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REMARKS ON THE STRONG MAXIMUM PRINCIPLE FOR NONLOCAL OPERATORS

JÉRÔME COVILLE

Abstract. In this note, we study the existence of a strong maximum principle for the nonlocal operator

$$\mathcal{M}[u](x) := \int_G J(g)u(x * g^{-1})d\mu(g) - u(x),$$

where G is a topological group acting continuously on a Hausdorff space X and $u \in C(X)$. First we investigate the general situation and derive a pre-maximum principle. Then we restrict our analysis to the case of homogeneous spaces (i.e., $X = G/H$). For such Hausdorff spaces, depending on the topology, we give a condition on J such that a strong maximum principle holds for $\mathcal{M}$. We also revisit the classical case of the convolution operator (i.e. $G = (\mathbb{R}^n, +), X = \mathbb{R}^n, d\mu = dy$).

1. Introduction and Main results

This note is devoted to the study of the strong maximum principle satisfied by an operator

$$\mathcal{M}[u] := \int_G J(g)u(x * g^{-1})d\mu(g) - u(x), \tag{1.1}$$

where $G, *, X, J, d\mu$ satisfy the following assumptions:

(H1) X is a Hausdorff space,
(H2) G is a topological group acting continuously on X with the operation $*$,
(H3) $d\mu$ is a Borel measure on G such that for all nonempty open sets $A \subset G$ we have $d\mu(A) > 0$,
(H4) $J \in C(G, \mathbb{R})$ is a non-negative function of unit mass with respect to $d\mu$.

Such kind of operators have been recently introduced in various models where long range interactions play an important role, see for example [1, 4, 5, 7, 10, 13]. A first example of such models is given by the well known nonlocal reaction diffusion equation below,

$$\frac{\partial u}{\partial t} = \int_{\mathbb{R}^n} J(x-y)u(y)\,dy - u + u(1-u) \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^n. \tag{1.2}$$

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The above equation models the evolution of a population density through a homogeneous environment with a constant rate of birth and death. In this case, we have $(G,*)=(\mathbb{R}^n,+)$, $X=\mathbb{R}^n$, $J\in C(\mathbb{R}^n)$ and $d\mu=dy$ is the Lebesgue measure. Such an equation, with a different type of nonlinearity, appears also in some Ising models and in ecology, see for example [1, 5, 6, 13] and their many references.

Other examples are given by the following two discrete versions of (1.2),

$$\frac{\partial u}{\partial t}=\frac{1}{2}[u(x+1)+u(x-1)-2u(x)]+f(u) \quad \text{in } \mathbb{R}^+\times\mathbb{R}, \tag{1.3}$$

$$\frac{\partial u}{\partial t}=\frac{1}{2}[u(p+1)+u(p-1)-2u(p)]+f(u) \quad \text{in } \mathbb{R}^+\times\mathbb{Z}. \tag{1.4}$$

In both situations the discrete diffusion operator can be reformulated in terms of a nonlocal operator $\mathcal{M}$ defined in (1.1). Indeed, in these two cases, by taking $(G,*)=(\mathbb{Z},+)$, $d\mu$ the counting measure and $J\in C(\mathbb{Z},\mathbb{R})$ defined as follows:

$$J(p):=\begin{cases}\frac{1}{2} & \text{if } p=-1 \text{ or } p=1,\\ 0 & \text{otherwise,}\end{cases}$$

it follows that for any x in the Hausdorff space $\mathbb{R}$ or $\mathbb{Z}$ we have

$$\frac{1}{2}[u(x+1)+u(x-1)-2u(x)]=\int_G J(g)u(x*g^{-1})d\mu(g)-u(x).$$

As for their continuous version (1.2), equations (1.3) and (1.4) appear in discrete reaction diffusion models describing a wide variety of phenomenon, ranging from combustion to nerve propagation and phase transitions. We point the interested reader to [4, 3, 9] and the many references cited therein.

Another example comes from the following size structured population model, recently introduced by Perthame *et al.* in [10, 11],

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}=\int_0^{+\infty}u(\frac{x}{y})b(y)dy-u(x) \quad \text{in } \mathbb{R}^+\times\mathbb{R}^+. \tag{1.5}$$

In such case, we have $(G,*)=(\mathbb{R}^+\setminus\{0\},\cdot)$, $X=\mathbb{R}^+$ and $d\mu(y)=dy$ is the Lebesgue measure.

In all these examples, depending on the group and the measure considered, the properties satisfied by the corresponding operator $\mathcal{M}$ show significant differences. However, as for the classical Laplace operator (Δ), they all satisfy the following positive maximum principle.

Definition 1.1 (Courrège Positive maximum principle [2]). *An operator $A\in\mathcal{L}(C(X))$ is said to satisfy the positive maximum principle if for all $f\in C(X)$ and $x\in X$ such that $f(x)=\sup(f)$ we have $A(f)(x)\le 0$.*

For the Laplace operator (Δ), in addition to the above property, it is well known, see [8, 12], that a sub-harmonic functions satisfies a strong maximum principle:

Theorem 1.2 (Elliptic Strong maximum principle). *Let $u\in C^2(\mathbb{R}^n)$ be such that $\Delta u\ge 0$ in $\mathbb{R}^n$. Then u cannot achieve a global maximum without being constant.*

In this note, we investigate the conditions on $(G,*),X,J$ and $d\mu$ in order to achieve such strong maximum principles for $\mathcal{M}$. More precisely , we are interested in finding simple conditions on $(G,*),X,J$ and $d\mu$ for the strong maximum principle to hold:

Theorem 1.3 (Strong maximum principle)**.** *Let $u \in C(X,\mathbb{R})$ be such that $\mathcal{M}[u] \geq 0$ in X. Then u cannot achieve a global maximum without being constant.*

In the analysis of nonlinear elliptic equations, the strong maximum principle plays a very important role in proving key *a priori* estimates. It is expected that such a strong maximum property for $\mathcal{M}$ will play a similar role in the analysis of nonlinear equations involving nonlocal operators. It is therefore of great interest to investigate the conditions on $G, X, d\mu$ and J in order that a strong maximum principle hold for $\mathcal{M}$.

In this direction, we first establish a generic result satisfied by all operators $\mathcal{M}$. More precisely, we show the following result.

Theorem 1.4 (Pre-maximum principle)**.** *Let $(G, *, X, J, d\mu)$ be such that $(H1-H4)$ are satisfied and let $u \in C(X,\mathbb{R})$ be such that*

$$\mathcal{M}[u] \geq 0 \qquad (\textit{resp.} \leq 0).$$

Assume that u achieves a global maximum (resp. minimum) at some point $x_0 \in X$ and let F_{x_0} denote the smallest closed subset of X such that

- $x_0 \in F_{x_0}$,
- $F_{x_0} * \{g^{-1} \in G | J(g) > 0\} \subset F_{x_0}$.

Then $u \equiv u(x_0)$ in F_{x_0}.

Our next result is a characterization of the set F_{x_0} defined in the above Theorem 1.4.

Proposition 1.5. *Let $(G, *, X, J, d\mu)$ be such that* (H1)–(H4) *are satisfied and let F_{x_0} be the set defined in Theorem 1.4. Then*

$$F_{x_0} = \overline{\bigcup_{n\in\mathbb{N}} F_n},$$

where the F_n are defined by induction as follows

$$F_0 = \{x_0\}, \quad \textit{and} \quad \forall n \geq 0 \quad F_{n+1} := F_n * \{g^{-1} \in G | J(g) > 0\}.$$

In view of the above generic result, in order to get a strong maximum principle for $\mathcal{M}$, we need to find conditions on $(G, *), X, d\mu$ and J which imply that $F_{x_0} = X$. Note that, from the characterization of the set F_{x_0}, the condition $F_{x_0} = X$ implies that $X = F_{x_0} \subset \overline{orb(x_0)} := \overline{\{x * g^{-1} | g \in G\}} \subset X$, which means that $orb(x_0)$ is a dense set in X.

Observe that for the discrete diffusion operator considered in (1.3), the set $orb(x)$ is never dense in $\mathbb{R}$. Therefore, we cannot expect to have a strong maximum principle in such situation. On the contrary, for the same diffusion operator considered in (1.4), the set $orb(x)$ is always dense in $\mathbb{Z}$. Moreover we can easily see that in this situation the discrete operator satisfies a strong maximum principle.

Considering the above remarks, in what follows we restrict our attention to the case of Hausdorff homogeneous spaces X (i.e. $X := G/H$, where H is a closed subgroup of G). For such Hausdorff spaces, the set $orb(x)$ is always dense in X and sufficient conditions on $(G, *), X, J$ and $d\mu$ for the strong maximum principle to hold reduce to find some simple conditions on J. In this direction, we first give a sufficient condition on J to ensure that $\mathcal{M}$ satisfies the strong maximum principle. Namely, we have the following result.

Theorem 1.6. *Let X be a connected homogeneous space and let $(G,*), J, d\mu$ be as in Theorem 1.4. Let e be the unit element of G and assume that $J(e) > 0$. Then $\mathcal{M}$ satisfies the strong maximum principle.*

When X is a compact connected homogeneous spaces, we can generalize the previous statement to the following result.

Theorem 1.7. *Let X be a connected compact homogeneous space and $(G,*), J, d\mu$ as in Theorem 1.4. Then $\mathcal{M}$ satisfies the strong maximum principle.*

Next, we state optimal condition on J in two special cases. Namely, we first retrieve the Markov necessary and sufficient condition for the convolution operator (i.e. $(G,*) = (\mathbb{R}^n,+), X = \mathbb{R}^n, d\mu = dy$), which is well known among experts in stochastic processes.

Theorem 1.8 (Markov condition)**.** *Assume that $(G,*) = (\mathbb{R}^n,+), X = \mathbb{R}^n$ and $d\mu = dy$. Then $\mathcal{M}$ satisfies the strong maximum principle iff the convex hull of $\{y \in \mathbb{R}^n \,|\, J(y) > 0\}$ contains 0.*

As a consequence of the above Markov condition, we derive the following optimal condition when $(G,*) = (\mathbb{R}^+ \setminus \{0\}, \bullet), X = \mathbb{R}^+$ and $d\mu = dy$:

Corollary 1.9. *Assume that $(G,*) = (\mathbb{R}^+ \setminus \{0\}, \bullet), X = \mathbb{R}^+$ and $d\mu = dy$. Then $\mathcal{M}$ satisfies the strong maximum principle iff there exists 2 points x_1 and x_2 such that $J(x_i) > 0$ and $0 < x_1 \le 1 \le x_2$).*

1.1. **General comments.** We first note that, provided an extra assumption on the sign of the maximum (minimum) is made, we can easily extend the above results to operators $\mathcal{M}[u] + c(x)u$ with non-positive zero order term (i.e. $c(x) \le 0$). As for $\mathcal{M}$, the operator $\mathcal{M} + c(x)$ satisfies a Courrèges positive maximum principle [2], which in this case state the following definition.

Definition 1.10 (Positive maximum principle)**.** An operator $A \in \mathcal{L}(C(X))$ is said to satisfy the positive maximum principle if for all $f \in C(X)$ and $x \in X$ such that $f(x) \ge 0$ and $f(x) = \sup(f)$ we have $A(f)(x) \le 0$.

In our investigation of homogeneous spaces, we also observe that to obtain a strong maximum principle for $\mathcal{M}$, we only need the inequality $\mathcal{M}[u] \ge 0$ at points where the function u achieves its global maximum. As a consequence, in the two situation investigated above (Theorems 1.6 and 1.7), we have the following characterization:

Proposition 1.11. *Let $(G,*), X, d\mu$ and J be as in Theorem 1.6 or 1.7. Then for all $u \in C(X)$ and $x \in X$ such that $u(x) = \sup(u)$ we have the following alternatives: Either*

- *there exists $y \in X$ such that $u(y) = u(x)$ and $\mathcal{M}[u](y) < 0$, or*
- *u is a constant.*

We also want to point out that, although the Markov condition is well known among experts in stochastic analysis, we present here a simple analytical proof, which we believe is new. Using such a point of view allows us to relate a simple recovering problem with the conditions for the strong maximum principle.

The outline of this note is the following. In the two first Sections (Sections 2 and 3), we recall some basic topological results and prove the pre-maximum principle and the characterization of F_x (Theorems 1.4 and Proposition 1.5). Then in Section 4, we establish the strong maximum principle (Theorems 1.6 and 1.7). Finally, in the last section, we prove the optimal conditions (Theorems 1.8 and 1.9).

2. Preliminaries

In this section, we first present some definitions and notation that we will use in this paper. Then we establish a useful proposition. Let us first define some notations:

- $\Sigma := \{g^{-1} \in G | J(g) > 0\}$.
- For a function u, we define $\Gamma_y := \{x \in X | u(y) = u(x)\}$.

Let us now introduce the following two definitions:

Definition 2.1. Let $A \subset X$ and $B \subset G$ be two sets, then we define $A * B \subset X$ as follows

$$A * B := \{a * b \mid a \in A \text{ and } b \in B\}.$$

Definition 2.2. Let $A \subset X$ and $B \subset G$ be two sets, then we say that A is $B*$ stable if

$$A * B \subset A.$$

Next, let us recall the following basic property of $*$ stable sets.

Proposition 2.3. *Let $A \subset X$ and $B \subset G$ be two sets. If A is $B*$ stable, then $\bar{A}$ is $B*$ stable, where $\bar{A}$ denotes the closure of A.*

Proof. Let $y \in \bar{A} * B$ and $V(y)$ be an open neighbourhood of y. By definition, we have $y := x_1 * b_1$ for some $x_1 \in \bar{A}$ and $b_1 \in B$. Since the operation $*$ is continuous, the following map T is continuous:

$$\begin{aligned} T : X &\to X \\ z &\mapsto z * b_1. \end{aligned}$$

Therefore, $T^{-1}(V(y))$ is a open neighbourhood of x_1. Since $\bar{A}$ is a closed set and $x_1 \in \bar{A}$, we have $T^{-1}(V(y)) \cap A \neq \emptyset$. By definition of $T^{-1}(V(y))$, using the stability of A, it follows that for all $z \in T^{-1}(V(y)) \cap A$, $z * b_1 \in A$. Therefore,

$$z * b_1 \in V(y) \cap A \quad \text{for all } z \in T^{-1}(V(y)) \cap A,$$

and yields $V(y) \cap A \neq \emptyset$.

The above argumentation, being independent of the choice of $V(y)$, shows that $y \in \bar{A}$. Now, since y is arbitrary, we end up with $\bar{A} * B \subset \bar{A}$. □

3. Pre-maximum principle and Characterizations of F_x

In this Section we prove Theorem 1.4 and Proposition 1.5. Let us first start with the proof of the pre-maximum principle.

Proof of Theorem 1.4. The proof is rather simple. Let us first recall the definition of Γ_{x_0}:

$$\Gamma_{x_0} := \{x \in X | u(x) = u(x_0)\}. \tag{3.1}$$

Since u is continuous, Γ_{x_0} is a closed subset of X. Now observe that Γ_{x_0} is $\Sigma*$ stable (i.e. $\Gamma_{x_0} * \Sigma \subset \Gamma_{x_0}$). Indeed, choose any $\bar{x} \in \Gamma_{x_0}$. At $\bar{x}, u$ satisfies

$$0 \leq \mathcal{M}[u](\bar{x}) = \int_G J(g) u(\bar{x} * g^{-1})\, d\mu - u(\bar{x}) = \int_G J(g)[u(\bar{x} * g^{-1}) - u(\bar{x})]\, d\mu \leq 0.$$

Therefore,

$$\int_G J(g)[u(\bar{x} * g^{-1}) - u(\bar{x})]\, d\mu = 0. \tag{3.2}$$

Using that $J \geq 0$ and that for all $g \in G, [u(\bar{x} * g) - u(\bar{x})] \leq 0$, (3.2) yields

$$u(\bar{x} * g^{-1}) = u(\bar{x}) \quad \text{for all} \quad g \in \Sigma.$$

Thus, we have

$$u(y) = u(x_0) \quad \text{for all} \quad y \in \{\bar{x}\} * \Sigma.$$

Hence, $\{\bar{x}\} * \Sigma \subset \Gamma_{x_0}$. Since this computation holds for any element $\bar{x}$ of Γ_{x_0}, we have $\Gamma_{x_0} * \Sigma \subset \Gamma_{x_0}$.

Recall now that F_{x_0} is the smallest closed subset of X such that

- $x_0 \in F_{x_0}$,
- $F_0 * \Sigma \subset F_{x_0}$.

Since Γ_{x_0} satisfies the above conditions, we then have $F_{x_0} \subset \Gamma_{x_0}$. □

Note that Γ_{x_0} is independent of the choice of the point where u takes its global maximum. Indeed, we easily see that $\Gamma_{x_0} = \Gamma_y$ for any $y \in \Gamma_{x_0}$. On the contrary, the set F_{x_0} strongly depends on x_0 and there is no reason to always have $F_{x_0} = F_y$. Indeed, for $X = G = \mathbb{R}$, if $\Sigma = \mathbb{R}^+$ then for $x_0 < y$, $F_y \subset_{\neq} F_{x_0}$.

Now, we give a characterization of the set F_{x_0} defined in Theorem 1.4 and prove Proposition 1.5. For the sake of clarity, let us first recall Proposition 1.5.

Proposition 3.1. *Let F_{x_0} be the set defined in Theorem 1.4, then*

$$F_{x_0} = \overline{\bigcup_{n \in \mathbb{N}} F_n},$$

where the F_n are defined by induction as follows: $F_0 = \{x_0\}$ and for $n \geq 0$, $F_{n+1} := F_n * \Sigma$.

Proof. Let us define the set

$$F_\infty := \bigcup_{n \in \mathbb{N}} F_n.$$

Using the definition of F_∞, we easily see that F_∞ is $\Sigma*$ stable. From Proposition 2.3, it follows that $\bar{F}_\infty$ is $\Sigma*$ stable. Therefore, by definition of F_{x_0}, we have $F \subset \bar{F}_\infty$.

Now, since $x_0 \in F_{x_0}$ and F_{x_0} is $\Sigma*$ stable, by induction we easily see that $\forall n \in \mathbb{N}, F_n \subset F_{x_0}$. Thus, $F_\infty \subset F_{x_0}$ and yields $F_{x_0} \subset \bar{F}_\infty \subset F_{x_0}$. □

Remark 3.2. As already mentioned in the introduction, to obtain a strong maximum principle for $\mathcal{M}$, we only need to find conditions on $X, d\mu$ and J such that $F_{x_0} = \Gamma_{x_0} = X$.

4. Strong maximum principle when X is an homogeneous space

In this Section, we treat the case of connected homogeneous space X and prove sufficient conditions on J (Theorems 1.6 and 1.7) in order to have a strong maximum principle f or $\mathcal{M}$. Let us start with the proof of Theorem 1.6.

Proof of Theorem 1.6. Again the proof is rather simple. We must check that for any $u \in C(X,\mathbb{R})$ such that

$$\mathcal{M}[u] \geq 0 \qquad (\text{resp. } \leq 0)$$

then u cannot achieve a global maximum (resp. minimum) in X without being constant. So consider $u \in C(X,\mathbb{R})$ such that u achieves a maximum at x_0 and satisfies $\mathcal{M}[u] \geq 0 \qquad (\text{resp. } \leq 0)$. By definition of Γ_x, we only need to show that $\Gamma_{x_0} = X$. To this end, we will prove that Γ_{x_0} is a closed and open set. By definition of Γ_{x_0}, Γ_{x_0} is a closed set of X. Now, let us show that Γ_{x_0} is open. Choose any $y \in \Gamma_{x_0}$. Then at this point

$$0 \leq \mathcal{M}[u](y) = \int_G J(g)u(y * g^{-1})\,d\mu - u(y) = \int_G J(g)[u(y * g^{-1}) - u(y)]\,d\mu(g) \leq 0.$$

Arguing as in the proof of Theorem 1.4, we have $u(y * g^{-1}) = u(y) = u(x_0)$ for all $g \in \Sigma$. Since $e \in \Sigma$, we have for some open neighbourhood $B(e)$ of e

$$u(y * g^{-1}) = u(x_0) \quad \text{for all } g^{-1} \in B(e).$$

Using that G is a topological group, $y * B(e)$ is then an open neighbourhood of y. Thus,

$$B(y) := y * B(e) \subset \Gamma_{x_0}.$$

Therefore Γ_{x_0} is an open set. Hence, $X = \Gamma_{x_0}$ since X is connected. $\square$

Let us now turn our attention to the case of compact homogeneous space and prove Theorem 1.7. First, let us prove the following technical Lemma.

Lemma 4.1. *For any $g \in X$ there exists a sequence of integers $(n_k)_{k\in\mathbb{N}}$ with $n_k \geq 1$ and $g^{n_k} \to e$ as $k \to +\infty$, where e is the unit element of G.*

Proof. Take $g \in X$ and let us consider the following sequence $(g^m)_{m\in\mathbb{N}}$. Since X is compact, $(g_m)_{m\in\mathbb{N}}$ has a convergent sub-sequence $(g_{m_k})_{k\in\mathbb{N}}$. Without any restriction, we can assume that $m_{k+1} \geq m_k + 1$. Consider now the following sequence, $w_k := g^{m_{k+1}-m_k}$. By construction, $w_k \to e$ and $m_{k+1} - m_k \in \mathbb{N}^*$. Hence, with $n_k := m_{k+1} - m_k$, $g^{n_k} \to e$. $\square$

We are now in a position to prove Theorem 1.7.

Proof of Theorem 1.7. As for Theorem 1.6 we have to check that for any $u \in C(X,\mathbb{R})$ such that

$$\mathcal{M}[u] \geq 0 \qquad (\text{resp. } \leq 0)$$

then u cannot achieve a global maximum (resp. minimum) in X without being constant. So consider $u \in C(X,\mathbb{R})$ such that u achieves a maximum at x_0 and satisfies $\mathcal{M}[u] \geq 0 \qquad (\text{resp. } \leq 0)$. By definition of Γ_x, we only need to show that $\Gamma_{x_0} = X$. Again, as in the proof of Theorem 1.6, we prove that Γ_{x_0} is an open and closed set and therefore $X = \Gamma_{x_0}$ since X is connected. By definition Γ_{x_0} is closed. Now let us show that Γ_{x_0} is open. Let $y \in \Gamma_{x_0}$ and F_y be the set defined in Theorem 1.4 with y instead of x_0. Using now the characterization of F_y given in Proposition 1.5 we have

$$F_y := \overline{\bigcup_{n\in\mathbb{N}} F_n} \subset \Gamma_{x_0}, \tag{4.1}$$

where $F_n := \{y\} * \Sigma^n$.

Choose now $g \in \Sigma$. According to Lemma 4.1 there exists a sequence $(n_k)_{k\in\mathbb{N}}$ such that $g^{n_k} \to e$. By assumption, Σ is an open subset of G. Therefore Σ^{n_k} is a sequence of open subset of G. Since $g^{n_k} \to e$, Σ^{n_k} is a open neighbourhood of e for k sufficiently large. Therefore,

$$\{y\} * \Sigma^{n_k} \subset F_y \subset \Gamma_{x_0}$$

Since Σ^{n_k} is a open neighbourhood of e for k sufficiently large, $\{y\} * \Sigma^{n_k}$ is then an open neighbourhood of y. Thus, Γ_{x_0} contains an open neighbourhood of y for any y in Γ_{x_0}. Hence, Γ_{x_0} is open. □

5. Some optimal conditions

In this section we prove the optimal Markov condition for the convolution operator (Theorem 1.8) and prove Theorem 1.9 .

The classical convolution case $(X = G = \mathbb{R}^n)$ **and** $d\mu = dy$**:** When $(X = G = \mathbb{R}^n)$ the operator $\mathcal{M}$ takes the form of the usual convolution; i.e.,

$$\mathcal{M}[u] := \int_{\mathbb{R}^n} J(y)u(x-y)\,dy - u.$$

For such a convolution operator, the optimal condition on J in order that $\mathcal{M}$ satisfy a strong maximum principle is the following. This condition is known as the Markov condition.

Theorem 5.1. *$\mathcal{M}$ satisfies a strong maximum principle if and only if the convex hull of $\{y \in \mathbb{R}^n | J(y) > 0\}$ contains 0.*

Proof. Let us start with the necessary condition. Assume that the Markov condition fails. We will show that $\mathcal{M}$ does not satisfy the strong maximum principle. To this end, we construct a non constant function u that achieves a global maximum and satisfies $\mathcal{M}[u] \ge 0$.

Let us denote $\operatorname{conv}(\{y \in \mathbb{R}^n | J(y) > 0\})$ the convex hull of $\{y \in \mathbb{R}^n | J(y) > 0\}$. By assumption, $0 \notin \operatorname{conv}(\{y \in \mathbb{R}^n | J(y) > 0\})$. Using the Hahn-Banach Theorem, there exists a hyperplane H such that $\operatorname{conv}(\{y \in \mathbb{R}^n | J(y) > 0\}) \subset H^+$, where $H^+ := \{x \in \mathbb{R}^n | x_n \ge 0\}$ in an orthonormal basis $(e_1; e_2; \dots; e_n)$. Let v be a non-increasing function that is constant in $\mathbb{R}^-$, and let us compute $\mathcal{M}[u]$ with $u(x) := v(x_n)$. Since the Lebesgue measure is invariant under rotation and $\operatorname{supp}(J) \subset H^+$ we have

$$\begin{aligned}
\mathcal{M}[u] &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} J(t, x_n - y_n)[v(y_n) - v(x_n)]\,dx_n\,dt \\
&= \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{x_n} J(t, x_n - y_n)[v(y_n) - v(x_n)]\,dx_n\,dt.
\end{aligned}$$

Therefore, since v is non increasing we end up with $\mathcal{M}[u] \ge 0$. Since u achieves a global maximum without being constant, u is our desired function.

Let us now turn our attention to the sufficient condition. Assume that $0 \in \operatorname{conv}(\{y \in \mathbb{R}^n | J(y) > 0\})$, then there exists a simplex $S(p_i)$ formed by $n+1$ points of $\mathbb{R}^n$ such that $0 \in S$ and $J(p_i) > 0$.

By continuity, we can always assume that $(p_1, \dots, p_n)$ is a basis of $\mathbb{R}^n$. Let us now rewrite x_0 in the basis $(p_1, \dots, p_n)$:

$$x_0 = -a_1 p_1 \cdots - a_n p_n \quad \text{with } a_i \ge 0.$$

Observe now that for $\mathbb{R}^n$ equipped with the sup norm associated to the base $(p_1,\dots,p_n)$, there exists $r>0$ so that $B(x_0,r)\subset\{J>0\}$. Now for all integer $m>0$, set $y_m=mp_0+[ma_1]p_1+\dots+[ma_n]p_n$, where $[\cdot]$ denotes the integer part. Now let u be a continuous function satisfying $\mathcal{M}[u]\geq 0$ and that achieves a global maximum at some point $z\in\mathbb{R}$. Without loss of generality, we may always assume that $z=0$. Indeed, if $z\neq 0$, we consider the function $u_z(x):=u(x-z)$, instead of u. We easily see that u_z achieves a global maximum at 0 and satisfies $\mathcal{M}[u_z]\geq 0$. Using now Theorem 1.4, we see that for all $m\in\mathbb{N}$,

$$\|y_m\|<1 \quad \text{and} \quad B(y_m;mr)\subset\Gamma_0.$$

Therefore,

$$\bigcup_{m\in\mathbb{N}} B(y_m;mr)\subset\Gamma_0.$$

Hence, $\mathbb{R}^n\subset\Gamma_0$. ◻

The above necessary and sufficient condition for the convolution operator can be weakened depending on the underlying topological structure of the space. In particular, we have in mind the following setting. Since $\mathcal{M}$ is translation invariant, $\mathcal{M}$ is also an operator on the set of periodic functions. On this set of functions, the strong maximum principle always holds. This condition is not so surprising since the additional periodic structure will in some sense compactify the homogeneous space $\mathbb{R}^n$.

Another special case: $X=\mathbb{R}^+,(G,*)=(\mathbb{R}^+\setminus\{0\},\cdot)$ **and** $d\mu=dy$**.** In this situation,

$$\mathcal{M}[u]:=\int_{\mathbb{R}^+} J(y)u\big(\frac{x}{y}\big)\,dy-u,$$

and the above operator has essentially the same property as the usual convolution operator. Indeed, let us make the following change of variables $x:=e^t$. Then we have

$$\mathcal{M}[u](e^t)=\int_{\mathbb{R}}\tilde{J}(t-s)u(e^s)\,ds-u(e^t),$$

where $\tilde{J}(t):=J(e^t)e^t$. Therefore, letting $v(t)=u(e^t)$, we end up with

$$\mathcal{R}[v](t)=\tilde{J}\star v(t)-v(t) \quad \text{in } \mathbb{R},$$

with $\int_{\mathbb{R}}\tilde{J}(t)dt=1$. Hence, the optimal condition to achieve a strong maximum principle for such a kind of operator will be of the same type as the one used for the convolution operator.

Namely, there exists two points $a<1<b$ such that $J(a)>0$ and $J(b)>0$. This condition corresponds to the one given for the convolution operator which is the existence of two points $a'<0<b'$ such that $\tilde{J}(a')>0$ and $\tilde{J}(b')>0$. The above observation proves Corollary 1.9.

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Jérôme Coville

Max Planck Institute for mathematical science, Inselstrasse 22, D-04103 Leipzig, Germany

E-mail address: coville@mis.mpg.de