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IMPULSIVE DYNAMIC EQUATIONS ON A TIME SCALE

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Abstract. Let $\mathbb{T}$ be a time scale such that $0, t_i, T \in \mathbb{T}$, $i = 1, 2, \dots, n$, and $0 < t_i < t_{i+1}$. Assume each t_i is dense. Using a fixed point theorem due to Krasnosel'skiĭ, we show that the impulsive dynamic equation

$$\begin{gathered} y^{\Delta}(t) = -a(t)y^{\sigma}(t) + f(t, y(t)), \quad t \in (0, T], \\ y(0) = 0, \\ y(t_i^+) = y(t_i^-) + I(t_i, y(t_i)), \quad i = 1, 2, \dots, n, \end{gathered}$$

where $y(t_i^{\pm}) = \lim_{t \to t_i^{\pm}} y(t)$, and y^{Δ} is the Δ-derivative on $\mathbb{T}$, has a solution. Under a slightly more stringent inequality we show that the solution is unique using the contraction mapping principle. Finally, with the aid of the contraction mapping principle we study the stability of the zero solution on an unbounded time scale.

1. Introduction

Let $\mathbb{T}$ be a time scale such that $0, t_i, T \in \mathbb{T}$, for $i = 1, 2, \dots, n$, $0 < t_i < t_{i+1}$, and assume that t_i is dense in $\mathbb{T}$ for each $i = 1, 2, \dots, n$. We will show the existence of solutions for the nonlinear impulsive dynamic equation

$$\begin{gathered} y^{\Delta}(t) = -a(t)y^{\sigma}(t) + f(t, y(t)), \quad t \in (0, T], \\ y(0) = 0, \\ y(t_i^+) = y(t_i^-) + I(t_i, y(t_i)), \quad i = 1, 2, \dots, n, \end{gathered} \tag{1.1}$$

where $y(t_i^{\pm}) = \lim_{t \to t_i^{\pm}} y(t)$, $y(t_i) = y(t_i^-)$, and $[0, T] = \{t \in \mathbb{T} : 0 \le t \le T\}$. Note, the intervals $[a, b), (a, b]$, and (a, b) are defined similarly.

In 1988, Stephan Hilger [10] introduced the theory of time scales (measure chains) as a means of unifying discrete and continuum calculi. Since Hilger's initial work there has been significant growth in the theory of dynamic equations on time scales, covering a variety of different problems; see [2, 3, 4] and references therein. The study of impulsive initial and boundary value problems is extensive. For the theory and classical results, we direct the reader to the monographs [1, 15, 16]. Recent works of D. Guo on the topic include [7, 8, 9] (and the references therein) and are devoted to the existence of solutions to integro-differential equations using the fixed point index of operators in ordered Banach spaces and other techniques.

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In Section 2 we present some preliminary material that we will need to show the existence of a solution of (1.1). We will state some facts about the exponential function on a time scale as well as a fixed point theorem due to Krasnosel'skiĭ. We present our main results in Section 3. In Section 4 we give sufficient conditions for the stability of the zero solution of (1.1).

2. Preliminaries

We assume the reader is familiar with the notation and basic results for dynamic equations on time scales. For a review of this topic we direct the reader to the monographs [3, 4]. We begin with a few definitions.

A function $p : \mathbb{T} \to \mathbb{R}$ is said to be *regressive* provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^\kappa$. The set of all regressive rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$.

Let $p \in \mathcal{R}$ and $\mu(t) \neq 0$ for all $t \in \mathbb{T}$. The *exponential function* on $\mathbb{T}$, defined by

$$e_p(t,s) = \exp\Big(\int_s^t \frac{1}{\mu(z)} \log(1+\mu(z)p(z))\,\Delta z\Big),$$

is the solution to the initial value problem $y^\Delta = p(t)y$, $y(s) = 1$. Other properties of the exponential function are given in the following lemma, [3, Theorem 2.36].

Lemma 2.1. *Let $p \in \mathcal{R}$. Then*

(i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$;
(ii) $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$;
(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$, *where* $\ominus p(t) = -\frac{p(t)}{1+\mu(t)p(t)}$;
(iv) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;
(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$;
(vi) $\big(\frac{1}{e_p(\cdot,s)}\big)^\Delta = -\frac{p(t)}{e_p^\sigma(\cdot,s)}$.

Lastly in this section, we state Krasnosel'skiĭ's fixed point theorem [14] which enables us to prove the existence of a periodic solution.

Theorem 2.2 (Krasnosel'skiĭ). *Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $\big(\mathcal{B}, \|\cdot\|\big)$. Suppose that*

(i) *the mapping $A : \mathbb{M} \to \mathcal{B}$ is completely continuous,*
(ii) *the mapping $B : \mathbb{M} \to \mathcal{B}$ is a contraction, and*
(iii) $x, y \in \mathbb{M}$, *implies* $Ax + By \in \mathbb{M}$.

Then the mapping $A + B$ has a fixed point in $\mathbb{M}$.

3. Existence Of Solutions

Define $t_{n+1} \equiv T$ and let $J_0 = [0, t_1]$ and for $k = 1, 2, \dots, n$, let $J_k = (t_k, t_{k+1}]$. Define

$$PC = \{y : [0,T] \to \mathbb{R} \,|\, y \in C(J_k),\ y(t_k^\pm) \text{ exist and } y(t_k^-) = y(t_k), k = 1, \dots, n\}$$

and

$$PC^1 = \{y : [0,T] \to \mathbb{R} \,|\, y \in C^1(J_k), k = 1, \dots, n\}$$

where $C(J_k)$ is the space of all real valued continuous functions on J_k and $C^1(J_k)$ is the space of all continuously delta-differentiable functions on J_k. The set PC is a Banach space when it is endowed with the supremum norm

$$\|u\| = \max_{0\le k\le n} \{\|u\|_k\},$$

where $\|u\|_k = \sup_{t\in J_k} |u(t)|$.

We will assume that the following conditions hold.

(A) $a \in \mathcal{R}$.

(F1) $f \in C(\mathbb{T}\times\mathbb{R},\mathbb{R})$.

(F2) There exist g and h with $\alpha := \max_{t\in[0,T]} \int_0^t |e_{\ominus a}(t,s)| g(s)\,\Delta s < \infty$, and $\beta := \max_{t\in[0,T]} \int_0^t |e_{\ominus a}(t,s)| h(s)\,\Delta s < \infty$, such that

$$|f(t,y)| \le g(t) + h(t)|y|, \quad t\in\mathbb{T},\ y\in\mathbb{R}.$$

(I) There exists a positive constant E such that

$$|I(t,x) - I(t,y)| \le E|x-y|, \quad \text{for } x,y\in\mathbb{R}.$$

Lemma 3.1. *The function $y \in PC^1$ is a solution of equation* (1.1) *if and only if $y \in PC$ is a solution of*

$$y(t) = \int_0^t e_{\ominus a}(t,s) f(s,y(s))\,\Delta s + \sum_{\{i:t_i<t\}} e_{\ominus a}(t,t_i) I\big(t_i, y(t_i)\big). \tag{3.1}$$

Proof. For $t\in J_0$, the solution of (1.1) satisfying $y(0)=0$ is

$$y(t) = \int_0^t e_{\ominus a}(t,0) f(s,y(s))\,\Delta s.$$

See [3] for details. To find the solution of (1.1) on J_1 we consider the initial value problem

$$\begin{gathered} y^{\Delta}(t) = -a(t)y^{\sigma}(t) + f(t,y(t)), \quad t\in J_1,\\ y(t_1^+) = \int_0^{t_1} e_{\ominus a}(t_1,s) f(s,y(s))\,\Delta s + I(t_1,y(t_1)). \end{gathered}$$

The solution to this initial value problem is

$$y(t) = e_{\ominus a}(t,t_1) I(t_1,y(t_1)) + \int_0^t e_{\ominus a}(t,s) f(s,y(s))\,\Delta s.$$

We proceed inductively to obtain that if $y\in PC^1$ is a solution of (1.1), then $y\in PC$ is a solution of

$$y(t) = \int_0^t e_{\ominus a}(t,s) f(s,y(s))\,\Delta s + \sum_{\{i:t_i<t\}} e_{\ominus a}(t,t_i) I\big(t_i, y(t_i)\big).$$

The converse statement follows trivially and the proof is complete. $\square$

Define the mapping $H : PC \to PC$ by

$$(H\varphi)(t) = \int_0^t e_{\ominus a}(t,s) f(s,\varphi(s))\,\Delta s + \sum_{\{i:t_i<t\}} e_{\ominus a}(t,t_i) I\big(t_i, \varphi(t_i)\big). \tag{3.2}$$

By Lemma 3.1, a fixed point of H is a solution of (1.1). The form of (3.2) suggests that we construct two mappings, one of which is completely continuous and the other is a contraction. We express equation (3.2) as

$$(H\varphi)(t) = (A\varphi)(t) + (B\varphi)(t)$$

where, A, B are given by

$$(A\varphi)(t) = \int_0^t e_{\ominus a}(t,s) f(s,\varphi(s))\,\Delta s, \tag{3.3}$$

$$(B\varphi)(t) = \sum_{\{i:t_i<t\}} e_{\ominus a}(t,t_i) I\big(t_i, \varphi(t_i)\big). \tag{3.4}$$

Lemma 3.2. *Suppose* (A), (F1), (F2) *hold. Then* $A : PC \to PC$*, as defined by* (3.3)*, is completely continuous.*

Proof. It is clear that $A : PC \to PC$. To see that A is continuous, let $\{\varphi_i\} \subset PC$ be such that $\varphi_i \to \varphi$ as $i \to \infty$. By (F2) and the continuity of f we have, for each $t \in [0,T]$,

$$\begin{aligned}
\lim_{i\to\infty} \big|A\varphi_i(t) - A\varphi(t)\big| &\le \lim_{i\to\infty} \int_0^T \big|e_{\ominus a}(t,s)\big(f(s,\varphi_i(s)) - f(s,\varphi(s))\big)\big|\,\Delta s \\
&\le \int_0^T \lim_{i\to\infty} |e_{\ominus a}(t,s)|\big|f(s,\varphi_i(s)) - f(s,\varphi(s))\big|\,\Delta s \\
&\to 0.
\end{aligned}$$

Thus A is continuous. A standard application of the Arzelà-Ascoli Theorem shows that A is compact. $\square$

Lemma 3.3. *Let* (A) *and* (I) *hold and let* B *be defined by* (3.4)*. Suppose that*

$$E \max_{t\in[0,T]} \sum_{i=1}^n |e_{\ominus a}(t,t_i)| \le \zeta < 1. \tag{3.5}$$

Then $B : PC \to PC$ *is a contraction.*

Proof. Since $e_{\ominus a}(t,t_i)$ is continuous for all $i = 1,\dots,n$, it follows trivially that $B : PC \to PC$. For $\varphi, \psi \in PC$, we have

$$\begin{aligned}
\|B\varphi - B\psi\| &= \max_{0\le i\le n} \Big\{ |B\varphi(t) - B\psi(t)| : t \in J_i \Big\} \\
&\le \max_{0\le i\le n} \Big\{ \sum_{\{i:t_i<t\}} |e_{\ominus a}(t,t_i)| \Big| I(t_i,\varphi(t_i)) - I(t_i,\psi(t_i)) \Big| : t \in J_i \Big\} \\
&\le \Big(E \max_{t\in[0,T]} \sum_{i=1}^n |e_{\ominus a}(t,t_i)| \Big) |\varphi(t_i) - \psi(t_i)| \\
&\le \zeta \|\varphi - \psi\|.
\end{aligned}$$

Hence B defines a contraction mapping with contraction constant ζ. $\square$

We now state and prove our first existence theorem.

Theorem 3.4. *Assume* $\eta := \max_{t\in[0,T]}\sum_{i=1}^{n}|e_{\ominus a}(t,t_i)|\,|I(t_i,0)| < \infty$. *Suppose* (A), (F1), (F2), (I) *and* (3.5) *hold. Let* J *be a positive constant satisfying the inequality*

$$\alpha+\eta+\Big(\beta+E\max_{t\in[0,T]}\sum_{i=1}^{n}|e_{\ominus a}(t,t_i)|\Big)J\le J. \tag{3.6}$$

Then (1.1) *has a solution* φ *such that* $\|\varphi\|\le J$.

Proof. Define $\mathbb{M}=\{\varphi\in PC:\|\varphi\|\le J\}$. By Lemma 3.2, $A:PC\to PC$ is completely continuous. Also, from Lemma 3.3, the mapping $B:PC\to PC$ is a contraction. The first and second conditions of Theorem 2.2 are satisfied.

We need to show that if $\varphi,\psi\in\mathbb{M}$, then $\|A\varphi+B\psi\|\le J$. Let $\varphi,\psi\in\mathbb{M}$. Then, $\|\varphi\|,\|\psi\|\le J$ and

$$\begin{aligned}
&|A\varphi(t)+B\psi(t)|\\
&\le\int_0^t e_{\ominus a}(t,s)|f(s,\varphi(s))|\,\Delta s+\sum_{\{i:t_i<t\}}|e_{\ominus a}(t,t_i)|\,|I(t_i,\psi(t_i))|\\
&\le\int_0^t|e_{\ominus a}(t,s)|g(s)\,\Delta s+\int_0^t|e_{\ominus a}(t,s)|h(s)\,\Delta s\,\|\varphi\|\\
&\quad+\sum_{i=1}^{n}|e_{\ominus a}(t,t_i)||I(t_i,0)|+\sum_{i=1}^{n}|e_{\ominus a}(t,t_i)||I(t_i,\psi(t_i))-I(t_i,0)|\\
&\le\alpha+\beta J+\eta+E\max_{t\in[0,T]}\sum_{i=1}^{n}|e_{\ominus a}(t,t_i)|J\le J.
\end{aligned}$$

Hence $\|A\varphi+B\psi\|\le J$ and so $A\varphi+B\psi\in\mathbb{M}$. All the conditions of Krasnosel'skiĭ's theorem are satisfied. Thus there exists a fixed point z in $\mathbb{M}$ such that $z=Az+Bz$. By Lemma 3.1, this fixed point is a solution of (1.1) and the proof is complete. □

The conditions (F2) and (I) are global conditions on the functions f and I. In the next theorem we replace these conditions with the following local conditions.

(F2') There exist g and h with $\alpha:=\max_{t\in[0,T]}\int_0^t|e_{\ominus a}(t,s)|g(s)\,\Delta s<\infty$, and $\beta:=\max_{t\in[0,T]}\int_0^t|e_{\ominus a}(t,s)|h(s)\,\Delta s<\infty$, such that
$$|f(t,y)|\le g(t)+h(t)|y|,\quad t\in\mathbb{T},|y|<J.$$

(I') There exists a positive constant E such that
$$|I(t,x)-I(t,y)|\le E|x-y|,\quad\text{for } |x|,|y|<J.$$

Theorem 3.5. *Assume* $\eta := \max_{t\in[0,T]}\sum_{i=1}^{n}|e_{\ominus a}(t,t_i)|\,|I(t_i,0)| < \infty$. *Suppose* (A), (F1)*, and* (3.5) *hold. Let* J *be a positive constant such that conditions* (F2') *and* (I') *hold and such that*

$$\alpha+\eta+\Big(\beta+E\max_{t\in[0,T]}\sum_{i=1}^{n}|e_{\ominus a}(t,t_i)|\Big)J\le J \tag{3.7}$$

is satisfied. Then (1.1) *has a solution* φ *such that* $\|\varphi\|\le J$.

The proof of Theorem 3.5 parallels that of Theorem 3.4 and hence is omitted. In our last theorem in this section, we give conditions for which the solution of (1.1) is unique.

Theorem 3.6. *Suppose* (A), (F1), (F2), (I) *and* (3.5) *hold. If* $\beta + \zeta < 1$, *then there exists a unique solution to the impulsive initial value problem* (1.1).

Proof. Let $\varphi, \psi \in PC$. For $t \in [0, T]$

$$\begin{aligned}
|H\varphi(t) - H\psi(t)| &\leq \Big| \int_0^t e_{\ominus a}(t,s)\Big(f(s,\varphi(s)) - f(s,\psi(s))\Big)\,\Delta s\Big| \\
&\quad + \sum_{\{i: t_i < t\}} |e_{\ominus a}(t,t_i)|\big|I(t_i,\varphi(t_i)) - I(t_i,\psi(t_i))\big| \\
&\leq \int_0^T |e_{\ominus a}(t,s)| h(s)\,\Delta s\,\|\varphi - \psi\| + \zeta\|\varphi - \psi\| \\
&\leq (\beta + \zeta)\|\varphi - \psi\|.
\end{aligned}$$

Hence $\|H\varphi - H\psi\| \leq (\beta + \zeta)\|\varphi - \psi\|$. By the contraction mapping principal, H has a fixed point in PC. By Lemma 3.1, this fixed point is a solution of (1.1) and the proof is complete. $\square$

4. Stability

Assume that $\mathbb{T}$ is unbounded above. In this section, we study the stability of the zero solution of the dynamic equation

$$\begin{gathered}
y^{\Delta}(t) = -a(t)y^{\sigma}(t) + f\big(t, y(t)\big), \\
y(0) = y_0, \\
y(t_i^+) = y(t_i^-) + I\big(t_i, y(t_i)\big), \quad i = 1, 2, \dots, n.
\end{gathered} \tag{4.1}$$

In addition to assumptions (A), (F1) and (I), we assume that a, I and f satisfy

$$I(t,0) = 0, \quad f(t,0) = 0, \tag{4.2}$$

for all $t \in \mathbb{T}$ and

$$e_{\ominus a}(t,0) \to 0, \quad \text{as } t \to \infty, \tag{4.3}$$

By Lemma 2.1 and (4.3) we have that $e_{\ominus a}(t,t_i) \to 0$ as $t \to \infty$.

We replace condition (F2) with the following condition.

(F3) There exist continuous functions g and h with

$$\begin{gathered}
\alpha := \max_{t \in \mathbb{T}} \int_0^t |e_{\ominus a}(t,s)| g(s)\,\Delta s < \infty, \\
\beta := \max_{t \in \mathbb{T}} \int_0^t |e_{\ominus a}(t,s)| h(s)\,\Delta s < \infty, \\
\int_0^t |e_{\ominus a}(t,s)| g(s)\,\Delta s \to 0, \text{ and} \\
\int_0^t |e_{\ominus a}(t,s)| h(s)\,\Delta s \to 0,
\end{gathered}$$

such that

$$|f(t,y)| \leq g(t) + h(t)|y|, \quad t \in \mathbb{T}, |y| < J.$$

Lastly, we assume that

$$\lim_{t \to \infty} \int_0^t |e_{\ominus a}(t,s)| g(s)\,\Delta s = 0. \tag{4.4}$$

Remark: Lyapunov's direct method has been used widely when $\mathbb{T}=\mathbb{R}$. However, the extension of the theory of Lyapunov functions to time scales has not been fully developed. When Lyapunov's direct method is used, one must impose pointwise conditions on the coefficients in order to get the derivative of the constructed Lyapunov function to be negative along the solutions of the differential equation of interest. Since we are using fixed point theory, our conditions on the functions a, g and h are of averaging type. For an excellent reference of collections of recent results on the use of fixed point theory in the study of stability, periodicity and boundedness, we refer the reader to the texts [5, 6].

As in Lemma 3.1, we can show that $y \in PC^1$ is a solution of (4.1) if and only if $y \in PC$ satisfies

$$y(t) = e_{\ominus a}(t,0)y_0 + \int_0^t e_{\ominus a}(t,s)f(s,y(s))\,\Delta s + \sum_{\{i:t_i<t\}} e_{\ominus a}(t,t_i)I\big(t_i,y(t_i)\big). \quad (4.5)$$

Define the set S by

$$S = \{\varphi \in PC : \varphi(0) = y_0, \varphi(t) \to 0 \text{ as } t \to \infty, \text{ and } \varphi \text{ is bounded}\}.$$

Then $(S, \|\cdot\|)$ is a complete metric space under the norm $\|y\| = \sup_{t\in\mathbb{T}} |y(t)|$.

Define the mapping H_2 by

$$(H_2\varphi)(t) = e_{\ominus a}(t,0)y_0 + \int_0^t e_{\ominus a}(t,s)f(s,\varphi(s))\,\Delta s + \sum_{\{i:t_i<t\}} e_{\ominus a}(t,t_i)I\big(t_i,\varphi(t_i)\big). \quad (4.6)$$

We say that the zero solution of (4.1) is stable if for each $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ and a $t^* > 0$ such that if $|y_0| < \delta$ then $|y(t)| < \varepsilon$ for all $t > t^*$.

Theorem 4.1. *Assume that* (A), (F1), (F3), (I), *and* (4.2)–(4.4) *hold. Suppose that*

$$\beta + E \max_{t\in\mathbb{T}} \sum_{\{i:t_i<t\}} |e_{\ominus a}(t,t_i)| < 1,$$

Then every solution $y(t)$ of (4.1) *with small initial value y_0 is bounded and goes to 0 as $t \to \infty$. Moreover, the zero solution is stable.*

Proof. We first show that $H_2 : S \to S$. Note that if $\varphi \in PC$ then $H_2\varphi \in PC$. Let $\varphi \in PC$ be such that $\|\varphi\| \le K$ and let $M = \max_{t\in\mathbb{T}} e_{\ominus a}(t,0)$. Then

$$\begin{aligned}
|H_2\varphi(t)| &\le |e_{\ominus a}(t,0)||y_0| + \int_0^t |e_{\ominus a}(t,s)||f(s,\varphi(s))|\,\Delta s \\
&\quad + \sum_{\{i:t_i<t\}} |e_{\ominus a}(t,t_i)||I\big(t_i,\varphi(t_i)\big)| \\
&\le |e_{\ominus a}(t,0)||y_0| + \int_0^t |e_{\ominus a}(t,s)|g(s)\,\Delta s \\
&\quad + \int_0^t |e_{\ominus a}(t,s)|h(s)|\varphi(s)|\,\Delta s + \sum_{\{i:t_i<t\}} E|e_{\ominus a}(t,t_i)||\varphi(t_i)| \\
&\le M|y_0| + \alpha + \beta K + E \max_{t\in\mathbb{T}} \sum_{\{i:t_i<t\}} |e_{\ominus a}(t,t_i)|K.
\end{aligned}$$

Since $\max_{t\in\mathbb{T}} \sum_{\{i:t_i<t\}} |e_{\ominus a}(t,t_i)| < \infty$, $H_2\varphi$ is bounded whenever φ is bounded.

Conditions (4.3) and (F_3) imply that $(H_2\varphi)(t) \to 0$ as $t \to 0$. Let $\varphi, \theta \in S$. Then

$$\begin{aligned}
|H_2\varphi(t) - H_2\theta(t)| &\le \int_0^t |e_{\ominus a}(t,s)|h(s)|\varphi(s) - \theta(s)|\,\Delta s \\
&\quad + \sum_{i:t_i<t} |e_{\ominus a}(t,t_i)|E|\varphi(t_i) - \theta(t_i)| \\
&\le \Big[\beta + E\max_{t\in\mathbb{T}} \sum_{\{i:t_i<t\}} |e_{\ominus a}(t,t_i)|\Big]\|\phi - \theta\|.
\end{aligned}$$

Since $\beta + E\max_{t\in\mathbb{T}}\sum_{\{i:t_i<t\}}|e_{\ominus a}(t,t_i)| < 1$ then H_2 is a contraction. By the Contraction Mapping Principal, there exists a unique fixed point in S which solves (4.1).

Since (4.4) holds then we can find $t^* \in \mathbb{T}$ such that if $t > t^*$ then

$$\int_0^t |e_{\ominus a}(t,s)|g(s)\,\Delta s < \frac{\varepsilon}{3}.$$

Fix $\varepsilon > 0$ and let $\varphi \in PC$ be such that $\|\varphi\| \le \max\{\varepsilon/3, \varepsilon/(3M)\}$. As above we have,

$$\begin{aligned}
|H_2\varphi(t)| &\le |e_{\ominus a}(t,0)||y_0| + \int_0^t |e_{\ominus a}(t,s)|g(s)\,\Delta s \\
&\quad + \int_0^t |e_{\ominus a}(t,s)|h(s)|\varphi(s)|\,\Delta s + \sum_{\{i:t_i<t\}} E|e_{\ominus a}(t,t_i)||\varphi(t_i)| \\
&\le \varepsilon.
\end{aligned}$$

The zero solution is stable and the proof is complete. □

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