we show that the system has a global H^2 solution provided that the initial data are in H^2, and small in the sense of (1.3). Finally in Section 4, we give some *a priori* estimates in H^k to deduce that the solution remains as smooth as the initial data for all time if the initial data are small in the sense of (1.3). In particular, if the solution is smooth, we get a global smooth solution for the system under consideration.

2. Local smooth solutions

In this section, we show that there exists a local smooth solution for the system we considered in this paper. For this purpose, we rewrite

$$\operatorname{div}(m \otimes (\nabla s \cdot m)) = (\Delta s \cdot m)m + \widetilde{\operatorname{div}}(m \otimes (\nabla s \cdot m)),$$

where $\widetilde{\operatorname{div}}(m\otimes(\nabla s\cdot m)) = \nabla m\cdot(\nabla s\cdot m)+(\nabla s\cdot\nabla m)m$. For the rest of this paper, we set

$$A = A(m) = \begin{pmatrix} 1-\beta m_1^2 & -\beta m_1 m_2 & -\beta m_1 m_3 \\ -\beta m_2 m_1 & 1-\beta m_2^2 & -\beta m_2 m_3 \\ -\beta m_3 m_1 & -\beta m_3 m_2 & 1-\beta m_3^2 \end{pmatrix}.$$

Since $0<\beta<1$, we know there two exists positive numbers $\lambda,\Lambda>0$, such that

$$\lambda|\xi|^2 \le \xi A(m)\xi^T \le \Lambda|\xi|^2, \quad \forall \xi\in\mathbb{R}^3,\ \xi\neq 0. \tag{2.1}$$

On the other hand, since $m\in\mathbb{S}^2$, we can write the equation for m in its equivalent form

$$(1+\alpha^2)\frac{\partial m}{\partial t} = -m\times(\Delta m+s) - \alpha m\times(m\times(\Delta m+s)). \tag{2.2}$$

Using this notation, system (1.1) can be rewritten as

$$\begin{gathered}\frac{\partial s}{\partial t} - A(m)\Delta s + s = -\operatorname{div}(m\otimes J_e) - \beta\widetilde{\operatorname{div}}(m\otimes(\nabla s\cdot m)) - s\times m \\ (1+\alpha^2)\frac{\partial m}{\partial t} - \alpha\Delta m = -m\times\Delta m - m\times s + \alpha|\nabla m|^2 m - \alpha m\times(m\times s).\end{gathered} \tag{2.3}$$

Obviously, this system is strongly parabolic [5, 8].

In the following, we use Hamilton's idea in [10] to derive the local existence of the solution of (1.1) with smooth initial data (s_0,m_0). As in [8], we prove the following result.

Lemma 2.1. *Let $1<p<\infty$. For every function $g_1,g_2\in L^p(\Omega\times[\gamma,\omega])$, there exists a unique solution (l,k) of*

$$\begin{pmatrix} \partial_t l \\ (1+\alpha^2)\partial_t k \end{pmatrix} = \begin{pmatrix} A(m)\Delta l + d(s,m)k + e(s,m,J_e)\nabla k + f(m)l + h(m)\nabla l + g_1 \\ \alpha\Delta k + m\times\Delta k + a(s,m)\nabla k + b(s,m)k + c(m)l + g_2 \end{pmatrix}$$

on $\Omega\times[\gamma,\omega]$, where a,b,c,d,e,f,s,m are given smooth functions.

Proof. Denote

$$\begin{gathered} H\begin{pmatrix} l \\ k\end{pmatrix} = \begin{pmatrix} \partial_t l - A(m)\Delta l \\ (1+\alpha^2)\partial_t k - \alpha\Delta k - m\times\Delta k\end{pmatrix}, \\ K\begin{pmatrix} l \\ k\end{pmatrix} = \begin{pmatrix} d(s)k + e(s,m,J_e)\nabla k + f(m)l + h(m)\nabla l \\ a(s,m)\nabla k + b(s,m)k + c(m)l\end{pmatrix}. \end{gathered}$$

Since H is a strong parabolic linear operator, the map $(l,k)\to H(l,k)$ is an isomorphism from $W^{2,p}(\Omega\times[\gamma,\omega])$ onto $L^p(\Omega\times[\gamma,\omega])$. On the other hand, $K: W^{2,p}(\Omega\times[\gamma,\omega])\to L^p(\Omega\times[\gamma,\omega])$ is compact. By the Fredholm theory, the indices of the map $(H-K)$ is 0 by definition of the indices of a Fredholm operator. Thus it suffices to show that its kernel is zero to deduce that it is an isomorphism.

Let $(l,k)\in W^{2,p}(\Omega\times[\gamma,\omega])$, and $f|_{\Omega\times\gamma}=0$. Since $(H-K)((l,k)^T)=0$, taking the scalar product of this equation with (l,k), we have respectively

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega |l|^2 + \lambda\int_\Omega|\nabla l|^2 &\le \int_\Omega |\nabla A(m)||\nabla l||l| + C_1\int_\Omega|\nabla l||l| \\ &\quad + C_2\int_\Omega|\nabla k||l| + C_3\int_\Omega |k||l| + C_4\int_\Omega|l|^2 \\ &\le \frac{\lambda}{4}\int_\Omega|\nabla l|^2 + \frac{\alpha}{4}\int_\Omega|\nabla k|^2 + C_5\int_\Omega |l|^2+|k|^2 \end{aligned}$$

and similarly,

$$\begin{aligned}(1+\alpha^2)\frac{1}{2}\frac{d}{dt}\int_\Omega |k|^2+\alpha\int_\Omega |\nabla k|^2 \le &\int_\Omega k\cdot(m\times\Delta k)+C_6\int_\Omega |\nabla k||k|\\ &+C_7\int_\Omega |k||l|+C_8\int_\Omega |k|^2\\ \le&\frac{\lambda}{4}\int_\Omega |\nabla l|^2+\frac{\alpha}{4}\int_\Omega |\nabla k|^2+C_9\int_\Omega |l|^2+|k|^2.\end{aligned}$$

In the last inequality, we used the fact that

$$\int_\Omega k\cdot(m\times\Delta k)=-\int_\Omega k\cdot\nabla m\times\nabla k.$$

Adding the two inequalities together we obtain a Gronwall type inequality

$$\frac{d}{dt}\int |l|^2+(1+\alpha^2)\int |k|^2+\lambda\int |\nabla l|^2+\alpha\int |\nabla k|^2\le C\int |l|^2+|k|^2.$$

Hence by Gronwall inequality, we have $(l,k)=0$, which shows that the kernel of the map $(H-K)$ is zero and finishes the proof the lemma. $\square$

Lemma 2.2. *Let (s_0,m_0) be a smooth map. Then there exists a positive number $\varepsilon>0$ and a unique local solution $(s,m)\in W^{2,p}(\Omega\times[0,\varepsilon])$ for $p>2$ strictly.*

Proof. Define a nonlinear map $\mathcal{L}: W^{2,p}(\Omega\times[0,\omega])\to L^p(\Omega\times[0,\omega])$ as follows:

$$\mathcal{L}\begin{pmatrix}s\\ m\end{pmatrix}=\begin{pmatrix}A(m)\Delta s-\operatorname{div}(m\otimes J_e)-\beta\widetilde{\operatorname{div}}(m\otimes(\nabla s\cdot m))-s-s\times m\\ \alpha\Delta m+\alpha|\nabla m|^2m-\alpha m\times(m\times s)-m\times(\Delta m+s)\end{pmatrix}.$$

Then for a smooth map (s,m), the derivative of L at (s,m) is given by

$$D\mathcal{L}(s,m)\begin{pmatrix}l\\ k\end{pmatrix}=\begin{pmatrix}A(m)\Delta l+d(s,m)k+e(s,m,J_e)\nabla k+f(m)l+h(m)\nabla l\\ \alpha\Delta k+m\times\Delta k+a(s,m)\nabla k+b(s,m)k+c(m)l\end{pmatrix},$$

where a,b,c,d,e,f are smooth matrix-valued functions. Denote

$$\mathcal{R}\begin{pmatrix}s\\ m\end{pmatrix}=\begin{pmatrix}\partial_t s\\ (1+\alpha^2)\partial_t m\end{pmatrix}-\mathcal{L}\begin{pmatrix}s\\ m\end{pmatrix}.$$

Let $(s_\omega,m_\omega):\Omega\times[0,\omega]\to\mathbb{R}^3$ be smooth maps with $(s_\omega,m_\omega)=(s_0,m_0)$ on $\Omega\times 0$, we denote $(s,m)=(s_\omega,m_\omega)+(s_*,m_*)$. The derivative of $\mathcal{R}$ at (s,m) is given by

$$D\mathcal{R}(s,m)\begin{pmatrix}l\\ k\end{pmatrix}=\begin{pmatrix}\partial_t l\\ (1+\alpha^2)\partial_t k\end{pmatrix}-D\mathcal{L}(s,m)\begin{pmatrix}l\\ k\end{pmatrix}. \tag{2.4}$$

Consider (s_*,m_*) as a variable function, then $(s_*,m_*)\to\mathcal{R}\begin{pmatrix}s_\omega+s_*\\ m_\omega+m_*\end{pmatrix}$ defines a continuously differential map of $W^{2,p}(\Omega\times[0,\omega])\to L^p(\Omega\times[0,\omega])$. Its derivative at $[(s_*,m_*)=0]$ is given by (2.4) with (s,m) replaced by (s_ω,m_ω) and from the above lemma we know it is an isomorphism from $W^{2,p}(\Omega\times[0,\omega])$ onto $L^p(\Omega\times[0,\omega])$.

Therefore by the inverse function theorem, the set of all $\mathcal{R}\begin{pmatrix}s_\omega+s_*\\ m_\omega+m_*\end{pmatrix}$ for (s_*,m_*) in a neighborhood $\mathscr{N}\subset W^{2,p}(\Omega\times[0,\omega])$ of 0 covers a neighborhood $\mathscr{O}$ of $\mathcal{R}(s_\omega,m_\omega)$ in $L^p(\Omega\times[0,\omega])$. If we choose $\varepsilon>0$ small enough, the function

$$(\tilde{l},\tilde{k})=\begin{cases}(0,0), & 0\le t\le\varepsilon;\\ \mathcal{R}(s_\omega,m_\omega), & \varepsilon<t\le\omega,\end{cases}$$

will be in this neighborhood $\mathscr{O}$. So $(s,m)|_{\Omega\times[0,\varepsilon]}$ solves the equation (1.1). On the other hand, from our choice $p>2$ and $m\in\mathbb{S}^2$, we can show the smoothness of the solution by standard bootstrap method. We stress that $p>2$ here plays an important role in the bootstrap method, because for $p>2$, we have $W^{k,p}(\Omega)\hookrightarrow W^{k-1,\infty}(\Omega)$ in space dimension 2. $\square$

3. A priori estimates

In this section, we show some *a priori* estimates for the solution. The following Lemma can be found in [6] and we omit its proof here.

Lemma 3.1. *Let $J_e\in(H^1(R^+\times\Omega))^3$ and $s(x,t)\in L^\infty(R^+,L^2(\Omega))$ and $m(x,t)\in L^\infty(R^+,H^1(\Omega))$ be a weak solution to the problem* (1.1)*. Then the solution* (s,m) *satisfies the following relations:*

$$m\in L^\infty(R^+,H^1(\Omega)),\quad \frac{\partial m}{\partial t}\in L^2(R^+,L^2(\Omega)),$$
$$s\in L^\infty(R^+,L^2(\Omega))\cap L^2(R^+,H^1(\Omega)),\quad \frac{\partial s}{\partial t}\in L^2(R^+,H^{-1}(\Omega))$$

with the following estimates: For any $T>0$,

$$\sup_{0\le t\le T}\int_\Omega|s(x,t)|^2+\int_0^T\int_\Omega|s|^2+\int_0^T\int_\Omega|\nabla s|^2\le C\int_\Omega|s_0|^2+C\int_0^T\int_\Omega|J_e|^2, \tag{3.1}$$

and

$$\int_0^T\int_\Omega|\frac{\partial m}{\partial t}|^2+\sup_{0\le t\le T}\int_\Omega|\nabla m(x,t)|^2\le C\int_\Omega|s_0|^2+C\int_0^T\int_\Omega|J_e|^2+C\int_\Omega|\nabla m_0|^2, \tag{3.2}$$

where C only depends on the coefficients (α,β) of the equation (1.1)*, in particular, C is independent of $T>0$.*

Remark 3.2. Because of the highly nonlinear terms and that the embedding $H^1(\Omega)\hookrightarrow L^\infty(\Omega)$ fails for dimension greater than one, we cannot expect more regularity than that stated in Lemma 3.1. One can also see from (3.2) that if the initial data are sufficiently small

$$C\|s_0\|^2+C\|J_e\|^2_{L^2(R^+\times\Omega)}+C\|\nabla m_0\|^2<\lambda_0, \tag{1.3}$$

one can keep $\|\nabla m(\cdot,t)\|^2<\lambda_0$ for all time $t>0$.

Below we will give the H^2 *a priori* estimates. We will use the following equivalent form for m,

$$(1+\alpha^2)\frac{\partial m}{\partial t}=-m\times(\Delta m+s)-\alpha m\times(m\times(\Delta m+s)).$$

Lemma 3.3. *Assume that $J_e\in H^2(R^+\times\Omega)$, $s_0\in H^2(\Omega)$ and $m_0\in H^2(\Omega)$. Then the smooth solution (s,m) satisfies*

$$\nabla^2 m\in L^\infty(R^+,L^2)\quad\textit{and}\quad\nabla^3 m\in L^2(R^+,L^2), \tag{3.3}$$

provided the smallness condition (1.3) *holds. Further more, one can get $\nabla m\in L^2(R^+,L^\infty)$ by sobolev embedding $H^2\hookrightarrow L^\infty$ in dimension 2.*

Proof. Denote $D^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}$, where α is a multi-index, $\alpha_1,\ \alpha_2$ are nonnegative integers and $|\alpha| = \alpha_1 + \alpha_2$. For simplicity, we will use D^k to denote any kind of the differential operator D^α with $|\alpha| = k$.

Differentiating equation (2.2) with D^2, taking inner product with D^2u and integrating over Ω, we have

$$\begin{aligned} \frac{1+\alpha^2}{2}\frac{d}{dt}\|D^2m\|^2 &= -\int_\Omega \langle D^2(m\times(\Delta m+s)), D^2m\rangle \\ &\quad + \alpha\int_\Omega \langle \Delta D^2m, D^2m\rangle + \alpha\int_\Omega\langle D^2(|\nabla m|^2m), D^2m\rangle \\ &\quad - \alpha\int_\Omega\langle D^2((m\cdot s)m), D^2m\rangle + \alpha\int_\Omega\langle D^2s, D^2m\rangle \\ &= I + II + III + IV + V. \end{aligned} \tag{3.4}$$

• Estimates of term $I =: VI + VII$.

$$\begin{aligned} VI &= |\int_\Omega \langle D^2(m\times\nabla m), \nabla D^2m\rangle| \\ &= |\int_\Omega\langle D^2m\times\nabla m + 2Dm\times D\nabla m + m\times\nabla D^2m, \nabla D^2m\rangle| \\ &\le C\|\nabla m\|_{L^4(\Omega)}\|\nabla^2m\|_{L^4(\Omega)}\|\nabla^3m\|. \end{aligned}$$

Using Gagliardo-Nirenberg inequality, we have (in dimension $d = 2$),

$$\|\nabla m\|_{L^4} \le C\|\nabla m\|_{L^2}^{\frac34}\|\nabla m\|_{H^2}^{\frac14} \tag{3.5}$$

$$\|D^2m\|_{L^4} \le C\|\nabla m\|_{L^2}^{\frac14}\|\nabla m\|_{H^2}^{\frac34}. \tag{3.6}$$

Then writing the H^2-norm explicitly and using the ε-Young's inequality, we can bound the term VI by

$$\begin{aligned} |VI| &\le (C\|\nabla m\|_{L^2} + \varepsilon)\|\nabla\Delta m\|_{L^2}^2 \\ &\quad + C\left(\|\nabla m\|_{L^2} + \|\nabla m\|_{L^2}^2\right)\|\Delta m\|_{L^2}^2 + C\|\nabla m\|_{L^2}^4, \end{aligned} \tag{3.7}$$

where the constant C can be different from line to line.

$$\begin{aligned} |VII| &= |\int_\Omega\langle D(m\times s), D^3m\rangle| \\ &\le \varepsilon\|D^3m\|^2 + C(\varepsilon)\|Ds\|^2 + C(\varepsilon)(\|Dm\|^4 + \|s\|^4) \\ &\le \varepsilon\|D^3m\|^2 + C(\varepsilon)(\|\nabla m\|^2\|\Delta m\|^2 + \|s\|^2\|\nabla s\|^2 \\ &\quad + \|\nabla s\|^2 + \|s\|_{L^2}^4 + \|\nabla m\|_{L^2}^4). \end{aligned} \tag{3.8}$$

Since $s, \nabla m \in L^\infty(0,T;L^2)$ and $\nabla s, \Delta m \in L^2(0,T;L^2)$, the last term on the right is integrable on $[0,T]$ for any $T > 0$.

• Estimates of term II. Since we consider the periodic case, no boundary terms appears, thus

$$\alpha\int_\Omega\langle\Delta D^2m, D^2m\rangle = -\alpha\|\nabla D^2m\|^2. \tag{3.9}$$

• Estimates of term III. Since we have

$$\|D(|\nabla m|^2m)\| \le \|\nabla m\|_{L^6(\Omega)}^3 + 2\|m\|_{L^\infty(\Omega)}\|\nabla m\|_{L^4(\Omega)}\|D\nabla m\|_{L^4(\Omega)},$$

the term III can be estimated by

$$
\begin{aligned}
|III| &= \Big|\int_\Omega < D((|\nabla m|^2 m)), D^3 m > \Big| \\
&\le (C\|\nabla m\| + \varepsilon)\|\nabla\Delta m\|^2 + C\left(\|\nabla m\|_{L^2} + \|\nabla m\|_{L^2}^2\right)\|\Delta m\|_{L^2}^2 \\
&\quad + C\|\nabla m\|_{L^2}^4,
\end{aligned}
\tag{3.10}
$$

thanks to the Gagliardo-Nirenberg inequality (GN).

• Estimates of the last two terms IV and V.

$$
\begin{aligned}
|IV| + |V| &\le \varepsilon\|D^3 m\|^2 + C(\varepsilon)\left(\|D((m\cdot s)m)\|_{L^2}^2 + \|Ds\|_{L^2}^2\right) \\
&\le \varepsilon\|D^3 m\|^2 + C(\varepsilon)(\|\nabla s\|_{L^2}^2 + \|s\|_{L^2}^4 + \|\nabla m\|_{L^2}^4) \\
&\le \varepsilon\|D^3 m\|^2 + C(\varepsilon)(\|\nabla s\|_{L^2}^2 + \|s\|_{L^2}^4 + \|s\|_{L^2}^2\|\nabla s\|_{L^2}^2 \\
&\quad + \|\nabla m\|_{L^2}^4 + \|\nabla m\|_{L^2}^2\|\nabla^2 m\|_{L^2}^2),
\end{aligned}
\tag{3.11}
$$

where in the last inequality, we used the Gagliardo-Nirenberg inequality (GN). Summarizing (3.4)-(3.11), we have

$$
\begin{aligned}
&\frac{1+\alpha^2}{2}\frac{d}{dt}\|\nabla^2 m\|_{L^2(\Omega)}^2 + \alpha\|\nabla^3 m\|_{L^2(\Omega)}^2 \\
&\le (C\|\nabla m\|_{L^2} + 4\varepsilon)\|\nabla^3 m\|_{L^2}^2 + C(\|\nabla m\|_{L^2} + \|\nabla m\|_{L^2}^2)\|\nabla^2 m\|_{L^2}^2 \\
&\quad + C(\|\nabla m\|_{L^2}^4 + \|\nabla s\|_{L^2}^2 + \|s\|_{L^2}^4 + \|s\|_{L^2}^2\|\nabla s\|_{L^2}^2).
\end{aligned}
\tag{3.12}
$$

By Remark 3.2, one can keep $\|\nabla m(t)\|_{L^2(\Omega)}^2 \le \lambda_0$ for all time $t > 0$ provided (1.3) holds. Thus there exists a $\lambda_0 > 0$ such that if (1.3) holds, we have

$$
C\|\nabla m\|_{L^2} < \frac{\alpha}{4}.
$$

Furthermore, set $4\varepsilon < \frac{\alpha}{4}$, we then have

$$
C\|\nabla m\|_{L^2} + 4\varepsilon < \frac{\alpha}{2}.
$$

This together with (3.12) insures us to deduce the Gronwall-type inequality for $\|\nabla^2 m\|_{L^2(\Omega)}^2$:

$$
\begin{aligned}
&\frac{d}{dt}\|\nabla^2 m\|_{L^2(\Omega)}^2 + \frac{\alpha}{2}\|\nabla^3 m\|_{L^2(\Omega)}^2 \\
&\le C(\|\nabla m\|_{L^2} + \|\nabla m\|_{L^2}^2)\|\nabla^2 m\|_{L^2}^2 \\
&\quad + C(\|\nabla m\|_{L^2}^4 + \|\nabla s\|_{L^2}^2 + \|s\|_{L^2}^4 + \|s\|_{L^2}^2\|\nabla s\|_{L^2}^2).
\end{aligned}
\tag{3.13}
$$

Since both the coefficient of $\|\nabla^2 m\|_{L^2(\Omega)}^2$ and the second term on the right hand side are integrable by Lemma 3.1, we deduce by Gronwall inequality that for any $T > 0$,

$$
\sup_{0\le t\le T}\|\nabla^2 m\|_{L^2(\Omega)}^2 \le C, \quad \forall T > 0. \tag{3.14}
$$

Then integrating (3.13), we have

$$
\|\nabla^3 m\|_{L^2(0,T;L^2(\Omega))}^2 \le C, \quad \forall T > 0, \tag{3.15}
$$

and by Sobolev embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, we get $\nabla m \in L^2(0,T;L^\infty(\Omega))$. □

Remark 3.4. It can be easily seen from this Lemma and the expression (2.2), that $\partial_t m \in L^\infty(0, R^+; L^2(\Omega))$ for any positive times $T > 0$. Furthermore, differentiating equation (2.2) with respect to spatial variable x, we have $\partial_t \nabla m \in L^2(0, R^+; L^2(\Omega))$. This observation is rather simple, thus we omit the details here.

Below, we will focus ourselves on the estimates for the s variable.

Lemma 3.5. *Under the conditions of lemma 3.3, we have*

$$\frac{\partial s}{\partial t} \in L^2(R^+, L^2(\Omega)), \quad s \in L^\infty(R^+, H^1(\Omega))$$

with the estimation: for any $T > 0$,

$$\begin{aligned} &\int_0^T \|\frac{\partial s}{\partial t}\|^2_{L^2(\Omega)} + \sup_{0\le t\le T} \|s\|^2_{H^1(\Omega)} \\ &\le Ce^{C\|\nabla m\|^2_{L^2(0,T;L^\infty)}} \Big(\|s_0\|^2_{H^1(\Omega)} + \sup_{0\le t\le T} \|\nabla m\|^2_{L^2(\Omega)} \|\nabla m\|^2_{L^2(0,T;L^\infty(\Omega))} \\ &\quad + \|\nabla J_e\|^2_{L^2([0,T]\times\Omega)}\Big). \end{aligned} \tag{3.16}$$

Proof. Taking the inner product of s-equation with $\frac{\partial s}{\partial t}$, and then integrating over Ω, we get

$$\int_\Omega |\frac{\partial s}{\partial t}|^2 + \frac{1}{2}\frac{d}{dt}\int_\Omega |s|^2 = -\int_\Omega \langle \operatorname{div} J_s, \frac{\partial s}{\partial t}\rangle - \int_\Omega \langle s\times m, \frac{\partial s}{\partial t}\rangle =: I + II. \tag{3.17}$$

Since the second term on the right can be easily controlled by

$$|II| \le \varepsilon \int_\Omega |\frac{\partial s}{\partial t}|^2 + C(\varepsilon)\int_\Omega |s|^2, \tag{3.18}$$

we need only to handle the first term I on the right hand side carefully. Using the notations introduced in Section 2, we can rewrite I as follows

$$I = -\int_\Omega \langle \operatorname{div}(m\otimes J_e), \frac{\partial s}{\partial t}\rangle + \int_\Omega \langle A(m)\Delta s, \frac{\partial s}{\partial t}\rangle - \beta\int_\Omega \langle \widetilde{\operatorname{div}}(m\otimes(\nabla s\cdot m)), \frac{\partial s}{\partial t}\rangle. \tag{3.19}$$

The first and the last term on the right side of (3.19) can be estimated by

$$|\int_\Omega \langle \operatorname{div}(m\otimes J_e), \frac{\partial s}{\partial t}\rangle| \le \varepsilon\int_\Omega |\frac{\partial s}{\partial t}|^2 + C(\varepsilon)\Big(|J_e|^2_{L^\infty(\Omega)}\int_\Omega |\nabla m|^2 + \int_\Omega |\nabla J_e|^2\Big), \tag{3.20}$$

and

$$|\int_\Omega \langle \widetilde{\operatorname{div}}(m\otimes(\nabla s\cdot m)), \frac{\partial s}{\partial t}\rangle| \le \varepsilon\int_\Omega |\frac{\partial s}{\partial t}|^2 + C(\varepsilon)|\nabla m|^2_{L^\infty}\int_\Omega |\nabla s|^2 \tag{3.21}$$

respectively, where $\nabla m \in L^\infty(0, T; L^2(\Omega))$ and $J_e \in H^2(R^+\times\Omega)$.

In the following, we focus on estimates of the second term of (3.19). For this purpose, write

$$A\Delta s = A\nabla\cdot\nabla s = \nabla\cdot(A\nabla s) - (\nabla\cdot A)\cdot\nabla s.$$

Then the second term on the right of (3.19) can be rewritten as

$$\begin{aligned} \int_\Omega \langle A\Delta s, \frac{\partial s}{\partial t}\rangle &= -\frac{1}{2}\frac{d}{dt}\int_\Omega \langle A\nabla s, \nabla s\rangle - \frac{1}{2}\int_\Omega \langle (\partial_t A)\nabla s, \nabla s\rangle - \int_\Omega \langle \nabla\cdot A\cdot\nabla s, \partial_t s\rangle \\ &=: III + IV + V \end{aligned}$$

• Since A is strictly positively definite, we can write III as

$$III = -\frac{1}{2}\frac{d}{dt}\int_\Omega \langle A^{1/2}\nabla S, A^{1/2}\nabla S\rangle \tag{3.22}$$

• Estimates of term IV. From the positivity of A in (2.1), we know that for any vector $\xi \in \mathbb{R}^3$, we have $\lambda^2|\xi|^2 \le |A\xi|^2$. This together with the s-equation in (2.3)

$$A\Delta s = \frac{\partial s}{\partial t} + \operatorname{div}(m \otimes J_e) + \beta\widetilde{\operatorname{div}}(m \otimes (\nabla s \cdot m)) + s + s \times m \tag{3.23}$$

implies

$$\begin{aligned}
\lambda^2\|\Delta s\|_{L^2}^2 &\le \|A\Delta s\|_{L^2}^2 \\
&\le \|\frac{\partial s}{\partial t}\|_{L^2}^2 + 2\|s\|_{L^2}^2 + C\|\nabla m\|_{L^2}^2 + \|\nabla J_e\|_{L^2}^2 + C\|\nabla m\|_{L^\infty}^2\|\nabla s\|_{L^2}^2.
\end{aligned}$$

On the other hand, by lemma 3.1, lemma 3.3 and (2.2), we can choose ε_1 small enough to satisfy $\varepsilon_1\|m_t\|_{L^2(\Omega)} < \frac{1}{8}\lambda^2$ for all $t \ge 0$. Therefore, we can estimate IV by

$$\begin{aligned}
|IV| &\le \int_\Omega |m_t||\nabla s|^2 \le \|m_t\|_{L^2}\|\nabla s\|_{L^4}^2 \\
&\le C\|m_t\|_{L^2}\|\nabla s\|_{H^1}\|\nabla s\|_{L^2} \\
&\le \varepsilon_1\|m_t\|_{L^2}\|\Delta s\|_{L^2}^2 + C(\varepsilon_1)\|m_t\|_{L^2}\|\nabla s\|_{L^2}^2 \\
&\le \frac{1}{8}\|\frac{\partial s}{\partial t}\|_{L^2}^2 + \left(\|s\|_{L^2}^2 + C\|\nabla m\|_{L^2}^2 + \|\nabla J_e\|_{L^2}^2\right) \\
&\quad + C\left(\|\nabla m\|_{L^\infty}^2 + \|m_t\|_{L^2}\right)\|\nabla s\|_{L^2}^2.
\end{aligned}$$

• Estimates of term V.

$$|V| \le \varepsilon\|\frac{\partial s}{\partial t}\|_{L^2}^2 + C(\varepsilon)\|\nabla m\|_{L^\infty}^2\|\nabla s\|_{L^2}^2. \tag{3.24}$$

Finally, since A is positively definite and can be bounded from above and below by

$$\sqrt{\lambda}\|\nabla s\|_X \le \|A^{1/2}\nabla s\|_X \le \sqrt{\Lambda}\|\nabla s\|_X, \tag{3.25}$$

we can regard $\|A^{1/2}\nabla s\|_X$ and $\|\nabla s\|_X$ as the same in the sequel. From (3.17) and the above estimates, if we set ε small enough, say $\varepsilon < \frac{1}{8}$, we can get the Gronwall inequality:

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\frac{\partial s}{\partial t}|^2 + \frac{1}{2}\frac{d}{dt}\int_\Omega |s|^2 + \frac{d}{dt}\int_\Omega |\nabla s|^2 \\
&\le C\left(\|\nabla m\|_{L^\infty}^2 + \|m_t\|_{L^2} + \|m_t\|_{L^2}\|\nabla m\|_{L^\infty}^2\right)\|\nabla s\|_{L^2}^2 \\
&\quad + C\|s\|_{L^2}^2 + C\Big(\|J_e\|_{L^\infty}^2\|\nabla m\|_{L^2}^2 + \|\nabla J_e\|_{L^2}^2 + \|m_t\|_{L^2}\|s\|_{L^2}^2 \\
&\quad + \|m_t\|_{L^2}\|\nabla m\|_{L^2}^2\Big).
\end{aligned} \tag{3.26}$$

By the previous lemmas, $|\nabla m| \in L^2(0,T;L^\infty)$, $\nabla m \in L^\infty(0,T;L^2)$ and $|m_t| \in L_t^\infty L_x^2 \cap L_t^2 L_x^2$, thus both the coefficients before $\|s\|_{L^2}^2$ and $\|\nabla s\|_{L^2}^2$ and the last term are integrable in time. Then the Gronwall inequality immediately implies

$$\frac{\partial s}{\partial t} \in L^2(R^+, L^2), \quad s \in L^\infty(R^+, H^1(\Omega)), \tag{3.27}$$

with the estimates (3.16). This concludes the proof. $\square$

Lemma 3.6. *Under the conditions of Lemma 3.3, we have $s \in L^2(R^+, H^2(\Omega))$. Indeed, by Lemma 3.1 and Lemma 3.5, we get that the right hand side of* (3.23) *belongs to $L^2(\Omega)$ for a.e. $0 \le t \le T$, which implies that $\Delta s \in L^2(\Omega)$ for a.e. $0 \le t \le T$. On the other hand*

$$
\begin{aligned}
\|\widetilde{\operatorname{div}}(m \otimes (\nabla s \cdot m))\|^2_{L^2(0,T;L^2)} &\le C \int_0^T \|\nabla m\|^2_{L^\infty} \|\nabla s\|^2_{L^2} dt \\
&\le C \|\nabla m\|^2_{L^2(0,T;L^\infty)} \sup_{0 \le t \le T} \|\nabla s\|^2_{L^2}.
\end{aligned}
$$

By equation (3.23)*, we immediately have $\|\Delta s\|_{L^2(\Omega)} \in L^2(0,T)$ for any $T > 0$. We can also get $s \in L^2(R^+, L^\infty(\Omega))$ by embedding $H^2 \hookrightarrow L^\infty$ in dimension 2.*

However we can expect more about the regularity of the s-variable. In the following lemma, we improve the regularity of the s-variable.

Lemma 3.7. *Suppose that J_e is smooth, and $(s_0, m_0) \in H^2(\Omega) \times H^2(\Omega)$ is small in the sense of* (1.3)*, then for any $T > 0$, the solution of* (1.1) *satisfy*

$$
\partial_t s \in L^\infty(0,T;L^2), \quad \partial_t \nabla s \in L^2(0,T;L^2), \tag{3.28}
$$

$$
s \in L^\infty(0,T;H^2), \quad s \in L^2(0,T;H^3). \tag{3.29}
$$

Also, for s'', we have

$$
s'' \in L^2(0,T;H^{-1}(\Omega)). \tag{3.30}
$$

Proof. 1. Differentiating the s-equation with respect to t, taking inner product with $\partial_t s$ and then integrating over Ω, we obtain

$$
\frac{1}{2}\frac{d}{dt}\|\partial_t s\|^2 = \int_\Omega \langle \partial_t J_s, \partial \nabla s \rangle - \|\partial_t s\|^2 - \int_\Omega \langle s \times \partial_t m, \partial_t s \rangle. \tag{3.31}
$$

The last term can be estimated as

$$
|\int_\Omega \langle s \times \partial_t m, \partial_t s \rangle| \le C\|\partial_t s\|^2 + C\|\partial_t m\|^2 \|s\|^2_{L^\infty(\Omega)}, \tag{3.32}
$$

where the second term on the right is integrable on $[0,T]$ since $s \in L^2(R^+, L^\infty(\Omega))$ by Lemma 3.6 and $\partial_t m \in L^\infty(0,T;L^2(\Omega))$ by Remark 3.2. The first term on the right of (3.31) can be estimated as

$$
\begin{aligned}
\int_\Omega \langle \partial_t J_s, \partial_t \nabla s \rangle &= \int_\Omega \langle \partial_t (m \otimes J_e), \partial_t \nabla s \rangle - \|\partial_t \nabla s\|^2 \\
&\quad + \beta \int_\Omega \langle \partial_t (m \otimes (\nabla s \cdot m)), \partial_t \nabla s \rangle \\
&\le -(1 - \beta - \varepsilon)\|\partial_t \nabla s\|^2 + C(\varepsilon)\|\partial_t (m \otimes J_e)\|^2 \\
&\quad + C(\varepsilon)\|\partial_t m\|^2_{L^2}\|\partial_t m\|^2_{H^1} + C(\varepsilon)\|\nabla s\|^2 \|\nabla s\|^2_{H^1},
\end{aligned} \tag{3.33}
$$

where in the last step, we used the Gagliardo-Nirenberg inequality, and $\|\partial_t (m \otimes J_e)\|^2$ can be bounded by

$$
\|\partial_t (m \otimes J_e)\|^2 \le \|\partial_t J_e\|^2 + C\|J_e\|^2_{L^\infty(\Omega)}\|\partial_t m\|^2. \tag{3.34}
$$

Combining (3.31)-(3.34), and setting ε small enough, say $\varepsilon = \frac{1-\beta}{2}$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\partial_t s\|^2 + \delta\|\partial_t \nabla s\|^2 + \frac{1}{2}\|\partial_t s\|^2 \\ &\le C\|\partial_t s\|^2 + C\Big(\|\partial_t m\|^2\|s\|^2_{L^\infty(\Omega)} + \|\partial_t J_e\|^2 + \|J_e\|^2_{L^\infty(\Omega)}\|\partial_t m\|^2 \\ &\quad + (\|\partial_t m\|^2\|\partial_t m\|^2_{H^1} + \|\nabla s\|^2\|\nabla s\|^2_{H^1}\Big), \end{aligned} \tag{3.35}$$

where $\delta = (1-\beta) - \varepsilon > 0$ and the terms in the parentheses $(\cdots)$ is integrable due to the above lemmas. Applying the Gronwall lemma and integrating on $[0,T]$, we have (3.28).

2. Differentiating the s-equation in (1.1) with Δ, taking inner product with Δs and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\|\Delta s\|^2 = \int \langle \Delta J_s, \nabla\Delta s\rangle - \|\Delta s\|^2 - \int \langle \nabla(s\times m), \nabla \Delta s\rangle. \tag{3.36}$$

The last term of the right hand side can be controlled by

$$|\int \langle \nabla(s\times m), \nabla\Delta s\rangle| \le \varepsilon\|\nabla\Delta s\|^2 + C(\varepsilon)(\|\nabla s\|^2 + \|s\|^2_{L^\infty}\|\nabla m\|^2), \tag{3.37}$$

where the second term on the right of which is integrable on $[0,T]$ thanks to lemma 3.6. For the first term on the right of (3.36), we have

$$\begin{aligned} &\int \langle \Delta J_s, \nabla \Delta s\rangle \\ &= \int \langle \Delta(m\otimes J_e, \nabla\Delta s)\rangle - \|\nabla\Delta s\|^2 + \beta\int\langle \Delta(m\otimes(\nabla s\cdot m)), \nabla\Delta s\rangle \\ &=: I + II + III \end{aligned} \tag{3.38}$$

For I, we have

$$\begin{aligned} |I| &\le \varepsilon\|\nabla\Delta s\|^2 + C(\varepsilon)(\|J_e\|^2_{L^\infty}\|\Delta m\|^2 + \|\Delta J_e\|^2) \\ &\quad + C(\varepsilon)(\|\nabla m\|^4_{L^4} + \|\nabla J_e\|^4_{L^4}) \\ &\le \varepsilon\|\nabla\Delta s\|^2 + C(\varepsilon)(\|J_e\|^2_{L^\infty}\|\Delta m\|^2 + \|\Delta J_e\|^2) \\ &\quad + C(\varepsilon)(\|\nabla m\|^2\|\nabla m\|^2_{H^1} + \|\nabla J_e\|^2\|\nabla J_e\|^2_{H^1}), \end{aligned} \tag{3.39}$$

where the last two terms are integrable on $[0,T]$. For III, we have

$$\begin{aligned} |III| &\le (\beta+\varepsilon)\|\nabla\Delta s\|^2 + C(\varepsilon)\Big(\|\nabla m\|^2_{L^\infty}\|\Delta s\|^2 + \|\nabla s\|^2\|\nabla s\|^2_{H^1} \\ &\quad + \|\Delta m\|^2\|\Delta m\|^2_{H^1} + \|\nabla m\|^2_{L^\infty}(\|\nabla s\|^2\|\nabla s\|^2_{H^1} + \|\nabla m\|^2\|\nabla m\|^2_{H^1})\Big). \end{aligned} \tag{3.40}$$

Setting ε small enough, say $\varepsilon = \frac{1-\beta}{4}$, from (3.36)–(3.40), we have

$$\frac{1}{2}\frac{d}{dt}\|\Delta s\|^2 + \delta\|\nabla\Delta s\|^2 \le C\Big(\|\nabla m\|^2_{L^\infty} + \|\nabla s\|^2 + \|\nabla m\|^2_{L^\infty}\|\nabla s\|^2\Big)\|\Delta s\|^2_{L^2} + R, \tag{3.41}$$

where $\delta = (1-\beta) - 3\varepsilon > 0$ and the remainder term R is integrable on $[0,T]$. Thus the Gronwall inequality implies our result (3.29).

3. Prove (3.30). Differentiating the s-equation with respect to t, we get

$$\frac{\partial^2 s}{\partial t^2} = -\operatorname{div}\partial_t J_s - \partial_t s - \partial_t(s\times m). \tag{3.42}$$

Then for any $v \in H_0^1(\Omega)$, we have

$$|\int \langle \partial_t s, v\rangle| \le \|\partial_t s\|\|v\|, \tag{3.43}$$

$$|\int \langle \partial_t (s\times m), v\rangle| \le (\|\partial_t s\| + \|s\|_{L^\infty(\Omega)}\|\partial_t m\|)\|v\|. \tag{3.44}$$

For the first term on the right of (3.42), we have

$$-\int \langle \operatorname{div} \partial_t J_s, v\rangle = \int \langle \partial_t J_s, \nabla v\rangle, \tag{3.45}$$

which can be bounded by

$$|\int \langle \partial_t J_s, \nabla v\rangle| \le C\|\nabla v\|_{L^2} \tag{3.46}$$

thanks to the above Lemmas 3.5, Remark 3.4 and the first two parts of this lemma. Thus $s'' \in L^2(0,T;H^{-1})$ for all $T>0$. This completes the proof of the lemma. □

Finally, These *a priori* estimates imply well-posedness of the problem in H^2. We can do this by approximating the initial data (s_0, m_0) by smooth data $(s_0^l, m_0^l) \in C^\infty$ such that

$$(s_0^l, m_0^l) \to (s_0, m_0) \quad \text{in } H^2 \times H^2$$

as $l \to \infty$. Thus the *a priori* estimates imply a uniform (in l) H^2 bound on the approximating solutions (s^l, m^l). Therefore we can extract a subsequence (if necessary) $\{(s^l, m^l)\}$ which converges weakly in H^2 to an H^2 solution (s,m) of the problem. The *a priori* estimates also implies that the solution is global.

For the reader's convenience, we summarize the above lemmas in the following theorem.

Theorem 3.8. *Suppose that $J_e \in H^2(R^+ \times \Omega)$. Let $(s_0, m_0) \in H^2(\Omega)$ satisfy the smallness condition* (1.3). *Then there exists a global solution in H^2 such that for any $T>0$, we have*

$$m \in L^\infty(0,T;H^2), \quad m \in L^2(0,T;H^3),$$
$$\partial_t m \in L^\infty(0,T;L^2), \quad \partial_t \nabla m \in L^2(0,T;L^2);$$

for s,

$$s \in L^\infty(0,T;H^2), \quad s \in L^2(0,T;H^3),$$
$$\partial_t s \in L^\infty(0,T;L^2), \quad \partial_t \nabla s \in L^2(0,T;L^2),$$

and for s'', we have $s'' \in L^2(0,T;H^{-1}(\Omega))$.

4. Conclusions

In the previous section, we obtained some *a priori* estimates of the solution to ensure that the solution is in $L^\infty(0,T;H^2\Omega)$ for all positive times $T>0$ for given initial data in $H^2(\Omega)$. In this section, we provide the higher order *a priori* estimates of the solution of (1.1) to show that the solution is indeed smooth. In the sequel, we will abuse the notation of D with ∇ and we get our result by induction.

Lemma 4.1. *Let $k \geq 3$. Suppose $J_e \in H^k(R^+ \times \Omega)$, and (s,m) is a smooth solution of the equation* (1.1) *with initial data (s_0, m_0). Furthermore, if the smallness condition* (1.3) *holds, then*

$$
\begin{gathered}
m \in L^\infty(0,T;H^k(\Omega)), \quad m \in L^2(0,T;H^{k+1}(\Omega)) \\
s \in L^\infty(0,T;H^k(\Omega)), \quad s \in L^2(0,T;H^{k+1}(\Omega)).
\end{gathered}
$$

Proof. We prove this lemma by induction. Let us first assume that for the initial data $s_0, m_0 \in H^{k-1}$, we have proved our result; i.e.,

$$
\begin{gathered}
m \in L^\infty(0,T;H^{k-1}(\Omega)), \quad m \in L^2(0,T;H^{k}(\Omega)) \\
s \in L^\infty(0,T;H^{k-1}(\Omega)), \quad s \in L^2(0,T;H^{k}(\Omega)).
\end{gathered}
$$

We divide this proof into two parts and we omit the constant α, which will not influence the result.

1. In this part, we prove the regularity for m. Differentiating (2.2) with D^k, and taking inner product with $D^k m$, then integrating over Ω leads us to

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|D^k m\|^2 + \|\nabla D^k m\|^2 \\
&= \langle D^k(m \times \nabla m), \nabla D^k m\rangle - \langle D^k(m \times s), D^k m\rangle \\
&+ \langle D^{k-1}(|\nabla m|^2 m), D^{k+1} m\rangle - \langle D^{k-1}((m\cdot s)m), D^{k+1}m\rangle + \langle D^k s, D^k m\rangle,
\end{aligned} \tag{4.1}
$$

where $\langle \cdot,\cdot\rangle$ denotes the inner product over Ω. Below we will bound the terms one by one. The last term can be estimated

$$
|\langle D^k s, D^k m\rangle| \leq \|D^k s\|^2 + \|D^k m\|^2,
$$

where the first term on the right is integrable on $[0,T]$. Recalling Kato's inequality, we have

$$
\|D^l(fg) - fD^l g\|_{L^p} \leq C\|Df\|_{L^\infty}\|D^{l-1}g\|_{L^p} + C\|D^l f\|_{L^p}\|g\|_{L^\infty}
$$

for any two functions f and g, see [8]. Then the second term can be estimated by

$$
\begin{aligned}
&|\langle D^k(m\times s), D^k m\rangle| \\
&\leq \|D^k m\|^2 + \|D^k(m\times s)\|^2 \\
&\leq C\|Ds\|_{L^\infty}^2\|D^{k-1}m\|^2 + C\|D^k s\|^2 + C\left(\|s\|_{L^\infty}^2 + 1\right)\|D^k m\|^2,
\end{aligned} \tag{4.2}
$$

where both the coefficients of $\|D^k m\|^2$ and the the first two terms are integrable on $[0,T]$ by the induction assumption. The fourth term can be estimated the same. Since

$$
D^k(m\times\nabla m) = D^k m \times \nabla m + m \times D^k\nabla m + \sum_{h=1}^{k-1} c_h D^h m \times D^{k-h}\nabla m, \tag{4.3}
$$

the first term can be estimated by

$$
\begin{aligned}
&|\langle D^k(m\times\nabla m), \nabla D^k m\rangle| \\
&\leq |\langle D^k m\times\nabla m, \nabla D^k m\rangle| + \sum_{h=1}^{k-1} c_h|\langle D^h m\times D^{k-h}\nabla m, \nabla D^k m\rangle| \\
&\leq \frac{1}{4}\|\nabla D^k m\|^2 + C\|\nabla^k m\|^2 + R,
\end{aligned}
$$

where the remainder term R depends on the lower order derivatives of m and is integrable on $[0, T]$.

Finally from (4.2), we have

$$\begin{aligned}&\|D^{k-1}(|\nabla m|^2 m)\|_{L^2}\\&\leq \|mD^{k-1}(|\nabla m|^2)\| + C\|\nabla m\|_{L^\infty}\|D^{k-2}|\nabla m|^2\| + C\|D^{k-1}m\|\||\nabla m|^2\|_{L^\infty}\\&\leq C_1\|\nabla^k m\| + C_2.\end{aligned}$$

Then the third term of (4.1) can be estimated by

$$|\langle D^{k-1}(|\nabla m|^2 m), D^{k+1}m\rangle| \leq \frac{1}{4}\|\nabla D^k m\|^2 + C\|\nabla^k m\|^2 + C. \tag{4.4}$$

Summarizing (4.1)-(4.4), we arrive at the Gronwall type inequality

$$\frac{d}{dt}\|D^k m\|^2 + \|D^{k+1}m\|^2 \leq C\|\nabla^k m\|^2 + C,$$

from which we deduce that for all $T > 0$,

$$\sup_{0\leq t\leq T}\|\nabla^k m(\cdot, t)\|_{L^2(\Omega)} \leq C, \quad \|\nabla^{k+1}m\|_{L^2(0,T;L^2(\Omega))} \leq C.$$

Furthermore, we have $\|\nabla m\|_{L^\infty(\Omega)} \leq C$.

2. In this part, we prove the regularity for s. This is similar with that in the first part. Differentiating the s equation with D^k, and then taking inner product with $D^k s$ over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\|D^k s\|^2 = \langle D^k J_s, \nabla D^k s\rangle - \|D^k s\|^2 - \langle D^k(s \times m), D^k s\rangle = I + II + III. \tag{4.5}$$

• Estimates of I. We expand I as

$$\langle D^k J_s, \nabla D^k s\rangle + \|\nabla D^k s\|^2 = \langle D^k(m \times J_e), \nabla D^k s\rangle + \beta\langle D^k(m \otimes (\nabla s \cdot m)), \nabla D^k s\rangle,$$

where the terms on right hand side can be estimated by

$$|\langle D^k(m \times J_e), \nabla D^k s\rangle| \leq \varepsilon\|\nabla D^k s\|^2 + C\|D^k(m \times J_e)\|^2 \leq \varepsilon\|\nabla D^k s\|^2 + C$$

and

$$|\beta\langle D^k(m \otimes (\nabla s \cdot m)), \nabla D^k s\rangle| \leq (\beta + \varepsilon)\|\nabla D^k s\|^2 + C\|D^k s\|^2 + C.$$

• Estimates of III:

$$|III| \leq \|D^k s\|^2 + C, \tag{4.6}$$

where C is integrable on $[0, T]$.

Letting ε sufficiently small, we can deduce from (4.5)-(4.6) the Gronwall type inequality

$$\frac{d}{dt}\|D^k s\|^2 + \|\nabla D^k s\|^2 \leq C\|D^k s\|^2 + C,$$

from which we can deduce

$$\nabla^k s \in L^\infty(0, T; L^2(\Omega)), \quad \nabla^{k+1}s \in L^2(0, T; L^2(\Omega)),$$

which concludes the proof. □

One can also give the regularity in time as what we did in Section 3, which is standard and we omit the details here. The local existence theorem in Section 2 and the *a priori* estimates then imply global existence of solutions for system (1.1) in H^k. In particular, the above estimates hold for any $k \geq 2$. Thus if the initial data is smooth, we have global smooth solutions as stated in Theorem 1.1.

Acknowledgements. The authors thank the anonymous referee for his/her useful suggestions and comments, as well as the journal editors for their hard work.

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