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SOME BASIC THEOREMS ON DIFFERENCE-DIFFERENTIAL EQUATIONS

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Abstract. In this paper some basic theorems on the existence, uniqueness and continuous dependence of solutions of a certain difference-differential equation are established. The well known Banach fixed point theorem and the Gronwall-Bellman integral inequality are used to establish these results.

1. Introduction

Let $\mathbb{R}^n$ denote the real n-dimensional Euclidean space with the corresponding norm $|\cdot|$. Let $\mathbb{R}_+ = [0, \infty)$ be a subset of the real numbers. Consider the difference-differential equation

$$x'(t) = f(t, x(t), x(t-1)), \tag{1.1}$$

for $t \in \mathbb{R}_+$ under the initial conditions

$$x(t-1) = \phi(t) \quad (0 \le t < 1), \quad x(0) = x_0, \tag{1.2}$$

with $c_0 = \phi(1-0)$, where $f \in C(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ and $\phi(t)$ is a continuous function for which $\lim_{t\to 1-0} \phi(t)$ exists. If we consider the solutions of (1.1) for $t \in \mathbb{R}_+$, we obtain a function $x(t-1)$ which is unable to define as a solution for $0 \le t < 1$. Hence, we have to impose some additional condition, for example the first condition in (1.2). In this case it is sufficient to consider the ordinary differential equation

$$x'(t) = f(t, x(t), \phi(t)),$$

for $0 \le t < 1$, with the second condition in (1.2). We note that, here it is essential to obtain the solutions of (1.1) for $0 \le t < \infty$. It is easy to observe that the integral equations which are equivalent to (1.1)-(1.2) are

$$x(t) = x_0 + \int_0^t f(s, x(s), \phi(s))ds,$$

for $0 \le t < 1$ and

$$x(t) = x_0 + \int_0^1 f(s, x(s), \phi(s))ds + \int_1^t f(s, x(s), x(s-1))ds,$$

for $0 \le t < \infty$.

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In 1960, Sugiyama [9] (see also [10]) studied the existence and uniqueness of solutions to (1.1) under the initial conditions (1.2) by using Tychonov's fixed point theorem, the method of successive approximations, and the comparison method. The main objective of the present paper is to study the existence, uniqueness and continuous dependence of solutions to the initial-value problem (1.1)-(1.2). The main tools are the applications of the Banach fixed point theorem [4, p. 37] and the Gronwall-Bellman integral inequality (see [8, p.11]).

2. Existence and uniqueness

Let S be the space of functions $z(t) \in \mathbb{R}^n$ which are continuous for $t \in \mathbb{R}_+$ and fulfill the condition

$$|z(t)| = O(\exp(\lambda t)), \tag{2.1}$$

for some positive constant $\lambda > 0$. In this space we define the norm (see [2, 7])

$$|z|_S = \sup_{t\in\mathbb{R}_+} [|z(t)| \exp(-\lambda t)]. \tag{2.2}$$

It is easy to see that S with the above norm is a Banach space. Note that condition (2.1) implies the existence of a nonnegative constant N such that $|z(t)| \le N \exp(\lambda t)$ for $t \in \mathbb{R}_+$. Using this fact in (2.2) we observe that

$$|z|_S \le N. \tag{2.3}$$

. We need the following integral inequality, often referred to as Gronwall Bellman inequality [8, p.11].

Lemma 2.1. *Let u and f be continuous functions defined on $\mathbb{R}_+$ and c be a nonnegative constant. If*

$$u(t) \le c + \int_0^t f(s)u(s)ds,$$

for $t \in \mathbb{R}_+$, then

$$u(t) \le c \exp\Big(\int_0^t f(s)ds\Big),$$

for $t \in \mathbb{R}_+$.

Now we shall prove the following main result of this section.

Theorem 2.2. *Assume that:*

(i) *The function f in (1.1) satisfies the condition*

$$|f(t, x, y) - f(t, \bar{x}, \bar{y})| \le h(t)[|x - \bar{x}| + |y - \bar{y}|], \tag{2.4}$$

for $(t, x, y), (t, \bar{x}, \bar{y}) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n$, where $h \in C(\mathbb{R}_+, \mathbb{R}_+)$,

(ii) *for λ as in (2.1):*

(a) there exists a nonnegative constant α such that $\alpha < 1$ and

$$\int_0^t [h(s) + h(s + 1)] \exp(\lambda s)ds \le \alpha \exp(\lambda t), \tag{2.5}$$

for $t \in \mathbb{R}_+$,

(b) there exists a nonnegative constant β such that

$$|x_0| + \int_0^1 h(s)|\phi(s)|ds + \int_0^t |f(s, 0, 0)|ds \le \beta \exp(\lambda t), \tag{2.6}$$

for $t \in \mathbb{R}_+$. Then the (1.1)-(1.2) has a unique solution on $\mathbb{R}_+$ in S.

Remark 2.3. We note that the linear systems with such a delay with bounded coefficients on $\mathbb{R}_+$ satisfy (2.4), (2.5), (2.6).

Proof of Theorem 2.2. Let $x(t) \in S$ and define the operator T by (see [9])

$$Tx(t) = x_0 + \int_0^t f(s, x(s), \phi(s))ds, \tag{2.7}$$

for $0 \le t < 1$, and

$$Tx(t) = x_0 + \int_0^1 f(s, x(s), \phi(s))ds + \int_1^t f(s, x(s), x(s-1))ds, \tag{2.8}$$

for $1 \le t < \infty$. First we shall show that Tx maps S into itself. Evidently, Tx is continuous on $\mathbb{R}_+$ and $Tx \in \mathbb{R}^n$. To verify that (2.1) is fulfilled, we consider the following two cases.

Case 1: $0 \le t < 1$. From (2.7), using the hypotheses and (2.3), we have

$$\begin{aligned}
|Tx(t)| &\le |x_0| + \int_0^t |f(s, x(s), \phi(s)) - f(s,0,0)|ds + \int_0^t |f(s,0,0)|ds \\
&\le |x_0| + \int_0^t h(s)[|x(s)| + |\phi(s)|]ds + \int_0^t |f(s,0,0)|ds \\
&\le |x_0| + \int_0^1 h(s)|\phi(s)|ds + \int_0^t |f(s,0,0)|ds + |x|_S \int_0^t h(s)\exp(\lambda s)ds \\
&\le [\beta + N\alpha]\exp(\lambda t).
\end{aligned} \tag{2.9}$$

Case 2: $1 \le t < \infty$ From (2.8), using the hypotheses and (2.3), we have

$$\begin{aligned}
|Tx(t)| &\le |x_0| + \int_0^1 |f(s, x(s), \phi(s)) - f(s,0,0)|ds + \int_0^1 |f(s,0,0)|ds \\
&\quad + \int_1^t |f(s, x(s), x(s-1)) - f(s,0,0)|ds + \int_1^t |f(s,0,0)|ds \\
&\le |x_0| + \int_0^1 h(s)[|x(s)| + |\phi(s)|]ds + \int_0^t |f(s,0,0)|ds \\
&\quad + \int_1^t h(s)[|x(s)| + |x(s-1)|]ds \\
&= |x_0| + \int_0^1 h(s)|\phi(s)|ds + \int_0^t |f(s,0,0)|ds \\
&\quad + \int_0^1 h(s)|x(s)|ds + \int_1^t h(s)|x(s)|ds + \int_1^t h(s)|x(s-1)|ds \\
&\le \beta\exp(\lambda t) + \int_0^t h(s)|x(s)|ds + I_1,
\end{aligned} \tag{2.10}$$

where

$$I_1 = \int_1^t h(s)|x(s-1)|ds. \tag{2.11}$$

By making the change of variable, we obtain

$$I_1 = \int_0^{t-1} h(\sigma+1)|x(\sigma)|d\sigma \le \int_0^t h(\sigma+1)|x(\sigma)|d\sigma. \tag{2.12}$$

Using (2.12) in (2.10), we get

$$\begin{aligned} |Tx(t)| &\le \beta\exp(\lambda t) + |x|_S \int_0^t [h(s)+h(s+1)]\exp(\lambda s)ds \\ &\le [\beta + N\alpha]\exp(\lambda t). \end{aligned} \tag{2.13}$$

From this inequality and (2.9), it follows that $Tx \in S$. This proves that T maps S into itself.

Next, we verify that the operator T is a contraction map. Let $x(t), y(t) \in S$. We consider the following two cases.

Case 1: $0 \le t < 1$**.** From (2.7) and using the hypotheses, we have

$$\begin{aligned} |Tx(t) - Ty(t)| &\le \int_0^t |f(s, x(s), \phi(s)) - f(s, y(s), \phi(s))|ds \\ &\le \int_0^t h(s)|x(s) - y(s)|ds \\ &\le |x-y|_S \int_0^t h(s)\exp(\lambda s)ds \\ &\le |x-y|_S \alpha\exp(\lambda t). \end{aligned} \tag{2.14}$$

Case 2: $1 \le t < \infty$**.** From (2.8) and using the hypotheses, we have

$$\begin{aligned} |Tx(t) - Ty(t)| &\le \int_0^1 |f(s, x(s), \phi(s)) - f(s, y(s), \phi(s))|ds \\ &\quad + \int_1^t |f(s, x(s), x(s-1)) - f(s, y(s), y(s-1))|ds \\ &\le \int_0^1 h(s)|x(s) - y(s)|ds \\ &\quad + \int_1^t h(s)[|x(s) - y(s)| + |x(s-1) - y(s-1)|]ds \\ &= \int_0^t h(s)|x(s) - y(s)|ds + I_2, \end{aligned} \tag{2.15}$$

where

$$I_2 = \int_1^t h(s)|x(s-1) - y(s-1)|ds. \tag{2.16}$$

By making the change of variable, we obtain

$$I_2 \le \int_0^t h(s+1)|x(s) - y(s)|ds. \tag{2.17}$$

Using this inequality and (2.15) we get

$$\begin{aligned} |Tx(t) - Ty(t)| &\le |x-y|_S \int_0^t [h(s)+h(s+1)]\exp(\lambda s)ds \\ &\le |x-y|_S \alpha\exp(\lambda t), \end{aligned} \tag{2.18}$$

for all $x, y \in S$. From (2.14) and (2.18), we observe that

$$|Tx - Ty|_S \le \alpha|x-y|_S.$$

Since $\alpha < 1$, it follows from Banach fixed point theorem [4, p. 37] that T has a unique fixed in S. The fixed point of T is however a solution of (1.1)-(1.2). The proof is complete. □

Remark 2.4. We note that the norm defined in (2.2) was first used by Bielecki [2] for proving the existence and uniqueness of global solutions for ordinary differential equations. For the developments related to Bielecki's method, see [3].

The following theorem shows the uniqueness of solutions to (1.1)-(1.2) without the existence part.

Theorem 2.5. *Assume that the function f in equation* (1.1) *satisfies* (2.4). *Then the* (1.1)-(1.2) *has at most one solution on* $\mathbb{R}_+$.

Proof. Let $x_1(t)$ and $x_2(t)$ be two solutions of (1.1)-(1.2) and $u(t) = |x_1(t) - x_2(t)|$, $t \in \mathbb{R}_+$. We consider the following two cases.
Case 1: $0 \le t < 1$**.** From the hypotheses, we have

$$\begin{aligned} u(t) &\le \int_0^t |f(s, x_1(s), \phi(s)) - f(s, x_2(s), \phi(s))| ds \\ &\le \int_0^t h(s)|x_1(s) - x_2(s)| ds \\ &= \int_0^t h(s)u(s) ds. \end{aligned} \tag{2.19}$$

Now a suitable application of Lemma 2.1 (with $c = 0$) yields

$$|x_1(t) - x_2(t)| \le 0. \tag{2.20}$$

Case 2: $1 \le t < \infty$**.** From the hypotheses, we have

$$\begin{aligned} u(t) \le &\int_0^1 |f(s, x_1(s), \phi(s)) - f(s, x_2(s), \phi(s))| ds \\ &+ \int_1^t |f(s, x_1(s), x_1(s-1)) - f(s, x_2(s), x_2(s-1))| ds \\ \le &\int_0^1 h(s)|x_1(s) - x_2(s)| ds \\ &+ \int_1^t h(s)[|x_1(s) - x_2(s)| + |x_1(s-1) - x_2(s-1)|] ds \\ = &\int_0^t h(s)|x_1(s) - x_2(s)| ds + I_3, \end{aligned} \tag{2.21}$$

where

$$I_3 = \int_1^t h(s)|x_1(s-1) - x_2(s-1)| ds.$$

By making a change of variable, we observe that

$$I_3 \le \int_0^t h(s+1)|x_1(s) - x_2(s)| ds.$$

Using this inequality in (2.21), we obtain

$$u(t) \le \int_0^t [h(s) + h(s+1)]u(s) ds.$$

Now a suitable application of Lemma 2.1 (with $c = 0$) yields

$$|x_1(t) - x_2(t)| \le 0.$$

From (2.20) and this inequality, we have $x_1(t) = x_2(t)$ for $t \in \mathbb{R}_+$. Thus there is at most one solution to (1.1)-(1.2) on $\mathbb{R}_+$. □

3. Continuous dependence

In this section we study the continuous dependence of solutions to (1.1) on the given initial data, and on the function f. Also we show the continuous dependence of solutions of equations of the form (1.1) on certain parameters.

First, we shall give the following theorem concerning the continuous dependence of solutions to (1.1) on the given initial data.

Theorem 3.1. *Assume that the function f in* (1.1) *satisfies the condition* (2.4). *Let $x_1(t)$ and $x_2(t)$ be the solutions of* (1.1) *with the initial conditions*

$$x_1(t-1) = \phi_1(t) \quad (0 \le t < 1), \quad x_1(0) = c_1, \tag{3.1}$$

$$x_2(t-1) = \phi_2(t) \quad (0 \le t < 1), \quad x_2(0) = c_2, \tag{3.2}$$

respectively, where c_1, c_2 are constants. Then

$$|x_1(t) - x_2(t)| \le c \exp\Big(\int_0^t h(s)ds\Big), \tag{3.3}$$

for $0 \le t < 1$ and

$$|x_1(t) - x_2(t)| \le c \exp\Big(\int_0^t [h(s) + h(s+1)]ds\Big), \tag{3.4}$$

for $1 \le t < \infty$, where

$$c = |c_1 - c_2| + \int_0^1 h(s)|\phi_1(s) - \phi_2(s)|ds. \tag{3.5}$$

Proof. Let $u(t) = |x_1(t) - x_1(t)$ for $t \in \mathbb{R}_+$. We consider the following two cases.
Case 1: $0 \le t < 1$**.** From the hypotheses, it follows that

$$\begin{aligned} u(t) &\le |c_1 - c_2| + \int_0^t h(s)|f(s, x_1(s), \phi_1(s)) - f(s, x_2(s), \phi_2(s))|ds \\ &\le |c_1 - c_2| + \int_0^t h(s)[||x_1(s) - x_2(s)| + |\phi_1(s) - \phi_2(s)|]ds \\ &\le |c_1 - c_2| + \int_0^1 h(s)|\phi_1(s) - \phi_2(s)|ds + \int_0^t h(s)|x_1(s) - x_2(s)|ds \\ &= c + \int_0^t h(s)u(s)ds. \end{aligned} \tag{3.6}$$

Now an application of Lemma 2.1 to (3.6), yields (3.2).
Case 2: $1 \le t < \infty$**.** By following a similar arguments as in the proof of Theorem 2.5 in case 2, from the hypotheses, it follows that

$$\begin{aligned} u(t) \le |c_1 - c_2| &+ \int_0^1 |f(s, x_1(s), \phi_1(s)) - f(s, x_2(s), \phi_2(s))|ds \\ &+ \int_1^t |f(s, x_1(s), x_1(s-1)) - f(s, x_2(s), x_2(s-1))|ds \end{aligned}$$

$$
\begin{aligned}
&\le |c_1 - c_2| + \int_0^1 h(s)[|x_1(s) - x_2(s)| + |\phi_1(s) - \phi_2(s)|]ds \\
&\quad + \int_1^t h(s)[|x_1(s) - x_2(s)| + |x_1(s-1) - x_2(s-1)|]ds \\
&= |c_1 - c_2| + \int_0^1 h(s)|\phi_1(s) - \phi_2(s)|ds + \int_0^t h(s)|x_1(s) - x_2(s)|ds \\
&\quad + \int_1^t h(s)|x_1(s-1) - x_2(s-1)|ds \\
&\le c + \int_0^t [h(s) + h(s+1)]u(s)ds. \qquad (3.7)
\end{aligned}
$$

Now an application of Lemma 2.1 yields (3.4). From (3.3) and (3.4), it follows that the solutions of equation (1.1) depends on the given initial data. □

Now, we consider (1.1)-(1.2) and the corresponding initial-value problem

$$
y'(t) = F(t, y(t), y(t-1)), \tag{3.8}
$$

for $t \in \mathbb{R}_+$ under the initial conditions

$$
y(t-1) = \psi(t), \quad y(0) = y_0, \tag{3.9}
$$

where $F \in C(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ and $\psi(t)$ is a continuous function for which the limit $\lim_{t\to 1-0} \psi(t)$ exists.

The following theorem shows the continuous dependence of solutions to (1.1)-(1.2) on the function $f/$

Theorem 3.2. *Assume that the function f in* (1.1) *satisfies* (2.4) *and*

$$
\begin{aligned}
&|x_0 - y_0| + \int_0^1 h(s)|\phi(s) - \psi(s)|ds \\
&+ \int_0^1 |f(s, y(s), \phi(s)) - F(s, y(s), \psi(s))|ds \le \varepsilon_1,
\end{aligned} \tag{3.10}
$$

$$
\int_1^t |f(s, y(s), y(s-1)) - F(s, y(s), y(s-1))|ds \le \varepsilon_2, \tag{3.11}
$$

where x_0, ϕ, f and y_0, ψ, F are as in (1.1)-(1.2) *and* (3.8)-(3.9)*, $\varepsilon_1, \varepsilon_2$ are nonnegative constants and $y(t)$ is a solution of* (3.8)-(3.9)*. Then the solution $x(t)$ of* (1.1)-(1.2) *depends continuously on the functions involved therein as given below by* (3.13) *and* (3.15).

Proof. Let $u(t) = |x(t) - y(t)|$ for $t \in \mathbb{R}_+$. We consider the following two cases.
Case 1: $0 \le t < 1$. From the hypotheses, we have

$$
\begin{aligned}
u(t) &\le |x_0 - y_0| + \int_0^t |f(s, y(s), \phi(s)) - F(s, y(s), \psi(s))|ds \\
&\le |x_0 - y_0| + \int_0^t |f(s, y(s), \phi(s)) - f(s, y(s), \psi(s))|ds \\
&\quad + \int_0^t |f(s, y(s), \psi(s)) - F(s, y(s), \psi(s))|ds \\
&\le |x_0 - y_0| + \int_0^t h(s)[|x(s) - y(s)| + |\phi(s) - \psi(s)|]ds
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \int_0^1 |f(s, y(s), \psi(s)) - F(s, y(s), \psi(s))| ds \\
&\le |x_0 - y_0| + \int_0^t h(s)|x(s) - y(s)| ds + \int_0^1 h(s)|\phi(s) - \psi(s)| ds \\
&+ \int_0^1 |f(s, y(s), \psi(s)) - F(s, y(s), \psi(s))| ds \\
&\le \varepsilon_1 + \int_0^t h(s) u(s) ds.
\end{aligned} \tag{3.12}
$$

Now an application of Lemma 2.1 yields that for $0 \le t < 1$,

$$
|x(t) - y(t)| \le \varepsilon_1 \exp\Big(\int_0^t h(s) ds\Big). \tag{3.13}
$$

Case 2: $1 \le t < \infty$. Following an arguments as in the proof of Theorem 2.5 in case 2, from the hypotheses, we have

$$
\begin{aligned}
u(t) &\le |x_0 - y_0| + \int_0^1 |f(s, x(s), \phi(s)) - F(s, y(s), \psi(s))| ds \\
&\quad + \int_1^t |f(s, x(s), x(s-1)) - F(s, y(s), y(s-1))| ds \\
&\le |x_0 - y_0| + \int_0^1 |f(s, x(s), \phi(s)) - f(s, y(s), \psi(s))| ds \\
&\quad + \int_0^1 |f(s, y(s), \psi(s)) - F(s, y(s), \psi(s))| ds \\
&\quad + \int_1^t |f(s, x(s), x(s-1)) - f(s, y(s), y(s-1))| ds \\
&\quad + \int_1^t |f(s, y(s), y(s-1)) - F(s, y(s), y(s-1))| ds \\
&\le |x_0 - y_0| + \int_0^1 h(s)[|x(s) - y(s)| + |\phi(s) - \psi(s)|] ds \\
&\quad + \int_0^1 |f(s, y(s), \psi(s)) - F(s, y(s), \psi(s))| ds \\
&\quad + \int_1^t h(s)[|x(s) - y(s)| + |x(s-1) - y(s-1)|] ds \\
&\quad + \int_1^t |f(s, y(s), y(s-1)) - F(s, y(s), y(s-1))| ds \\
&\le |x_0 - y_0| + \int_0^1 h(s)|\phi(s) - \psi(s)| ds \\
&\quad + \int_0^1 |f(s, y(s), \psi(s)) - F(s, y(s), \psi(s))| ds \\
&\quad + \int_0^t h(s)|x(s) - y(s) ds| + \int_0^t h(s+1)|x(s) - y(s)| ds \\
&\quad + \int_1^t |f(s, y(s), y(s-1)) - F(s, y(s), y(s-1))| ds
\end{aligned}
$$

$$\leq (\varepsilon_1 + \varepsilon_2) + \int_0^t [h(s) + h(s+1)]u(s)ds. \tag{3.14}$$

Now an application of Lemma 2.1 yields that for $1 \leq t < \infty$,

$$|x(t) - y(t)| \leq (\varepsilon_1 + \varepsilon_2) \exp\Big(\int_0^t [h(s) + h(s+1)]ds\Big). \tag{3.15}$$

From this inequality and (3.13), it follows that (1.1)-(1.2) depends continuously on the functions involved therein. □

Next, we consider the difference-differential equations

$$x'(t) = g(t, x(t), x(t-1), \mu), \tag{3.16}$$

$$x'(t) = g(t, x(t), x(t-1), \mu_0), \tag{3.17}$$

for $t \in \mathbb{R}_+$, where $g \in C(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \times R, \mathbb{R}^n)$, and with the initial conditions given by (1.2),

The following theorem states the continuous dependency of solutions to (3.16)-(1.2) and (3.17)-(1.2) on parameters.

Theorem 3.3. *Assume that the function g satisfy the conditions*

$$|g(t, x, y, \mu) - g(t, \bar{x}, \bar{y}, \mu)| \leq p(t)[|x - \bar{x}| + |y - \bar{y}|], \tag{3.18}$$

$$|g(t, x, y, \mu) - g(t, x, y, \mu_0)| \leq q(t)|\mu - \mu_0|, \tag{3.19}$$

where $p, q \in C(\mathbb{R}_+, \mathbb{R}_+)$, Let $x_1(t)$ and $x_2(t)$ be the solutions of (3.16)-(1.2) *and* (3.17)-(1.2) *respectively. Then*

$$|x_1(t) - x_2(t)| \leq \Big(|\mu - \mu_0| \int_0^1 q(s)\Big) \exp\Big(\int_0^t p(s)ds\Big), \tag{3.20}$$

for $0 \leq t < 1$ and

$$|x_1(t) - x_2(t)| \leq \Big(|\mu - \mu_0| \int_0^t q(s)\Big) \exp\Big(\int_0^t [p(s) + p(s+1)]ds\Big), \tag{3.21}$$

for $1 \leq t < \infty$.

Proof. Let $u(t) = |x_1(t) - x_2(t)$ for $t \in \mathbb{R}_+$. We consider the following two cases.
Case 1: $0 \leq t < 1$**.** From the hypotheses, we have

$$\begin{aligned}
u(t) &\leq \int_0^t |g(s, x_1(s), \phi(s), \mu) - g(s, x_2(s), \phi(s), \mu_0)|ds \\
&\leq \int_0^t |g(s, x_1(s), \phi(s), \mu) - g(s, x_2(s), \phi(s), \mu)|ds \\
&\quad + \int_0^t |g(s, x_2(s), \phi(s), \mu) - g(s, x_2(s), \phi(s), \mu_0)|ds \\
&\leq \int_0^t p(s)|x_1(s) - x_2(s)|ds + \int_0^t q(s)|\mu - \mu_0|ds \\
&\leq |\mu - \mu_0| \int_0^1 q(s)ds + \int_0^t p(s)u(s)ds.
\end{aligned}$$

Now a suitable application of Lemma 2.1 yields (3.20).

Case 2: $1 \le t < \infty$**.** By following the arguments in the proof of Theorem 2.5 in case 2, from the hypotheses, we have

$$
\begin{aligned}
u(t) &\le \int_0^1 |g(s,x_1(s),\phi(s),\mu) - g(s,x_2(s),\phi(s),\mu_0)|ds \\
&\quad + \int_1^t |g(s,x_1(s),x_1(s-1),\mu) - g(s,x_2(s),x_2(s-1),\mu_0)|ds \\
&\le \int_0^1 |g(s,x_1(s),\phi(s),\mu) - g(s,x_2(s),\phi(s),\mu)|ds \\
&\quad + \int_0^1 |g(s,x_2(s),\phi(s),\mu) - g(s,x_2(s),\phi(s),\mu_0)|ds \\
&\quad + \int_1^t |g(s,x_1(s),x_1(s-1),\mu) - g(s,x_2(s),x_2(s-1),\mu)|ds \\
&\quad + \int_1^t |g(s,x_2(s),x_2(s-1),\mu) - g(s,x_2(s),x_2(s-1),\mu_0)|ds \\
&\le \int_0^1 p(s)|x_1(s)-x_2(s)|ds + \int_0^1 q(s)|\mu-\mu_0|ds \\
&\quad + \int_1^t p(s)[|x_1(s)-x_2(s)| + |x_1(s-1)-x_2(s-1)|]ds + \int_1^t q(s)|\mu-\mu_0|ds \\
&= \int_0^t q(s)|\mu-\mu_0|ds + \int_0^t p(s)|x_1(s)-x_2(s)|ds \\
&\quad + \int_1^t p(s)|x_1(s-1)-x_2(s-1)|ds \\
&\le |\mu-\mu_0| \int_0^t q(s)ds + \int_0^t [p(s)+p(s+1)]u(s)ds.
\end{aligned}
$$

Now a suitable application of Lemma 2.1 yields (3.21). From (3.20) and (3.21), it follows that the solutions to (3.16)-(1.2) and (3.17)-(1.2) depend continuously on the parameter μ. $\square$

Note that there are many papers and monographs concerning the existence, uniqueness and other properties of solutions of (1.1); see [8, 9], [1, p. 342], [5, p.308], [6, p. 18] and the references cited therein. We believe that the results given here, using elementary analysis, present some useful basic results for future reference.

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