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OPTIMIZATION OF THE PRINCIPAL EIGENVALUE OF THE ONE-DIMENSIONAL SCHRÖDINGER OPERATOR

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Abstract. In this paper we consider two optimization problems related to the principal eigenvalue of the one dimensional Schrödinger operator. These optimization problems are formulated relative to the rearrangement of a fixed function. We show that both problems have unique solutions, and each of these solutions is a fixed point of an appropriate function.

1. Introduction

In this paper we investigate two optimization problems related to the following one-dimensional Schrödinger eigenvalue problem:

$$\begin{gathered} -u'' + \alpha f(x) u = \lambda u \quad \text{in } (-1,1) \\ u(-1) = u(1) = 0, \end{gathered} \tag{1.1}$$

where α is a non-negative constant, $f \in L^\infty(-1,1)$ is a non-negative potential, and λ denotes an eigenvalue.

We fix $f_0 \in L^\infty(-1,1)$, and consider the rearrangement class $\mathcal{R}$ generated by f_0, see the next section for definition. By $\lambda_1(\alpha, f)$ we denote the principal eigenvalue of (1.1) for a given α and $f \in L^\infty(-1,1)$. We are interested in the following optimization problems:

$$\inf_{f\in\mathcal{R}} \lambda_1(\alpha, f) \quad \text{and} \quad \sup_{f\in\mathcal{R}} \lambda_1(\alpha, f) \tag{1.2}$$

Note that (1.1) is a scaled version of the one dimensional steady state Schrödinger eigenvalue equation governing a particle of mass m, moving in a potential $V(x)$:

$$\begin{gathered} -(\overline{h}/2m)\ u'' + V(x)\ u = \Lambda\ u \quad \text{in } (-1,1) \\ u(-1) = u(1) = 0, \end{gathered} \tag{1.3}$$

where $\bar{\text{h}}$ stands for the Planck's constant. It is well known that for (1.3) there exists a countable collection of solutions u_n (quantum states), and real numbers Λ_n (energies). Therefore a physical interpretation of the problems (1.2) is: we seek potentials that minimize (maximize) the principal energy corresponding to the state equations (1.1) relative to $\mathcal{R}$.

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There is another physical motivation for considering the problems (1.2) which we describe as follows. If we assume $f_0 = \chi_{D_0}$, the characteristic function of a given set $D_0 \subset (-1,1)$, then it turns out that $\mathcal{R}$ can be identified with the set

$$S = \{D \subset (-1,1) : \ |D| = |D_0|\},$$

where $|\cdot|$ denotes the one dimensional Lebesgue measure. Therefore, the problems (1.2) can be reformulated as

$$\inf_{D\subset(-1,1), D\in S} \lambda_1(\alpha, \chi_D) \quad \text{and} \quad \sup_{D\subset(-1,1), D\in S} \lambda_1(\alpha, \chi_D). \tag{1.4}$$

The minimization problem in (1.4) is addressed in [4], where the problem is posed in any space dimension. There, amongst other results, it is shown that the optimal solution $\hat{D}$ provides an answer to the following physical problem, specialized to one dimension:

Problem. Build a string out of two given materials (of varying densities) in such a way that the string has a prescribed mass and so that the basic frequency of the resulting string (with fixed ends) is as small as possible.

To avoid redundancy in (1.2) we impose a geometric condition on f_0; namely, we assume that f_0 has no flat sections on its graph where it is positive, see the next section for precise formulation of this condition. Clearly, taking into account this geometric stipulation rules out the identification between $\mathcal{R}$ and S, hence the problems (1.2) and (1.4) are no longer equivalent.

We would like to mention that recently Bonder [6] investigated a similar problem where the state equation is a nonlinear Schödinger eigenvalue problem. However, in that case the admissible set considered is a bounded, closed and convex subset of an appropriate Lebesgue integrable function space. The problems (1.2) cannot be considered in the context of [6], since the set of rearrangements $\mathcal{R}$ lacks both convexity and closedness in $L^\infty(-1,1)$.

Let us end this section by mentioning that we have also considered (1.2) in higher dimensions and the results will be reported in a follow up paper. We particularly want to present the work in one dimensional setting to ensure the presentation is elementary and essentially self contained.

2. Preliminaries and statement of the main result

In this section we recall the main properties of the principal eigenvalue and eigenfunction for (1.1). We also state the definition of a rearrangement class, and some well known rearrangement inequalities.

For problem (1.1), $\lambda \in \mathbb{R}$ is called an eigenvalue provided there exists a non-zero $u \in H_0^1(-1,1)$ that satisfies the integral equation

$$\int_{-1}^1 u'\phi'\,dx + \alpha \int_{-1}^1 fu\phi\,dx = \lambda \int_{-1}^1 u\phi\,dx, \quad \forall\, \phi \in C_0^\infty(-1,1)$$

The function u is called an eigenfunction corresponding to λ.

The principal (first) eigenvalue of (1.1) denoted $\lambda_1(\alpha, f)$, to emphasis dependence on α and f, is variationally formulated as follows:

$$\lambda_1(\alpha, f) = \inf_{u\in H_0^1(-1,1)\setminus\{0\}} \frac{\int_{-1}^1 u'^2\,dx + \alpha\int_{-1}^1 fu^2\,dx}{\int_{-1}^1 u^2\,dx}. \tag{2.1}$$

While it is well known that $\lambda_1(\alpha, f)$ is positive and simple, we provide a short proof of the simplicity of the principal eigenvalue based on ideas introduced in [1]. Suppose u_1 and u_2 are eigenfunctions corresponding to $\lambda_1(\alpha, f) \equiv \lambda$. Thus both of these functions are positive, and we have

$$\lambda = \frac{\int_{-1}^{1} {u_i'}^2\,dx + \alpha \int_{-1}^{1} f u_i^2\,dx}{\int_{-1}^{1} u_i^2\,dx}, \quad i = 1, 2. \tag{2.2}$$

Clearly $v \equiv (u_1^2 + u_2^2)^{1/2} \in H_0^1(-1,1)$. Thus $v' = (u_1^2 + u_2^2)^{-1/2}(u_1, u_2) \cdot (u_1', u_2')$, where the dot denotes the usual dot product in $\mathbb{R}^2$. Note that from the Cauchy-Schwarz inequality we get the strict inequality $|(u_1, u_2) \cdot (u_1', u_2')|^2 < (u_1^2 + u_2^2)({u_1'}^2 + {u_2'}^2)$, unless (u_1, u_2) and (u_1', u_2') are colinear. Therefore by easy calculations, and (2.2), we find

$$\frac{\int_{-1}^{1} {v'}^2\,dx + \alpha \int_{-1}^{1} f v^2\,dx}{\int_{-1}^{1} v^2\,dx} < \lambda;$$

which contradicts (2.1). Hence (u_1, u_2) and (u_1', u_2') must be colinear, so there exists a nonzero constant k such that $(u_1, u_2) = k(u_1', u_2')$. This, in turn, implies $(u_1' u_2 - u_2' u_1)/u_2^2 = 0$. Thus $d/dx(u_1/u_2) = 0$, hence $u_1 = k u_2$. This proves that the principal eigenvalue is simple.

For a non-negative measurable f defined on the interval $(-1,1)$, the distribution function $\xi_f : [0,\infty) \to [0,\infty)$ is defined as

$$\xi_f(\gamma) = |\{x \in (-1,1) : \ f(x) \geq \gamma\}|.$$

We say the non-negative measurable functions $f, g : (-1,1) \to \mathbb{R}$ are rearrangements of each other if and only if

$$\xi_f(\gamma) = \xi_g(\gamma), \quad \forall \gamma \in [0,\infty).$$

If $f \in L^\infty(-1,1)$, and g is a rearrangement of f, then $g \in L^\infty(-1,1)$ and $\|f\|_\infty = \|g\|_\infty$.

We fix a non-negative function $f_0 \in L^\infty(-1,1)$ satisfying the following geometric condition:

(H) f_0 has negligible flat sections over its strong support; that is,

$$|f_0^{-1}(\{C\}) \cap \{f_0 > 0\}| = 0, \quad \forall\, C > 0.$$

We define the rearrangement class $\mathcal{R}$, generated by f_0, as follows

$$\mathcal{R} = \{g : (-1,1) \to [0,\infty) : \ f_0 \text{ and } g \text{ are rearrangements of each other}\}.$$

For a non-negative function h defined on $(-1,1)$, the decreasing rearrangement of h, denoted h^Δ, is defined on $(-1,1)$ as follows:

$$h^\Delta(s) = \sup\{\gamma : \ \xi_h(\gamma) \geq s\},$$

and the symmetric decreasing and increasing rearrangements of h, denoted h^*, h_*, respectively, are given by:

$$h^*(x) = h^\Delta(2|x|), \quad h_*(x) = h^\Delta(2 - 2|x|).$$

The following inequality is standard:

$$\int_{-1}^{1} f_*(x) g^*(x) dx \leq \int_{-1}^{1} f(x) g(x) dx \leq \int_{-1}^{1} f^*(x) g^*(x) dx. \tag{2.3}$$

It is also known that if $u \in H_0^1(-1,1)$ is a non-negative function, then $u^* \in H_0^1(-1,1)$ and

$$\int_{-1}^{1} {u^*}'^2 dx \le \int_{-1}^{1} u'^2 dx, \tag{2.4}$$

see for example [8]. Now we state the main results of this paper

Theorem 2.1 (Minimization). *For every $\alpha \ge 0$, f_* is the unique solution of*

$$\inf_{f\in\mathcal{R}} \lambda_1(\alpha, f),$$

that is, $\lambda_1(\alpha, f_*) = \inf_{f\in\mathcal{R}} \lambda_1(\alpha, f)$.

Theorem 2.2 (Maximization). *There is $\hat{\alpha} > 0$ such that for $\alpha \le \hat{\alpha}$, f^* is the unique solution of*

$$\sup_{f\in\mathcal{R}} \lambda_1(\alpha, f),$$

that is, $\lambda_1(\alpha, f^*) = \sup_{f\in\mathcal{R}} \lambda_1(\alpha, f)$.

3. Proofs of Theorems

In this section we give proofs of our main results.

Proof of Theorem 2.1. Let us first show that f_* is a solution. For any $\alpha \ge 0$ and $f \in \mathcal{R}$, let u denote the unique eigenfunction corresponding to $\lambda_1(\alpha, f) \equiv \lambda(f)$, satisfying the condition

$$\int_{-1}^{1} u^2(x)\, dx = 1. \tag{3.1}$$

Then from (2.1), (2.3) and (2.4) we deduce

$$\lambda(f) = \int_{-1}^{1} u'^2\, dx + \alpha \int_{-1}^{1} f u^2\, dx \ge \int_{-1}^{1} {u^*}'^2\, dx + \alpha \int_{-1}^{1} f_* {u^*}^2\, dx \ge \lambda(f_*). \tag{3.2}$$

Since $f_* = (f_0)_* \equiv F_0$, we infer $\lambda(f) \ge \lambda(F_0)$. This completes the existence part of the theorem.

To prove uniqueness we assume g is also a minimizer. Denoting the normalized eigenfunction corresponding to $\lambda(g) \equiv \lambda_1(\alpha, g)$ by u, as in (3.2) , we obtain

$$\begin{aligned} \lambda(g) &= \int_{-1}^{1} u'^2\, dx + \alpha \int_{-1}^{1} g u^2\, dx \\ &\ge \int_{-1}^{1} {u^*}'^2\, dx + \alpha \int_{-1}^{1} g u^2\, dx \\ &\ge \int_{-1}^{1} {u^*}'^2\, dx + \alpha \int_{-1}^{1} g_* {u^*}^2\, dx \\ &\ge \lambda(g_*) \ge \lambda(g). \end{aligned} \tag{3.3}$$

Thus all inequalities in (3.3) in fact equalities. In particular we deduce

(a) $\int_{-1}^{1} {u^*}'^2\, dx = \int_{-1}^{1} u'^2\, dx$
(b) $\int_{-1}^{1} g u^2\, dx = \int_{-1}^{1} g_* {u^*}^2\, dx$

At this stage we show that the graph of u has no significant flat sections. Let us consider the differential equation satisfied by u

$$-u'' + \alpha g u = \lambda(g) u. \tag{3.4}$$

Restricting (3.4) to the set where g is zero yields $-u'' = \lambda(g)u$, hence if $u = c$, c is a constant, then $c\lambda(g) = 0$, which is not possible, since $\lambda(g) > 0$, and c is positive by the positivity of the eigenfunctions. On the other hand, if we restrict (3.4) to the set where g is positive, then, assuming $u = c$, we find $\alpha c g = \lambda(g)c$, so $g = \lambda(g)/\alpha$, which is again impossible since g satisfies the geometric condition (H).

Since u has no significant flat sections on its graph, the set on which u' vanishes must have measure zero. This result in conjunction with (a) above and [7, Theorem 1.1] imply $u = u^*$ almost everywhere in $(-1, 1)$, see also [9]. So from (b) above we deduce

$$\int_{-1}^{1} g {u^*}^2 \, dx = \int_{-1}^{1} g_* {u^*}^2 \, dx.$$

Now recalling (2.3), we deduce that g and $g_* = F_0$ are both minimizers of the linear functional $\mathcal{L} : L^\infty(-1, 1) \to \infty$, defined by:

$$\mathcal{L}(h) = \int_{-1}^{1} h \, {u^*}^2 \, dx,$$

relative to $h \in \mathcal{R}$. Since the graph of u has no significant flat sections we can use [3, Lemma 2.9], [2, Theorem 3] and [5, Lemma 2.2] which guarantees that $\mathcal{L}$ has a unique minimizer relative to $\mathcal{R}$. This obviously implies $g = F_0$, as desired. $\square$

Remark 3.1. *From Burton's theory one can also establish the following functional relation*

$$F_0 = \phi_\alpha(u),$$

which holds pointwise almost everywhere in $(-1, 1)$, where ϕ_α is a decreasing function in terms of u^Δ and g_Δ. This relation shows that the largest parts of F_0 are concentrated where u is the smallest. Hence the largest parts of u must be concentrated near zero.

We now proceed to prove Theorem 2.2. Henceforth $u_{\alpha,f}$ denotes the normalized eigenfunction, see (3.1), corresponding to $\lambda_1(\alpha, f)$. We need the following lemmas.

Lemma 3.2. *If $f \in L^\infty(-1, 1)$ is a non-negative even function, then $u_{\alpha,f}$ is even.*

Proof. For simplicity we set $u \equiv u_{\alpha,f}$, and introduce $w(x) = u(-x)$. Note that w satisfies the normality condition (3.1). Moreover, observe that

$$\int_{-1}^{1} {u'}^2 \, dx = \int_{-1}^{1} {w'}^2 \, dx \quad \text{and} \quad \int_{-1}^{1} f u^2 \, dx = \int_{-1}^{1} f w^2 \, dx,$$

where in the second equation we have used $f(x) = f(-x)$. Thus we derive

$$\lambda(f) = \lambda_1(\alpha, f) = \int_{-1}^{1} {w'}^2 \, dx + \alpha \int_{-1}^{1} f w^2 \, dx.$$

Hence w is also an eigenfunction corresponding to $\lambda_1(\alpha, f)$, so by uniqueness we obtain $u = w$, as desired. $\square$

Lemma 3.3. *Suppose $f \in L^\infty(-1, 1)$ is a non-negative function such that $f = f^*$. There exists $\hat{\alpha} > 0$ such that, if $\alpha < \hat{\alpha}$, then $u_{\alpha,f} = u^*_{\alpha,f}$.*

Proof. Note that $u \equiv u_{\alpha,f}$ is an even function by Lemma 3.2. Hence to complete the proof of the lemma it suffices to show u' is non-positive on $(0,1)$. We first show that $\lambda_1(\cdot,f)$ is right continuous at zero; that is, $\lim_{\alpha\to 0^+}\lambda_1(\alpha,f)=\lambda$, where λ is the principal eigenvalue of (1.1) with $f=0$ (or $\alpha=0$). From (2.1), for any $\alpha>0$, we have

$$\lambda_1(\alpha,f)=\int_{-1}^{1} {u'}^2\,dx+\alpha\int_{-1}^{1} fu^2\,dx \ge \int_{-1}^{1} {u'}^2\,dx \ge \lambda.$$

Therefore, $\liminf_{\alpha\to 0^+}\lambda_1(\alpha,f)\ge\lambda$. Also, for $\epsilon>0$, there exists $v\in H_0^1(-1,1)$ normalized such that

$$\lambda+\epsilon \ge \int_{-1}^{1} {v'}^2\,dx \ge \lambda_1(\alpha,f)-\alpha\int_{-1}^{1} fv^2\,dx, \quad \forall\,\alpha>0.$$

Therefore we obtain $\lambda+\epsilon\ge\limsup_{\alpha\to 0^+}\lambda_1(\alpha,f)$. Since ϵ is arbitrary we get $\lambda\ge\limsup_{\alpha\to 0^+}\lambda_1(\alpha,f)$, as desired.

Next, we fix a non-negative $\phi\in H_0^1(-1,1)$. Multiplying the differential equation:

$$-u''+\alpha fu=\lambda_1(\alpha,f)u,$$

by ϕ and integrating over $(-1,1)$ yields

$$\int_{-1}^{1} u'\phi'\,dx=\int_{-1}^{1}(\lambda_1(\alpha,f)-\alpha f)u\phi\,dx. \tag{3.5}$$

Now, since $f=f^*$,

$$\lambda_1(\alpha,f)-\alpha f\ge\lambda_1(\alpha,f)-\alpha f(0), \tag{3.6}$$

and since $\lambda_1(\cdot,f)$ is right-continuous at zero, the function $q(\alpha)\equiv\lambda_1(\alpha,f)-\alpha f(0)$ is also right-continuous at zero. But since $q(0)>0$ we infer the existence of $\hat\alpha>0$ such that q is positive on $[0,\hat\alpha]$. With $\alpha\in[0,\hat\alpha]$, from (3.6) and (3.5) we deduce

$$\int_{-1}^{1} u'\phi'\,dx=\int_{-1}^{1}(\lambda_1(\alpha,f)-\alpha f)u\phi\,dx\ge q(\alpha)\int_{-1}^{1} u\phi\,dx\ge 0.$$

Hence $u''\le 0$ in the sense of distributions. Note that since $(\lambda_1(\alpha,f)-\alpha f)u\in L^\infty(-1,1)$, $u\in W^{2,p}(-1,1)$ for every $p\in[1,\infty)$. This implies that in fact $u''\le 0$ holds almost everywhere in $(-1,1)$. Therefore the following inequality holds:

$$\int_0^x u''\,dt\le 0,$$

for $x\in(0,1)$. Since u' is absolutely continuous on $[-1,1]$, we can apply the Fundamental Theorem of Calculus to deduce

$$u'(x)-u'(0)\le 0, \quad x\in(0,1)$$

Recall that u is an even function, thus clearly we have

$$(u(x)-u(-x))'=0, \quad x\in(-1,1).$$

Thus, in particular, we find $u'(0)=0$. Hence we obtain $u'(x)\le 0$ for almost every x in $(0,1)$. In fact, since u' is continuous we have $u'(x)\le 0$ for every x in $(0,1)$. This completes the proof of the lemma. □

From Lemmas 3.2 and 3.3, the following result is immediate.

Lemma 3.4. *If $\alpha \le \hat\alpha$ and $f \in L^\infty(-1,1)$, then u_{α,f^*} satisfies*
$$u_{\alpha,f^*} = (u_{\alpha,f^*})^*,$$
almost everywhere in $(-1,1)$.

Proof of Theorem 2.2. Let $\hat\alpha$ be given by Lemma 3.3, and let $\alpha \le \hat\alpha$. Fix $f \in \mathcal{R}$, and set $\lambda(f) \equiv \lambda_1(\alpha, f)$, $u_f \equiv u_{\alpha,f}$ for simplicity. Then from (2.1), (2.3) and Lemma 3.4 we have
$$\begin{aligned}\lambda(f) &\le \int_{-1}^1 u_{f^*}'^2\,dx + \alpha\int_{-1}^1 f{u_{f^*}}^2\,dx\\ &\le \int_{-1}^1 u\prime 2_{f^*}\,dx + \alpha\int_{-1}^1 f^*(u_{f^*})^{*2}\,dx\\ &= \lambda(f^*) = \lambda(F^0),\end{aligned}$$
where $F^0 \equiv f_0^*$. This completes the existence part of the theorem. To prove uniqueness, assume $g \in \mathcal{R}$ is another maximizer besides F^0. Then
$$\lambda(g) \le \int_{-1}^1 u_{g^*}'^2\,dx + \alpha\int_{-1}^1 g u_{g^*}^2\,dx \le \int_{-1}^1 u_{g^*}'^2\,dx + \alpha\int_{-1}^1 g^* u_{g^*}^2\,dx = \lambda(g^*) \le \lambda(g).$$
Thus all inequalities above are in fact equalities, hence in particular we infer
$$\int_{-1}^1 g^* u_{g^*}^2\,dx = \int_{-1}^1 g u_{g^*}^2\,dx. \tag{3.7}$$
As in the minimization case, we show that the graph of u_{g^*} has no significant flat sections. Recall that u_{g^*} satisfies the following differential equation
$$-u_{g^*}'' + \alpha g^* u_{g^*} = \lambda_1(\alpha, g^*)u_{g^*}. \tag{3.8}$$
Assume that $u_{g^*} = c$ on a subset of $E \subseteq \{g^* = 0\}$, where E has positive measure. Whence restricting (3.8) to E yields $c\lambda_1(\alpha, g^*) = 0$, which is impossible since both c and $\lambda_1(\alpha, g^*)$ are positive constants. On the other hand if we assume $u_{g^*} = c$ on a subset $K \subseteq \{g^* > 0\}$, where K has positive measure, then restricting (3.8) to K implies $\alpha c g^* = c\lambda_1(\alpha, g^*)$, from which it follows that the graph of g^* has a flat section on a set of positive measure which is not possible because g^* satisfies the condition (H). This shows that the graph of u_{g^*} has no significant flat sections.

From (3.7) and (2.3) it follows that g and g^* both maximize the linear functional $\mathcal{N} : L^\infty(-1,1) \to \mathbb{R}$ defined by
$$\mathcal{N}(h) = \int_{-1}^1 h u_{g^*}^2\,dx,$$
relative to $h \in \mathcal{R}$. However, by Burton's theory [2], u_{g^*} not having any significant flat sections on its graph, $\mathcal{N}$ has a unique maximizer relative to $\mathcal{R}$, hence we must have $g = g^* = F^0$, as desired. $\square$

Remark 3.5. As in the case of the minimization, the Burton's theory provides a functional relation as follows:
$$F^0 = \psi_\alpha(u),$$
almost everywhere in $(-1,1)$, where $u \equiv u_{F^0}$, and ψ_α is an increasing function.

4. F_0 and F^0 as fixed points

In this section we show the solutions to the optimization problems (1.2); namely, F_0 and F^0 are fixed points of appropriate operators. We begin by proving a continuity result for the principal eigenvalue.

Lemma 4.1. *For every $\alpha \geq 0$, the function $\lambda_1(\alpha,\cdot):\ L^\infty(-1,1) \to \mathbb{R}$ is continuous.*

Proof. Let us fix $\alpha \geq 0$, and assume $f_n \to f$ in $L^\infty(-1,1)$. We want to show $\lambda_1(\alpha, f_n) \to \lambda_1(\alpha, f)$, as $n \to \infty$. For simplicity we use the notation $\lambda(f_n) \equiv \lambda_1(\alpha, f_n)$ and $\lambda(f) \equiv \lambda_1(\alpha, f)$; also $u_n \equiv u_{\alpha,f_n}$ and $u \equiv u_{\alpha,f}$. From (2.1), we get

$$\lambda(f_n) \leq \int_{-1}^1 {u'}^2\,dx + \alpha \int_{-1}^1 f_n u^2\,dx, \quad n \in \mathbb{N}$$

Therefore

$$\limsup_{n\to\infty} \lambda(f_n) \leq \int_{-1}^1 {u'}^2\,dx + \alpha \int_{-1}^1 f u^2\,dx = \lambda(f). \tag{4.1}$$

On the other hand for every $n \in \mathbb{N}$ we have

$$\lambda(f_n) = \int_{-1}^1 u_n'^2\,dx + \alpha \int_{-1}^1 f_n u_n^2\,dx.$$

Thus, since $\{\lambda(f_n)\}$ and $\{f_n\}$ are bounded, it follows that $\{u_n\}$ is bounded in $H_0^1(-1,1)$. Hence there exist a subsequence of $\{u_n\}$, still denoted $\{u_n\}$, and a function $\hat{u} \in H_0^1(-1,1)$ such that

$$u_n \rightharpoonup \hat{u}, \quad \text{in } H_0^1(-1,1), \quad u_n \to \hat{u}, \text{ uniformly on } (-1,1).$$

Whence

$$\begin{aligned} \liminf_{n\to\infty} \lambda(f_n) &\geq \liminf_{n\to\infty} \Big(\int_{-1}^1 u_n'^2\,dx + \alpha \int_{-1}^1 f_n u_n^2\,dx \Big) \\ &\geq \int_{-1}^1 (\hat{u}')^2\,dx + \alpha \int_{-1}^1 f \hat{u}^2\,dx \geq \lambda(f). \end{aligned} \tag{4.2}$$

Clearly, (4.1), (4.2) and the fact that the sequence $\{f_n\}$ is arbitrary prove the assertion of the lemma. ∎

Lemma 4.2. *Let $\alpha \geq 0$, and $f_n \to f$ in $L^\infty(-1,1)$. Then $u_{\alpha,f_n} \to u_{\alpha,f}$ uniformly on $(-1,1)$.*

Proof. From Lemma 4.1, $\lambda_n \to \lambda$ as $n \to \infty$. Hence $\{\lambda_n\}$ is bounded; which implies that $\{u_n\}$ is bounded in $H_0^1(-1,1)$. Thus there exist a subsequence, still denoted $\{u_n\}$, and a function $\hat{u} \in H_0^1(-1,1)$ such that

$$u_n \rightharpoonup \hat{u}, \quad \text{in } H_0^1(-1,1), \quad u_n \to \hat{u}, \quad \text{uniformly on } (-1,1). \tag{4.3}$$

We show that $\hat{u} = u$; this proves the assertion of the lemma, since u is the only accumulation point of $\{u_n\}$. From (2.1), (4.3) and the fact that the H_0^1-norm is weakly sequentially lower semi-continuous we deduce

$$\begin{aligned}\lambda &\le \int_{-1}^{1} \hat{u}'^2\,dx + \alpha\int_{-1}^{1} f\hat{u}^2\,dx\\ &\le \liminf_{n\to\infty}\Big(\int_{-1}^{1} {u_n'}^2\,dx + \alpha\int_{-1}^{1} f_n u_n^2\,dx\Big)\\ &= \liminf_{n\to\infty}\lambda_n = \lambda.\end{aligned}\tag{4.4}$$

Thus we obtain

$$\lambda = \int_{-1}^{1} \hat{u}'^2\,dx + \alpha\int_{-1}^{1} f\hat{u}^2\,dx,$$

which shows that $\hat{u}$ is also an eigenfunction corresponding to λ. However, since $\hat{u}$ satisfies (3.1), it follows from uniqueness that $u = \hat{u}$, as desired. □

To state the next result we need some more notation. The principal eigenvalue of (1.1) is a map $\lambda_1 : \mathbb{R}\times L^\infty(-1,1) \to \mathbb{R}$. The Fréchet derivative of λ_1 with respect to the second variable, if it exists, denoted $\partial_2\lambda_1(\alpha,\cdot)$. For a function $f \in L^\infty(-1,1)$, $\partial_2\lambda_1(\alpha,f)$ would be a linear functional from $L^\infty(-1,1)$ into the reals. By $\partial_2\lambda_1(\alpha,f)[h]$ we denote the Fréchet derivative at f applied to $h \in L^\infty(-1,1)$.

Lemma 4.3. *Let $\alpha \ge 0$, and f and h be functions in $L^\infty(-1,1)$. Then*

$$\partial_2\lambda_1(\alpha,f)[h] = \alpha\int_{-1}^{1} h u_{\alpha,f}^2\,dx,\tag{4.5}$$

hence $\partial_2\lambda_1(\alpha,f)$ can be identified with $\alpha u_{\alpha,f}^2$.

Proof. From (2.1) we have

$$\begin{aligned}\lambda_1(\alpha,f+h) &\le \int_{-1}^{1} {u'}_{\alpha,f}^2\,dx + \alpha\int_{-1}^{1}(f+h)u_{\alpha,f}^2\,dx\\ &= \lambda_1(\alpha,f) + \alpha\int_{-1}^{1} h u_{\alpha,f}^2\,dx\end{aligned}\tag{4.6}$$

Thus we derive

$$\lambda_1(\alpha,f+h) - \lambda_1(\alpha,f) - \alpha\int_{-1}^{1} h u_{\alpha,f}^2\,dx \le 0.\tag{4.7}$$

On the other hand, again from (2.1), we have

$$\begin{aligned}\lambda_1(\alpha,f) &\le \int_{-1}^{1} {u'}_{\alpha,f+h}^2\,dx + \alpha\int_{-1}^{1} f u_{\alpha,f+h}^2\,dx\\ &= \lambda_1(\alpha,f+h) - \alpha\int_{-1}^{1} h u_{\alpha,f+h}^2\,dx.\end{aligned}$$

Thus

$$\lambda_1(\alpha,f+h) - \lambda_1(\alpha,f) - \alpha\int_{-1}^{1} h u_{\alpha,f}^2\,dx \ge \alpha\int_{-1}^{1} h(u_{\alpha,f+h}^2 - u_{\alpha,f}^2)\,dx.\tag{4.8}$$

From (4.7) and (4.8) we deduce

$$\begin{aligned} |\lambda_1(\alpha, f+h) - \lambda_1(\alpha, f) - \alpha \int_{-1}^{1} h u_{\alpha,f}^2 \, dx| &\le \alpha \|h\|_\infty \int_{-1}^{1} |u_{\alpha,f+h}^2 - u_{\alpha,f}^2| \, dx \\ &\le 2\alpha \|h\|_\infty \|u_{\alpha,f+h} - u_{\alpha,f}\|_2 \\ &\le 4\alpha \|h\|_\infty \|u_{\alpha,f+h} - u_{\alpha,f}\|_\infty, \end{aligned} \tag{4.9}$$

where in the second inequality we used the Hölder inequality, and in the third one the fact that $u_{\alpha,f+h}$ and $u_{\alpha,f}$ both satisfy (3.1). Note that from Lemma 4.2, we infer that $\|u_{\alpha,f+h} - u_{\alpha,f}\|_\infty = o(1)$, as $\|h\|_\infty \to 0$. This result coupled with (4.9) imply that

$$\lambda_1(\alpha, f+h) - \lambda_1(\alpha, f) - \alpha \int_{-1}^{1} h u_{\alpha,f}^2 \, dx = o(\|h\|_\infty),$$

as $\|h\|_\infty \to 0$. This, in turn, verifies (4.5). □

To state the next result we introduce the functional $\Phi_\alpha : L^\infty(-1,1) \to \mathbb{R}$ as

$$\Phi_\alpha(g) = \sqrt{\frac{1}{\alpha} \partial_2 \lambda_1(\alpha, g)}, \quad \alpha > 0.$$

Note that Φ_α is well-defined since from Lemma 4.3, $\partial_2 \lambda_1(\alpha, g)$ can be identified with the positive function $\alpha u_{\alpha,g}^2$.

Theorem 4.4. (a) *For every $\alpha > 0$, we have $F_0 = \phi_\alpha \circ \Phi_\alpha(F_0)$, almost everywhere in $(-1,1)$.*

(b) *For $0 < \alpha \le \hat{\alpha}$, we have $F^0 = \psi_\alpha \circ \Phi_\alpha(F^0)$, almost everywhere in $(-1,1)$.*

Where the functions ϕ_α and ψ_α are the functions stated in Remarks 3.1, and 3.5, respectively.

The proof of the above theorem is straightforward; so we omit it.

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