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POSITIVE SOLUTIONS FOR A HIGH-ORDER MULTI-POINT BOUNDARY-VALUE PROBLEM IN BANACH SPACES

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ABSTRACT. Using the fixed point theory of strict set contractions, we study the existence of at least one, two, and multiple positive solutions for higher order multiple point boundary-value problems in Banach spaces. Our result extends some of the existing results.

1. Introduction

In the previous 30 years, the theory of ordinary differential equations in Banach spaces has become a new important branch (see, for example, [2, 5, 6, 13] and references cited therein). In 1988, Guo and Lakshmikantham [8] discussed the existence of multiple solutions for two-point boundary-value problem of ordinary differential equations in Banach spaces. Since then, nonlinear second-order multi-point boundary-value problems in Banach spaces have been studied by several authors (see, for example, [4, 14, 15] and references cited therein). On the other hand, recently, high-order multi-point boundary-value problems for scalar ordinary differential equations have received a great deal of attention in the literature (see, for instance, [3, 7, 12] and references cited therein). However, to the best of our knowledge, no one has considered the existence of multiple positive solutions (at least three or more) for high-order multi-point boundary-value problems in Banach spaces. We will fill this gap in the literature. In this paper, we shall discuss the existence of at least one, two, and multiple positive solutions for the nth-order m-point boundary-value problem value problem

$$y^{(n)}(t) + f(t, y) = \theta, \quad 0 < t < 1, \tag{1.1}$$

$$y(0) = y'(0) = \cdots = y^{(n-2)}(0) = \theta, \quad y(1) = \sum_{i=1}^{m-2} k_i y(\xi_i) \tag{1.2}$$

in a real Banach space E, where $n \geq 2$, θ is the zero element of E, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $k_i > 0$, $i = 1, 2, \dots, m-2$. In the scalar case, the existence of positive solutions to (1.1)-(1.2) had been solved in [3, 7]; So our result extends those

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results, to some degree. The key tool in our approach is the following fixed point theorem of strict-set-contractions.

Theorem 1.1 ([1, 16]). *Let K be a cone of the real Banach space X and $K_{r,R} = \{x \in K : r \le \|x\| \le R\}$ with $R > r > 0$. Suppose that $A : K_{r,R} \to K$ is a strict set contraction such that one of the following two conditions is satisfied*

(i) *$Ax \nleq x$, for all $x \in K$, $\|x\| = r$ and $Ax \ngeq x$, for all $x \in K$, $\|x\| = R$.*
(ii) *$Ax \ngeq x$, for all $x \in K$, $\|x\| = r$ and $Ax \nleq x$, for all $x \in K$, $\|x\| = R$.*

Then A has at least one fixed point $x \in K$ satisfying $r < \|x\| < R$.

Let the real Banach space E with norm $\|\cdot\|$ be partially ordered by a normal cone P of E; i.e., $x \le y$ if and only if $y - x \in P$, and P^* denotes the dual cone of P. Denote the normal constant of P by N; i.e., $\theta \le x \le y$ implies $\|x\| \le N\|y\|$. Take $I = [0, 1]$. For any $x \in C[I, E]$, evidently, $(C[I, E], \|\cdot\|_c)$ is a Banach space with $\|x\|_c = \max_{t\in I} \|x(t)\|$, and $Q = \{x \in C[I, E] : x(t) \ge \theta \text{ for } t \in I\}$ is a cone of the Banach space $C[I, E]$. A function $x \in C^n[I, E]$ is called a positive solution of the boundary-value problem (1.1)-(1.2) if it satisfies (1.1)-(1.2) and $x \in Q,\ x(t) \not\equiv \theta$.

For a bounded set S in a Banach space, we denote $\alpha(S)$ the Kuratowski measure of non-compactness. In this paper, we denote $\alpha(\cdot)$ the Kuratowski measure of non-compactness of a bounded set in E and $C[I, E]$. The closed balls in spaces E and $C[I, E]$ are denoted by $T_r = \{x \in E : \|x\| \le r\}(r > 0)$ and $B_r = \{y \in C[I, E] : \|y\|_c \le r\}(r > 0)$, respectively.

For convenience, we set

$$a_0 = \sum_{i=1}^{m-2} k_i \xi_i^{n-1}, \quad a_1 = \sum_{i=1}^{m-2} k_i \xi_i^{n-1}(1 - \xi_{m-2})^n.$$

In this paper, we assume the following conditions.

(H1) $n \ge 2,\ k_i > 0,\ i = 1, 2, \dots, m-2,\ 0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$, $0 < a_0 < 1$.
(H2) P is a normal cone of E and N is the normal constant; $f : I \times P \to P$, $f(t, \theta) = \theta$, for all $t \in I$; for any $r > 0$, $f(t, x)$ is uniformly continuous and bounded on $I \times (P \cap T_r)$ and there exists a constant L_r with $0 \le L_r < \frac{(n-1)!(1-a_0)}{4}$ such that
$$\alpha(f(I \times D)) \le L_r \alpha(D), \quad \forall D \subset P \cap T_r.$$

2. Preliminary lemmas

Lemma 2.1. *Suppose $a_0 \ne 1$, then for $h(t) \in C[I, E]$, the problem*

$$y^{(n)}(t) + h(t) = \theta, \quad 0 < t < 1, \tag{2.1}$$

$$y(0) = y'(0) = \dots = y^{(n-2)}(0) = \theta, \quad y(1) = \sum_{i=1}^{m-2} k_i y(\xi_i) \tag{2.2}$$

has a unique solution

$$\begin{aligned} y(t) = &-\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} h(s)ds + \frac{t^{n-1}}{1-a_0} \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} h(s)ds \\ &- \frac{t^{n-1}}{1-a_0} \sum_{i=1}^{m-2} k_i \int_0^{\xi_i} \frac{(\xi_i - s)^{n-1}}{(n-1)!} h(s)ds. \end{aligned} \tag{2.3}$$

The proof of the above lemma is easy, so we omit it.

Lemma 2.2. *Let* (H1) *hold. If $h \in Q$, then the unique solution y of* (2.1)-(2.2) *satisfies $y(t) \geq \theta$, $t \in I$, that is $y \in Q$.*

Proof. By (2.3), we get
$$\begin{aligned} y(t) &= -\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}h(s)ds + \frac{t^{n-1}}{1-a_0}\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}h(s)ds \\ &\quad - \frac{t^{n-1}}{1-a_0}\sum_{i=1}^{m-2} k_i \int_0^{\xi_i} \frac{(\xi_i-s)^{n-1}}{(n-1)!}h(s)ds \\ &\geq \frac{t^{n-1}}{1-a_0}\sum_{i=1}^{m-2} k_i \xi_i^{n-1} \int_{\xi_i}^1 \frac{(1-s)^{n-1}}{(n-1)!}h(s)ds \geq \theta. \end{aligned}$$
The proof is complete. □

Lemma 2.3. *Assume* (H1). *If $h \in Q$, then the unique solution y of* (2.1)-(2.2) *satisfies*
$$y(t) \geq \gamma y(s), \quad \forall t \in [\xi_{m-2}, 1],\ s \in I,$$
where
$$\gamma = \min\{\frac{k_{m-2}(1-\xi_{m-2})}{1-k_{m-2}\xi_{m-2}},\ k_{m-2}\xi_{m-2}^{n-1},\ k_1\xi_1^{n-1},\ \xi_{m-2}^{n-1}\}.$$

Proof. For any $\varphi \in P^*$, we have $\varphi(h(t)) \geq 0,\ t \in I$. It follows from
$$\begin{aligned} \varphi(y(t)) &= -\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\varphi(h(s))ds + \frac{t^{n-1}}{1-a_0}\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}\varphi(h(s))ds \\ &\quad - \frac{t^{n-1}}{1-a_0}\sum_{i=1}^{m-2} k_i \int_0^{\xi_i} \frac{(\xi_i-s)^{n-1}}{(n-1)!}\varphi(h(s))ds \end{aligned}$$
and [7, Lemma 3.2] that
$$\varphi(y(t)) \geq \gamma\varphi(y(s)), \quad \forall t \in [\xi_{m-2}, 1],\ s \in I.$$
So, we have
$$\varphi(y(t) - \gamma y(s)) \geq 0, \quad \forall t \in [\xi_{m-2}, 1],\ s \in I.$$
Since $\varphi \in P^*$ is arbitrary, we get
$$y(t) - \gamma y(s) \geq \theta, \quad \forall t \in [\xi_{m-2}, 1],\ s \in I.$$
The proof is complete. □

Define an operator $A : Q \to C[I, E]$ as follows
$$\begin{aligned} A(y(t)) &:= -\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}f(s, y(s))ds + \frac{t^{n-1}}{1-a_0}\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}f(s, y(s))ds \\ &\quad - \frac{t^{n-1}}{1-a_0}\sum_{i=1}^{m-2} k_i \int_0^{\xi_i} \frac{(\xi_i-s)^{n-1}}{(n-1)!}f(s, y(s))ds. \end{aligned} \tag{2.4}$$
By Lemmas 2.1 and 2.2, we get that $A : Q \to C^n[I, E] \cap Q$, and $y(t)$ is a positive solution of (1.1)-(1.2) if and only if $y(t) \in C^n[I, E] \cap Q$ and $y(t) \not\equiv \theta$ is a fixed point of the operator A.

Lemma 2.4. *Suppose* (H1)–(H2) *hold. Then, for any $r > 0$, the operator A is a strict set contraction on $Q \cap B_r$.*

Proof. Since $f(t, x)$ is uniformly continuous and bounded on $I \times (P \cap T_r)$, we see from (2.4) that A is continuous and bounded on $Q \cap B_r$. For any $S \subset Q \cap B_r$, by (2.4), we can easily show that the functions $A(S) = \{Ay | y \in S\}$ are uniformly bounded and equicontinuous. By [13], we have

$$\alpha(A(S)) = \sup_{t \in I} \alpha(A(S(t))), \tag{2.5}$$

where $A(S(t)) = \{Ay(t) : y \in S,\ t \in I \text{ is fixed}\}$. For any $y \in C[I, E]$, $g \in C[I, I]$, by $\int_0^t g(s)y(s)ds \in \overline{co}(\{g(t)y(t)|t \in I\} \cup \{\theta\}) \subset \overline{co}(\{y(t)|t \in I\} \cup \{\theta\})$, we get

$$\begin{aligned}
\alpha(A(S(t))) &= \alpha(\{-\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s, y(s))ds + \frac{t^{n-1}}{1-a_0} \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} f(s, y(s))ds \\
&\quad - \frac{t^{n-1}}{1-a_0} \sum_{i=1}^{m-2} k_i \int_0^{\xi_i} \frac{(\xi_i - s)^{n-1}}{(n-1)!} f(s, y(s))ds | y \in S\}) \\
&\le \frac{1}{(n-1)!} \alpha(\overline{co}(\{f(s, y(s))| s \in I,\ y \in S\} \cup \{\theta\})) \\
&\quad + \frac{1}{(1-a_0)(n-1)!} \alpha(\overline{co}(\{f(s, y(s))| s \in I,\ y \in S\} \cup \{\theta\})) \\
&\quad + \frac{a_0}{(1-a_0)(n-1)!} \alpha(\overline{co}(\{f(s, y(s))| s \in I,\ y \in S\} \cup \{\theta\})) \\
&= \frac{2}{(1-a_0)(n-1)!} \alpha(\{f(s, y(s))| s \in I,\ y \in S\}) \\
&\le \frac{2}{(1-a_0)(n-1)!} \alpha(f(I \times B)),
\end{aligned}$$

where $B = \{y(s) : s \in I,\ y \in S\} \subset P \cap T_r$. By (H2), we get

$$\alpha(A(S(t))) \le \frac{2}{(1-a_0)(n-1)!} L_r \alpha(B). \tag{2.6}$$

For each $\varepsilon > 0$, there exists a partition $S = \bigcup_{j=1}^{l} S_j$ such that

$$\operatorname{diam}(S_j) < \alpha(S) + \frac{\varepsilon}{3}, \quad j = 1, 2, \dots, l. \tag{2.7}$$

Now, choose $y_j \in S_j$, $j = 1, 2, \dots, l$ and a partition $0 = t_0 < t_1 < \dots < t_k = 1$ such that

$$\|y_j(t) - y_j(\bar{t})\| < \frac{\varepsilon}{3}, \quad \forall t,\ \bar{t} \in [t_{i-1}, t_i],\ j = 1, 2, \dots, l,\ i = 1, 2, \dots, k. \tag{2.8}$$

Obviously, $B = \cup_{j=1}^{l} \cup_{i=1}^{k} B_{ij}$, where $B_{ij} = \{y(t) : y \in S_j, t \in [t_{i-1},\ t_i]\}$. For any $y(t)$, $\overline{y}(\bar{t}) \in B_{ij}$, by (2.7) and (2.8), we obtain

$$\begin{aligned}
\|y(t) - \overline{y}(\bar{t})\| &\le \|y(t) - y_j(t)\| + \|y_j(t) - y_j(\bar{t})\| + \|y_j(\bar{t}) - \overline{y}(\bar{t})\| \\
&\le \|y - y_j\|_c + \frac{\varepsilon}{3} + \|y_j - \overline{y}\|_c \\
&\le 2 \operatorname{diam}(S_j) + \frac{\varepsilon}{3} < 2\alpha(S) + \varepsilon,
\end{aligned}$$

which implies $\operatorname{diam}(B_{ij}) \leq 2\alpha(S)+\varepsilon$, and so, $\alpha(B) \leq 2\alpha(S)+\varepsilon$. Since ε is arbitrary, we get

$$\alpha(B) \leq 2\alpha(S). \tag{2.9}$$

It follows from (2.5), (2.6) and (2.9) that

$$\alpha(A(S)) \leq \frac{4}{(n-1)!(1-a_0)} L_r \alpha(S), \quad \forall S \subset Q \cap B_r.$$

By (H2), we get that A is a strict set contraction on $Q \cap B_r$. □

3. Main results

Let $K = \{y \in Q : y(t) \geq \gamma y(s),\ \forall t \in [\xi_{m-2}, 1],\ s \in I\}$. Clearly, $K \subset Q$ is a cone of $C[I, E]$. By Lemma 2.2 and Lemma 2.3, we get $AQ \subset K$. So, $AK \subset K$.

For convenience, for any $x \in P$ and $\varphi \in P^*$, we set

$$f^0 = \limsup_{\|x\| \to 0} \sup_{t \in I} \frac{\|f(t,x)\|}{\|x\|}, \quad f^\infty = \limsup_{\|x\| \to \infty} \sup_{t \in I} \frac{\|f(t,x)\|}{\|x\|},$$
$$f_0^\varphi = \liminf_{\|x\| \to 0} \inf_{t \in I} \frac{\varphi(f(t,x))}{\varphi(x)}, \quad f_\infty^\varphi = \liminf_{\|x\| \to \infty} \inf_{t \in I} \frac{\varphi(f(t,x))}{\varphi(x)}.$$

Then we list the following assumptions:

- (H3) There exists $\varphi \in P^*$ with $\varphi(x) > 0$, for all $x > \theta$ such that $\frac{n!(1-a_0)}{\gamma a_1} < f_0^\varphi \leq \infty$.
- (H4) There exists $\varphi \in P^*$ with $\varphi(x) > 0$, for all $x > \theta$ such that $\frac{n!(1-a_0)}{\gamma a_1} < f_\infty^\varphi \leq \infty$.
- (H5) $0 \leq f^0 < \frac{n!(1-a_0)}{N}$.
- (H6) $0 \leq f^\infty < \frac{n!(1-a_0)}{N}$.
- (H7) There exists $r_0 > 0$ such that $\sup_{t \in I,\, x \in P \cap T_{r_0}} \|f(t,x)\| < \frac{n!(1-a_0)}{N} r_0$.
- (H8) There exist $R_0 > 0$ and $\varphi \in P^*$ with $\varphi(x) > 0$ for any $x > \theta$ such that
$$\inf_{t \in [\xi_{m-2},1],\, x \in P,\, \gamma R_0/N \leq \|x\| \leq R_0} \frac{\varphi(f(t,x))}{\varphi(x)} > \frac{n!(1-a_0)}{\gamma a_1}.$$

Theorem 3.1. *Suppose* (H1)–(H2) *hold. In addition suppose* (H4) *and* (H5) *or* (H3) *and* (H6) *are satisfied. Then* (1.1)-(1.2) *has at least one positive solution.*

Proof. (i) Suppose (H4) and (H5) hold. By (H4), there exist constants

$$M > \frac{n!(1-a_0)}{\gamma a_1} \tag{3.1}$$

and $r_1 > 0$ such that

$$\varphi(f(t,x)) \geq M\varphi(x), \quad \forall t \in I, x \in P,\ \|x\| > r_1. \tag{3.2}$$

For $R > \frac{N}{\gamma} r_1$, we will show that

$$Ay \not\leq y,\ \forall y \in K,\ \|y\|_c = R. \tag{3.3}$$

In fact, if not, there exists $y_0 \in K$, $\|y_0\|_c = R$ such that $Ay_0 \leq y_0$. By

$$y_0(t) \geq \gamma y_0(s) \geq \theta, \quad \forall t \in [\xi_{m-2}, 1],\ s \in I, \tag{3.4}$$

we have

$$\|y_0(t)\| \geq \frac{\gamma}{N} \|y_0\|_c > r_1, \quad \forall t \in [\xi_{m-2}, 1]. \tag{3.5}$$

By (2.4), for any $t \in I$, we have

$$\begin{aligned}
A(y_0(t)) &= -\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s, y_0(s))ds + \frac{t^{n-1}}{1-a_0}\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} f(s, y_0(s))ds \\
&\quad - \frac{t^{n-1}}{1-a_0}\sum_{i=1}^{m-2} k_i \int_0^{\xi_i} \frac{(\xi_i - s)^{n-1}}{(n-1)!} f(s, y_0(s))ds. \\
&\geq \frac{a_0 t^{n-1}}{1-a_0}\int_{\xi_{m-2}}^1 \frac{(1-s)^{n-1}}{(n-1)!} f(s, y_0(s))ds.
\end{aligned}$$

This, together with (3.2), (3.4) and (3.5), implies

$$\begin{aligned}
\varphi(Ay_0(1)) &\geq \frac{a_0}{1-a_0}\int_{\xi_{m-2}}^1 \frac{(1-s)^{n-1}}{(n-1)!} M\gamma\varphi(y_0(1))ds \\
&= \frac{a_1}{n!(1-a_0)} M\gamma\varphi(y_0(1)).
\end{aligned}$$

Considering $Ay_0 \leq y_0$, we get

$$\varphi(y_0(1)) \geq \frac{\gamma a_1}{n!(1-a_0)} M\varphi(y_0(1)). \tag{3.6}$$

It is easy to see that $\varphi(y_0(1)) > 0$ (In fact, if $\varphi(y_0(1)) = 0$, by (3.4), we get $\varphi(y_0(1)) \geq \gamma\varphi(y_0(s)) \geq 0$, for all $s \in I$. So, we have $\varphi(y_0(s)) \equiv 0$, for all $s \in I$, that is $y_0(s) \equiv \theta$. This is a contradiction with $\|y_0\|_c = R$). So, (3.6) contradicts with (3.1). Therefore, (3.3) is true.

On the other hand, by (H5) and $f(t, \theta) = \theta$, we get that there exist constants $\varepsilon \in (0, n!(1-a_0)/N)$ and $0 < r_2 < R$ such that

$$\|f(t, x)\| \leq \varepsilon\|x\|, \quad \forall t \in I,\ x \in P,\ \|x\| < r_2. \tag{3.7}$$

For any $0 < r < r_2$, we now prove

$$Ay \not\geq y,\ \forall y \in K,\ \|y\|_c = r. \tag{3.8}$$

In fact, if not, there exists $y_0 \in K$, $\|y_0\|_c = r$ such that $Ay_0 \geq y_0$. Since (2.4) implies

$$Ay_0(t) \leq \frac{t^{n-1}}{1-a_0}\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} f(s, y_0(s))ds, \quad \forall t \in I, \tag{3.9}$$

we have

$$\theta \leq y_0(t) \leq \frac{t^{n-1}}{1-a_0}\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} f(s, y_0(s))ds, \quad \forall t \in I.$$

This, together with (3.7), implies

$$\|y_0(t)\| \leq \frac{N\varepsilon}{1-a_0}\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}\|y_0(s)\|ds \leq \frac{N\varepsilon\|y_0\|_c}{n!(1-a_0)}, \quad \forall t \in I.$$

Therefore, we get $\varepsilon \geq n!(1-a_0)/N$. This is a contradiction. So, (3.8) is true.

By (3.3), (3.8), Lemma 2.4 and Theorem 1.1, we get that the operator A has at least one fixed point $y \in K$ satisfying $r < \|y\|_c < R$.

(ii) Suppose (H3) and (H6) hold. By (H3), in the same way as establishing (3.3) we can assert that there exists $r_2 > 0$ such that for any $0 < r < r_2$,

$$Ay \not\leq y, \quad \forall y \in K,\ \|y\|_c = r. \tag{3.10}$$

On the other hand, by (H6), we get that there exist constants $0 < \varepsilon < n!(1-a_0)/N$ and $r_1 > 0$ such that

$$\|f(t,x)\| \le \varepsilon\|x\|, \quad \forall t \in I,\ x \in P,\ \|x\| > r_1.$$

By (H2), we get

$$\sup_{t\in I,\, x\in P\cap T_{r_1}} \|f(t,x)\| := b < \infty.$$

So, we have

$$\|f(t,x)\| \le \varepsilon\|x\| + b, \quad \forall t \in I,\ x \in P. \tag{3.11}$$

Take $R > \max\{r_2, \frac{Nb}{n!(1-a_0)-N\varepsilon}\}$. We will prove that

$$Ay \ngeq y, \quad \forall y \in K,\ \|y\|_c = R. \tag{3.12}$$

In fact, if there exists $y_0 \in K$, $\|y_0\|_c = R$ such that $Ay_0 \ge y_0$. Then, by (3.9) and (3.11), we get

$$\|y_0(t)\| \le \frac{Nt^{n-1}}{1-a_0}\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}(\varepsilon\|y_0(s)\| + b)ds \le \frac{N}{n!(1-a_0)}(\varepsilon\|y_0\|_c + b),\ \forall t \in I.$$

So, we have

$$\|y_0\|_c \le \frac{Nb}{n!(1-a_0) - N\varepsilon} < R.$$

A contradiction. Therefore, (3.12) holds.

By (3.10), (3.12), Lemma 2.4 and Theorem 1.1, the operator A has at least one fixed point $y \in K$ satisfying $r < \|y\|_c < R$. The proof is complete. $\square$

Theorem 3.2. *Suppose* (H1) *and* (H2) *hold. In addition suppose that one of the following conditions is satisfied*

(i) (H3), (H4), (H7) *hold.*
(ii) (H5), (H6), (H8) *hold.*

Then (1.1)-(1.2) *has at least two positive solutions.*

Proof. (i) By (H3), (H4) and the proof of Theorem 3.1, there exist r, R with $0 < r < r_0 < R$ such that

$$\begin{gathered} Ay \nleq y, \quad \forall y \in K,\ \|y\|_c = r. \\ Ay \nleq y, \quad \forall y \in K,\ \|y\|_c = R. \end{gathered} \tag{3.13}$$

Now, we will prove

$$Ay \ngeq y, \quad \forall y \in K,\ \|y\|_c = r_0. \tag{3.14}$$

In fact, if there exists $y_0 \in K$, $\|y_0\|_c = r_0$ such that $Ay_0 \ge y_0$. By (3.9) and (H7), we get

$$\|y_0\|_c < \frac{N}{1-a_0}\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}\cdot\frac{n!(1-a_0)}{N} r_0 ds = r_0.$$

A contradiction. So, (3.14) is true. By (3.12)-(3.14), Lemma 2.4 and Theorem 1.1, we get that the operator A has at least two fixed points y_1, $y_2 \in K$ satisfying

$$r < \|y_1\|_c < r_0 < \|y_2\|_c < R.$$

(ii) By (H5), (H6) and the proof of Theorem 3.1, there exist r, R with $0 < r < R_0 < R$ such that

$$Ay \ngeq y, \quad \forall y \in K,\ \|y\|_c = r. \tag{3.15}$$

$$Ay \ngeq y, \quad \forall y \in K,\ \|y\|_c = R. \tag{3.16}$$

On the other hand, by (H8) and the methods used in the proof of (3.3), we can prove that

$$Ay \not\leq y, \quad \forall y \in K, \quad \|y\|_c = R_0. \tag{3.17}$$

By (3.15)-(3.17), Lemma 2.4 and Theorem 1.1, the operator A has at least two fixed points $y_1, y_2 \in K$ satisfying

$$r < \|y_1\|_c < R_0 < \|y_2\|_c < R.$$

The proof is complete. □

Similar to the proofs of Theorem 3.1 and Theorem 3.2, we can easily get the following corollaries.

Corollary 3.3. *Suppose* (H1), (H2) *hold. In addition suppose that one of the following conditions is satisfied*

(i) (H4), (H5), (H7), (H8) *hold with* $R_0 < r_0$.
(ii) (H3), (H6), (H7), (H8) *hold with* $r_0 < R_0$.

Then (1.1)-(1.2) *has at least three positive solutions.*

Corollary 3.4. *Suppose* (H1), (H2) *hold. In addition suppose that one of the following conditions is satisfied*

(i) (H5)–(H7) *hold, and there exist* $R_i > 0$, $\varphi_i \in P^*$ *with* $\varphi_i(x) > 0$ *for* $x > \theta$, $i = 1, 2$ *such that*

$$\inf_{t\in[\xi_{m-2},1],\, x\in P,\, \gamma R_i/N\leq\|x\|\leq R_i} \frac{\varphi_i(f(t,x))}{\varphi_i(x)} > \frac{n!(1-a_0)}{\gamma a_1}, \quad i = 1, 2,$$

where $R_1 < r_0 < R_2$.

(ii) (H3), (H4), (H8) *hold, and there exist* $r_1, r_2 > 0$ *such that*

$$\sup_{t\in I,\, x\in P\cap T_{r_i}} \|f(t,\ x)\| < \frac{n!(1-a_0)}{N} r_i, \quad i = 1, 2,$$

where $r_1 < R_0 < r_2$.

Then (1.1)-(1.2) *has at least four positive solutions.*

We can easily obtain the existence of multiple positive solutions for (1.1)-(1.2).

4. Examples

In this section, we give some examples to illustrate our results.

Example 4.1. *Consider the boundary value problem*

$$y_i'''(t) + f_i(t, y_1, y_2, \dots, y_l) = 0, \quad 0 < t < 1, \tag{4.1}$$

$$y_i(0) = y_i'(0) = 0, \quad y_i(1) = y_i(\frac{1}{2}), \quad i = 1, 2, \dots, l, \tag{4.2}$$

where $f_i(t, y_1, y_2, \dots, y_l) = y_{i+1}^{\frac{2}{3}} + e^{-t} y_{i+2}^{\frac{3}{2}}$, $i = 1, 2, \dots, l-2$, $f_{l-1}(t, y_1, y_2, \dots, y_l) = y_l^{\frac{2}{3}} + e^{-t} y_1^{\frac{3}{2}}$, $f_l(t, y_1, y_2, \dots, y_l) = y_1^{\frac{2}{3}} + e^{-t} y_2^{\frac{3}{2}}$.

Conclusion. The problem (4.1)-(4.2) has at least two positive solutions.

Proof. Let $E = R^l = \{y = (y_1, y_2, \dots, y_l) | y_i \in R,\ i = 1, 2, \dots, l\}$ with the norm $\|y\| = \max_{1\le i\le l} |y_i|$, and $P = \{y = (y_1, y_2, \dots, y_l) | y_i \ge 0,\ i = 1, 2, \dots, l\}$. Then P is a normal cone in E and the normal constant $N = 1$. Corresponding to (1.1)-(1.2), we get $n = m = 3,\ k_1 = 1,\ \xi_1 = \frac{1}{2},\ a_0 = \frac{1}{4},\ a_1 = \frac{1}{32},\ \gamma = \frac{1}{4}$. Obviously, (H1) is satisfied. Set $\theta = (0, 0, \dots, 0)$ and $f = (f_1, f_2, \dots, f_l)$. Then $f : I \times P \to P$ is continuous and $f(t, \theta) = \theta$, for all $t \in I$. It is clear that $\alpha(f(D)) = 0$ for any $D \subset P \cap T_r$. So, (H2) holds. It is easy to see that $P^* = P$. So, we choose $\varphi = (1, 1, \dots, 1)$, and then

$$\frac{\varphi(f(t,y))}{\varphi(y)} = \frac{\sum_{i=1}^{l} f_i(t, y_1, y_2, \dots, y_l)}{\sum_{i=1}^{l} y_i}.$$

We now prove that the conditions (H3) and (H4) are satisfied.

For any $y \in P$, $y \neq \theta$, we can easily get $\varphi(y) > 0$ and

$$\frac{\varphi(f(t,y))}{\varphi(y)} = \frac{\sum_{i=1}^{l} y_i^{\frac{2}{3}} + e^{-t} \sum_{i=1}^{l} y_i^{\frac{3}{2}}}{\sum_{i=1}^{l} y_i} \ge \frac{\max_{1\le i\le l} y_i^{\frac{2}{3}}}{n \max_{1\le i\le l} y_i} = \frac{1}{n} \frac{1}{\max_{1\le i\le l} y_i^{\frac{1}{3}}} \to \infty,\ (\|y\| \to 0).$$

and

$$\frac{\varphi(f(t,y))}{\varphi(y)} = \frac{\sum_{i=1}^{l} y_i^{\frac{2}{3}} + e^{-t} \sum_{i=1}^{l} y_i^{\frac{3}{2}}}{\sum_{i=1}^{l} y_i} \ge \frac{e^{-1} \max_{1\le i\le l} y_i^{\frac{3}{2}}}{n \max_{1\le i\le l} y_i} = \frac{1}{ne} \max_{1\le i\le l} y_i^{\frac{1}{2}} \to \infty,\ (\|y\| \to \infty).$$

So, (H3) and (H4) hold. Finally, we will show (H7) is satisfied.

Since $\frac{n!(1-a_0)}{N} r_0 = 4.5 r_0$, taking $r_0 = 1$, we get

$$\sup_{t\in I,\, y\in P\cap T_{r_0}} \|f(t,y)\| \le \max_{1\le i\le l} y_i^{\frac{2}{3}} + \max_{1\le i\le l} y_i^{\frac{3}{2}} \le 2.$$

Therefore, (H7) holds. By Theorem 3.2 (i), we get that the problem (4.1)-(4.2) has at least two positive solutions. □

Example 4.2. *The boundary value problem*

$$y_i^{'''}(t) + e^{at} \sin^2(\frac{\pi}{2} y_i) = 0, \quad 0 < t < 1, \tag{4.3}$$

$$y_i(0) = y_i'(0) = 0, \quad y_i(1) = y_i(\frac{1}{2}), \quad i = 1, 2, \dots, l. \tag{4.4}$$

has at least two positive solutions, where $a > 2 \ln \frac{576 l}{\sin^2 \frac{\pi}{8}}$.

Proof. Let E, $\|\cdot\|$, P, θ, φ be the same as in Example 4.1. Take $f = (e^{at} \sin^2(\frac{\pi}{2} y_1), e^{at} \sin^2(\frac{\pi}{2} y_2), \dots, e^{at} \sin^2(\frac{\pi}{2} y_l))$. Then $f : I \times P \to P$ is continuous and $f(t, \theta) = \theta$, for all $t \in I$. Similar to the proof of Example 4.1, we get that (H1) and (H2) are satisfied. Now, we prove that (H5) and (H6) are satisfied. Because

$$\frac{\|f(t,y)\|}{\|y\|} = \frac{\max_{1\le i\le l} e^{at} \sin^2(\frac{\pi}{2} y_i)}{\max_{1\le i\le l} y_i} \le e^a \frac{\max_{1\le i\le l} \sin^2(\frac{\pi}{2} y_i)}{\max_{1\le i\le l} y_i} \to 0, \quad (\|y\| \to 0),$$

and

$$\frac{\|f(t,y)\|}{\|y\|} = \frac{\max\limits_{1\le i\le l} e^{at}\sin^2(\frac{\pi}{2}y_i)}{\max\limits_{1\le i\le l} y_i} \le e^a \frac{\max\limits_{1\le i\le l} \sin^2(\frac{\pi}{2}y_i)}{\max\limits_{1\le i\le l} y_i} \to 0, \quad (\|y\|\to\infty),$$

(H5) and (H6) hold. Now, we prove that (H8) is satisfied. Since $\frac{n!(1-a_0)}{\gamma a_1} = 576$ (where, n, a_0, a_1 and γ are the same as in Example 4.1), taking $R_0 = 1$, for $t\in[\frac{1}{2},1]$, $y\in P$, $\frac{1}{4}\le\|y\|\le 1$, we have

$$\frac{\varphi(f(t,y))}{\varphi(y)} = \frac{\sum\limits_{i=1}^{l} e^{at}\sin^2(\frac{\pi}{2}y_i)}{\sum\limits_{i=1}^{l} y_i} \ge e^{\frac{a}{2}}\frac{\sin^2(\frac{\pi}{8})}{l} > 576.$$

So, (H8) holds. By Theorem 3.2 (ii), we get that the problem (4.3)-(4.4) has at least two positive solutions. □

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