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ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO FUNCTIONAL INTEGRAL EQUATION WITH DEVIATING ARGUMENTS

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Abstract. This article presents results on the existence and asymptotic behavior of solutions of a functional integral equation with deviating arguments. The proof of our main result uses the classical Schauder fixed point theorem and the technique of measures of noncompactness.

1. Introduction

The theory of functional integral equations with deviating argument is very important and significant branch of nonlinear analysis. It is worthwhile mentioning that these theories find numerous applications in physics, mechanics, control theory, biology, ecology, economics, theory of nuclear reactors, engineering, natural sciences and so on [7, 8, 10]. One of the basic problems considered in the theory of functional integral equations with deviating arguments is to establish convenient conditions guaranteeing the existence of solutions of those equations. It is well known that existence of solutions of equations of such a type depends strongly on the size of the delay arguments involved in those equations.

In this article we will examine the functional integral equation

$$x(t) = f\Big(t, x(\sigma_1(t)), \int_0^t k(t,s)g\big(s, x(\sigma_2(s))\big)ds\Big), \quad t \geq 0. \tag{1.1}$$

The functional integral equation of the above form contains a lot of special types of functional integral equations. The differential equations with transformed argument or differential equations of neutral type can also be transformed to functional integral equations. Such type of equations were investigated in lots of papers [5, 12].

The aim of this paper is to investigate the existence and asymptotic behavior of solutions of (1.1). The main tools used in our considerations are the concept of a measure of noncompactness and the classical Schauder fixed point principle. The investigations of the paper are placed in the space of continuous and tempered functions on the real line. The result obtained here generalizes several ones obtained earlier by many authors [1, 8, 9, 10, 11].

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2. Notation and auxiliary results

Let E be a real Banach space with the norm $\|\cdot\|$ and the zero element θ. Denote by $B(x,r)$ the closed ball centered at x and with radius r. The ball $B(\theta,r)$ will be denoted by B_r.

If X is a subset of E then $\overline{X}$ denotes the closure, and $\operatorname{Conv} X$ denotes the convex closure of X. We use the standard notation $X+Y$, λX to denote the usual algebraic operations on subsets X, Y of the space E. Further, let $\mathcal{M}_E$ denote the family of all nonempty and bounded subsets of E and $\mathcal{N}_E$ its subfamily consisting of all relatively compact sets.

We will accept the following definition of measure of noncompactness [6].

Definition 2.1. A mapping $\mu : \mathcal{M}_E \to \mathbb{R}_+ = [0,+\infty)$ is said to be a measure of noncompactness in the space E if it satisfies the following conditions:

(i) The family $\ker\mu = \{X \in \mathcal{M}_E : \mu(X) = 0\}$ is nonempty and $\ker\mu \subset \mathcal{N}_E$;
(ii) $X \subset Y$ implies $\mu(X) \le \mu(Y)$;
(iii) $\mu(\operatorname{Conv} X) = \mu(X)$;
(iv) $\mu(\overline{X}) = \mu(X)$;
(v) $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$;
(vi) If (X_n) is a sequence of sets from $\mathcal{M}_E$ such that $X_{n+1} \subset X_n$, $\overline{X}_n = X_n$ $(n = 1,2,3,\dots)$ and if $\lim_{n\to\infty}\mu(X_n) = 0$, then the intersection $X_\infty = \cap_{n=1}^{\infty} X_n$ is nonempty.

The family $\ker\mu$ defined in axiom (i) is called the kernel of the measure of noncompactness μ.

Remark 2.2. Note that the intersection set X_∞ described in axiom (vi) is a member of the kernel of the measure of noncompactness μ. In fact, the inequality $\mu(X_\infty) \le \mu(X_n)$ for $n = 1,2,\dots$ implies that $\mu(X_\infty) = 0$. Hence $X_\infty \in \ker\mu$. This property of the set X_∞ will be very important in our investigations.

Now, let us assume that $p = p(t)$ is a given function defined and continuous on the interval $\mathbb{R}_+$ with real positive values. Denote by $C\big(\mathbb{R}_+, p(t)\big) = C_p$ the Banach space consisting of all real functions $x = x(t)$ defined and continuous on $\mathbb{R}_+$ and such that

$$\sup\{|x(t)|p(t) : t \ge 0\} < \infty.$$

The space C_p is furnished with the standard norm

$$\|x\| = \sup\{|x(t)|p(t) : t \ge 0\}.$$

Further we recall the definition of the measure of noncompactness in the space C_p which will be used in our considerations [2, 6]. Let X be a nonempty and bounded subset of the space C_p. Fix positive number $T > 0$. For an arbitrary function $x \in X$ and $\epsilon > 0$ denote by $\omega^T(x,\epsilon)$ the modulus of continuity of the function x, tempered by the function p, on the interval $[0,T]$; i.e.

$$\omega^T(x,\epsilon) = \sup\{|x(t)p(t) - x(s)p(s)| : t,\ s \in [0,T],\ |t-s| \le \epsilon\}.$$

Further, let us put

$$\omega^T(X,\epsilon) = \sup\{\omega^T(x,\epsilon) : x \in X\},$$
$$\omega_0^T(X) = \lim_{\epsilon\to 0}\omega^T(X,\epsilon), \quad \omega_0(X) = \lim_{T\to\infty}\omega_0^T(X).$$

Also, we put

$$b(X) = \lim_{T\to\infty} \Big\{ \sup_{x\in X} \big\{ \sup\{|x(t)|p(t) : t \geq T\}\big\}\Big\}.$$

Finally, we define the function μ on the family $\mathcal{M}_{C_p}$ by putting $\mu(X) = \omega_0(X) + b(X)$.

It may be shown that the function μ is the measure of noncompactness in the space C_p [2]. The kernel $\ker\mu$ is the family of all nonempty and bounded sets X such that functions belonging to X are locally equicontinuous on $\mathbb{R}_+$ and $\lim_{t\to\infty} x(t)p(t) = 0$ uniformly with respect to the set X, i.e. for each $\epsilon > 0$ there exists $T > 0$ with the property that $|x(t)|p(t) \leq \epsilon$ for $t \geq T$ and for $x \in X$. This property will be crucial in our further study.

Finally, let us assume that x is a real function defined and continuous on $\mathbb{R}_+$. Fix $T > 0$ and denote by $\nu^T(x,\epsilon)$ the usual modulus of continuity of the function x on the interval $[0,T]$:

$$\nu^T(x,\epsilon) = \sup\{|x(t) - x(s)| :\ t,\ s \in [0,T],\ |t-s| \leq \epsilon\}.$$

3. Main result

We will consider the nonlinear functional-integral equation (1.1) under the following assumptions:

(H1) $f : \mathbb{R}_+ \times R \times \mathbb{R} \to R$ is a continuous function and there exists a constant $K \geq 0$ such that

$$|f(t,x_1,y) - f(t,x_2,y)| \leq K|x_1 - x_2|$$

for all $t \in \mathbb{R}_+$, x_1, x_2, $y \in \mathbb{R}$;

(H2) There exists a continuous function $L_0 : \mathbb{R}_+ \to \mathbb{R}_+$ and a continuous nonincreasing function $L_1 : \mathbb{R}_+ \to \mathbb{R}$ such that

$$|f(t,0,y)| \leq L_0(t) + |y| \exp L_1(t)$$

for all $t \in \mathbb{R}_+$ and $y \in \mathbb{R}$;

(H3) $k : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is a continuous function and there exists a continuous function $a : \mathbb{R}_+ \to \mathbb{R}_+$ and a continuous nondecreasing function $b : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|k(t,s)| \leq a(t)b(s) \quad \text{for all } t,s \in \mathbb{R}_+;$$

(H4) $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there exists a continuous nondecreasing function $p : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|g(s,x)| \leq p(s)|x| \quad \text{for all } s \in \mathbb{R}_+ \text{ and } x \in \mathbb{R};$$

(H5) $\sigma_1, \sigma_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions such that $\sigma_1(t) \leq t$ and $\sigma_2(t) \leq t$;

(H6) $\lim_{t\to\infty} ta(t)b(t)p(t) = 0$ and $\lim_{t\to\infty} L_0(t) \exp\Big(-\int_0^t L_0(s)ds\Big) = 0$;

Let $A = \sup\{a(t) :\ t \in \mathbb{R}_+\}$ and

$$L(t) = \int_0^t [L_0(s) + b(s)p(s) \exp L_1(s)]ds.$$

Obviously the function $L(t)$ is nondecreasing and continuous on $\mathbb{R}_+$. Denote by C_L, the space $C\big(\mathbb{R}_+, \exp(-ML(t))\big)$, where $M > 1$ is an arbitrarily fixed constant.

Theorem 3.1. *Assume* (H1)-(H6) *and that* $(K + A/M) < 1$. *Then* (1.1) *has at least one solution* $x \in C_L$ *such that* $x(t) = o(\exp(ML(t))$ *as* $t \to \infty$.

Proof. Consider the operator F defined on the space C_L by the formula

$$(Fx)(t) = f\Big(t, x(\sigma_1(t)), \int_0^t k(t,s)g\big(s, x(\sigma_2(s))\big)ds\Big), \quad t \geq 0.$$

Observe that the operator F is well defined on the space C_L and the function Fx is continuous on $\mathbb{R}_+$. Next, in view of our assumptions, for arbitrarily fixed $x \in C_L$ and $t \in \mathbb{R}_+$, we get

$$\begin{aligned}
&|(Fx)(t)|\exp(-ML(t))\\
&\leq \big|f\Big(t, x(\sigma_1(t)), \int_0^t k(t,s)g\Big(s, x(\sigma_2(s))\Big)ds\ \Big)\\
&\quad - f\Big(t, 0, \int_0^t k(t,s)g\Big(s, x(\sigma_2(s))\Big)ds\ \Big)\big|\exp(-ML(t))\\
&\quad + \big|f\Big(t, 0, \int_0^t k(t,s)g\Big(s, x(\sigma_2(s))\Big)ds\ \Big)\big|\exp(-ML(t))\\
&\leq K|x(\sigma_1(t))|\exp(-ML(t)) + L_0(t)\exp(-ML(t))\\
&\quad + \exp L_1(t)\int_0^t \big|k(t,s)\big|\ \big|g\Big(s, x(\sigma_2(s))\Big)\big|ds\exp(-ML(t))\\
&\leq K\|x\|\exp(M(L(\sigma_1(t) - L(t)))) + L_0(t)\exp(-ML(t))\\
&\quad + \exp L_1(t)\int_0^t a(t)b(s)p(s)|x(\sigma_2(s))|ds\exp(-ML(t))\\
&\leq K\|x\| + L_0(t)\exp(-ML(t))\\
&\quad + a(t)\|x\|\int_0^t b(s)p(s)\exp L_1(s)\exp(ML(s))ds\exp(-ML(t))\\
&\leq K\|x\| + L_0(t)\exp\Big(-\int_0^t L_0(s)ds\Big)\\
&\quad + a(t)\|x\|\int_0^t [L_0(s) + b(s)p(s)\exp L_1(s)]\exp(ML(s))ds\exp(-ML(t))\\
&\leq K\|x\| + \frac{A}{M}\|x\| + B,
\end{aligned}$$

where

$$B = \sup\big\{L_0(t)\exp\Big(-\int_0^t L_0(s)ds\Big) : t \in \mathbb{R}_+\big\}.$$

Obviously $B < \infty$ by virtue of assumption (H6).

The above obtained estimate shows that Fx is bounded on $\mathbb{R}_+$. This implies that the operator F is a self mapping of the space C_L. Moreover for $r = B/(1 - K - A/M)$, the operator F transforms the ball B_r into itself.

Let us take an arbitrary nonempty subset X of the ball B_r. Fix $T > 0$ and take an arbitrary function $x \in X$. Then evaluating similarly as before, for a fixed t, $t \geq T$, we get

$$\begin{aligned}
&|(Fx)(t)|\exp(-ML(t))\\
&\leq K|x(\sigma_1(t))|\exp(-ML(t)) + L_0(t)\exp(-ML(t))
\end{aligned}$$

$$
\begin{aligned}
&+ \exp L_1(t) \int_0^t a(t)b(s)p(s)|x(\sigma_2(s))|ds \cdot \exp(-ML(t)) \\
&\le K|x(\sigma_1(t))| \exp(-ML(\sigma_1(t))) + L_0(t)\exp(-ML(t)) \\
&\quad + a(t)\exp L_1(t)\|x\| \int_0^t b(s)p(s)\exp(ML(s))ds \exp(-ML(t)) \\
&\le K|x(\sigma_1(t))| \exp(-ML(\sigma_1(t))) + L_0(t)\exp(-ML(t)) \\
&\quad + rta(t)b(t)p(t)\exp L_1(t) \\
&\le K|x(\sigma_1(t))| \exp(-ML(\sigma_1(t))) + L_0(t)\exp\Big(-\int_0^t L_0(s)ds\Big) \\
&\quad + rta(t)b(t)p(t)\exp L_1(0).
\end{aligned}
$$

Hence in view of the assumption (H6), we infer that

$$
b(FX) \le Kb(X). \tag{3.1}
$$

where $b(X)$ was defined previously.

Next, let us fix $T > 0$ and $\epsilon > 0$. Take arbitrary $t,\ s \in [0,T]$ with $|t-s| \le \epsilon$. Then we derive the following chain of inequalities.

$$
\begin{aligned}
|(Fx)(t) - (Fx)(s)| &\le \Big|f\Big(t, x(\sigma_1(t)), \int_0^t k(t,\tau)g\big(\tau, x(\sigma_2(\tau))\big)d\tau\Big) \\
&\quad - f\Big(t, x(\sigma_1(s)), \int_0^t k(t,\tau)g\big(\tau, x(\sigma_2(\tau))\big)d\tau\Big)\Big| \\
&\quad + \Big|f\Big(t, x(\sigma_1(s)), \int_0^t k(t,\tau)g\big(\tau, x(\sigma_2(\tau))\big)d\tau\Big) \\
&\quad - f\Big(s, x(\sigma_1(s)), \int_0^s k(s,\tau)g\big(\tau, x(\sigma_2(\tau))\big)d\tau\Big)\Big| \\
&\le K|x(\sigma_1(t)) - x(\sigma_1(s))| \\
&\quad + \Big|f\Big(t, x(\sigma_1(s)), \int_0^t k(t,\tau)g\big(\tau, x(\sigma_2(\tau))\big)d\tau\Big) \\
&\quad - f\Big(s, x(\sigma_1(s)), \int_0^t k(t,\tau)g\big(\tau, x(\sigma_2(\tau))\big)d\tau\Big)\Big| \\
&\quad + \Big|f\Big(s, x(\sigma_1(s)), \int_0^t k(t,\tau)g\big(\tau, x(\sigma_2(\tau))\big)d\tau\Big) \\
&\quad - f\Big(s, x(\sigma_1(s)), \int_0^s k(s,\tau)g\Big(\tau, x(\sigma_2(\tau))\Big)d\tau\Big)\Big| \\
&\le K|x(\sigma_1(t)) - x(\sigma_1(s))| + \nu_1^T(f,\epsilon) + \nu_2^T(f,\epsilon),
\end{aligned} \tag{3.2}
$$

where

$$
\begin{aligned}
\nu_1^T(f,\epsilon) = \sup\big\{&|f(t,x,y) - f(s,x,y)| : t,\ s \in [0,T],\ |t-s| \le \epsilon, \\
&|x| \le r\exp(ML(T)),\ |y| \le N_1\big\}, \\
\nu_2^T(f,\epsilon) = \sup\big\{&|f(t,x,y_1) - f(t,x,y_2)| : t \in [0,T],\ |x| \le r\exp(ML(T)), \\
&|y_1|, |y_2| \le N_1,\ |y_1 - y_2| \le N_2\big\}
\end{aligned}
$$

while the constants N_1 and N_2 appearing above are defined in the following way

$$N_1 = r \sup \big\{ a(t) \int_0^t b(s)p(s)\exp(ML(s))ds : s,\ t \in [0,T] \big\},$$

$$\begin{aligned}
N_2 &\le \Big| \int_0^t k(t,\tau) g\big(\tau, x(\sigma_2(\tau))\big) d\tau - \int_0^t k(s,\tau) g\big(\tau, x(\sigma_2(\tau))\big) d\tau \Big| \\
&\quad + \Big| \int_0^t k(s,\tau) g\big(\tau, x(\sigma_2(\tau))\big) d\tau - \int_0^s k(s,\tau) g\big(\tau, x(\sigma_2(\tau))\big) d\tau \Big| \\
&\le \int_0^t \big| k(t,\tau) - k(s,\tau) \big| \big| g\big(\tau, x(\sigma_2(\tau))\big) \big| d\tau \\
&\quad + \epsilon \sup \big\{ \big|k(t,\tau)\big| \big|g\big(\tau, x(\sigma_2(\tau))\big)\big| : t,\ \tau \in [0,T] \big\} \\
&\le \nu^T(k,\epsilon) T \sup \big\{ \big|g\big(\tau, x(\sigma_2(\tau))\big)\big| : \tau \in [0,T] \big\} \\
&\quad + \epsilon \sup \big\{ \big|k(t,\tau)\big|\ \big|g\big(\tau, x(\sigma_2(\tau))\big)\big| :\ t,\ \tau \in [0,T] \big\}.
\end{aligned}$$

Now, let us denote

$$q(\epsilon) = \nu_1^T(f,\epsilon) + \nu_2^T(f,\epsilon).$$

From the uniform continuity of the function $f(t,x,y)$ on compact subsets of $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$, we deduce that $q(\epsilon) \to 0$ as $\epsilon \to 0$. Further, from (3.2), we have

$$\begin{aligned}
&|(Fx)(t)\exp(-ML(t)) - (Fx)(s)\exp(-ML(s))| \\
&\le |(Fx)(t)\exp(-ML(t)) - (Fx)(s)\exp(-ML(t))| \\
&\quad + |(Fx)(s)\exp(-ML(t)) - (Fx)(s)\exp(-ML(s))| \\
&\le |(Fx)(t) - (Fx)(s)|\exp(-ML(t)) \\
&\quad + |(Fx)(s)|\ |\exp(-ML(t)) - \exp(-ML(s))| \\
&\le K|x(\sigma_1(t)) - x(\sigma_1(s))|\exp(-ML(t)) + q(\epsilon)\exp(-ML(t)) \\
&\quad + |(Fx)(s)|\ |\exp(-ML(t)) - \exp(-ML(s))| \\
&\le K|x(\sigma_1(t))\exp(-ML(t)) - x(\sigma_1(s))\exp(-ML(s))| \\
&\quad + K|x(\sigma_1(s))\exp(-ML(s)) - x(\sigma_1(s))\exp(-ML(t))| \\
&\quad + q(\epsilon)\exp(-ML(t)) + |(Fx)(s)|\ |\exp(-ML(t)) - \exp(-ML(s))| \\
&\le K|x(\sigma_1(t))\exp(-ML(\sigma_1(t))) - x(\sigma_1(s))\exp(-ML(\sigma_1(s)))| \\
&\quad + K|x(\sigma_1(s))|\ |\exp(-ML(\sigma_1(s))) - \exp(-ML(\sigma_1(t)))| \\
&\quad + q(\epsilon)\exp(-ML(t)) + |(Fx)(s)||\exp(-ML(t)) - \exp(-ML(s))| \\
&\le K\omega^T\Big(x, \nu^T(\sigma_1,\epsilon)\Big) + Kr\exp(ML(\sigma_1(T))) \\
&\quad \nu^T\Big(\exp(-ML(\sigma_1(t))),\epsilon\Big) + q(\epsilon)\exp(-ML(t)) \\
&\quad + r\nu^T\Big(\exp(-ML(t)),\epsilon\Big)\Big[Kr\exp(ML(T)) + \sup\{L_0(t) : t \in [0,T]\} \\
&\quad + r\sup\{t\exp(L_1(t))a(t)b(t)p(t)\exp(ML(t)) : t \in [0,T]\}\Big].
\end{aligned}$$

Keeping in mind, the uniform continuity of the functions $t \to \exp(-ML(t))$ and $t \to \exp(-ML(\sigma_1(t)))$ on the interval $[0,T]$, from the above estimate we infer that

$$\omega_0^T(FX) \le K\omega_0^T(X).$$

Consequently,

$$\omega_0(FX) \le K\omega_0(X). \tag{3.3}$$

Now linking (3.1) and (3.3), we get

$$\mu(FX) \le K\mu(X) \tag{3.4}$$

where μ denotes the measure of noncompactness defined earlier.

Further let us consider the sequence (B_r^n), where $B_r^1 = \operatorname{Conv} F(B_r)$, $B_r^2 = \operatorname{Conv} F(B_r^1), \dots$. Obviously all sets of this sequence are nonempty, bounded, convex and closed. Apart from this we have that $B_r^{n+1} \subset B_r^n \subset B_r$ for $n = 1, 2, 3, \dots$. Thus, keeping in mind that $K < 1$ and taking into account of (3.4), we infer that $\lim_{n\to\infty} \mu(B_r^n) = 0$. Hence, in view of the axiom (vi) of Definition 2.1, we deduce that the set $Y = \cap_{n=1}^{\infty} B_r^n$ is nonempty, bounded, convex and closed. Moreover, in the light of Remark 2.2 we have that $Y \in \ker \mu$. Let us also observe that the operator F maps the set Y into itself.

Next we show that F is continuous on the set Y. Let us fix $\epsilon > 0$ and take arbitrary functions $x, y \in Y$ such that $\|x - y\| \le \epsilon$. Taking into account the fact that $Y \in \ker \mu$ and the description of sets from $\ker \mu$ we can find $T > 0$ such that for each $z \in Y$ and $t \ge T$ we have that $|z(t)| \exp(-ML(t)) \le \epsilon/2$.

Observe that based on our assumptions, for an arbitrarily fixed $t \in \mathbb{R}_+$, we have

$$\begin{aligned}
&|(Fx)(t) - (Fy)(t)| \exp(-ML(t)) \\
&\le \Big| f\Big(t, x(\sigma_1(t)), \int_0^t k(t,s) g\Big(s, x(\sigma_2(s))\Big) ds\ \Big) \\
&\quad - f\Big(t, y(\sigma_1(t)), \int_0^t k(t,s) g\Big(s, x(\sigma_2(s))\Big) ds\ \Big)\Big| \exp(-ML(t)) \\
&\quad + \Big| f\Big(t, y(\sigma_1(t)), \int_0^t k(t,s) g\Big(s, x(\sigma_2(s))\Big) ds\ \Big) \\
&\quad - f\Big(t, y(\sigma_1(t)), \int_0^t k(t,s) g\Big(s, y(\sigma_2(s))\Big) ds\ \Big)\Big| \exp(-ML(t)) \\
&\le K\epsilon + \Big| f\Big(t, y(\sigma_1(t)), \int_0^t k(t,s) g\Big(s, x(\sigma_2(s))\Big) ds\ \Big) \\
&\quad - f\Big(t, y(\sigma_1(t)), \int_0^t k(t,s) g\Big(s, y(\sigma_2(s))\Big) ds\ \Big)\Big| \exp(-ML(t)).
\end{aligned} \tag{3.5}$$

Now, let us assume that $t \in [0, T]$, where T is chosen as above. Then we obtain

$$\begin{aligned}
&\Big| f\Big(t, y(\sigma_1(t)), \int_0^t k(t,s) g\Big(s, x(\sigma_2(s))\Big) ds\ \Big) \\
&- f\Big(t, y(\sigma_1(t)), \int_0^t k(t,s) g\Big(s, y(\sigma_2(s))\Big) ds\ \Big)\Big| \exp(-ML(t)) \\
&\le \nu^T(f, \epsilon) \exp(-ML(t)) \\
&\le \nu^T(f, \epsilon)
\end{aligned} \tag{3.6}$$

where $\nu^T(f, \epsilon)$ is defined as

$$\begin{aligned}
\nu^T(f, \epsilon) = \sup \big\{ &|f(t, v, x) - f(t, v, y)| : t \in [0, T],\ |v| \le r \exp(ML(T)), \\
&|x|,\ |y| \le N_1,\ |x - y| \le N_3 \big\}
\end{aligned}$$

while the constant N_3 appearing above is defined by the formula

$$\begin{aligned} N_3 &= \sup\Big\{\int_0^t k(t,s)\big|g\big(s,x(\sigma_2(s))\big)-g\big(s,y(\sigma_2(s))\big)\big|ds : t,s\in[0,T]\Big\}\\ &= T\nu^T(g,\epsilon)\sup\big\{|k(t,s)| : t,s\in[0,T]\big\}. \end{aligned}$$

Observe that in view of the uniform continuity of the function $f(t,v,x)$ on the set $[0,T]\times[-r\exp(ML(T)), r\exp(ML(T))]\times[-N_1,N_1]$, we conclude that $\nu^T(f,\epsilon)\to 0$ as $\epsilon\to 0$.

Next, we assume that $t\geq T$. Then keeping in mind that $x,y\in Y$ and $F:Y\to Y$, we derive easily the estimate

$$\begin{aligned} &|(Fx)(t)-(Fy)(t)|\exp(-ML(t))\\ &\leq |(Fx)(t)|\exp(-ML(t))+|(Fy)(t)|\exp(-ML(t))\leq\epsilon. \end{aligned}\tag{3.7}$$

Now linking (3.5)-(3.7), we conclude that the operator F is continuous on the set Y.

Finally, taking into account the properties of the set Y and the operator $F : Y\to Y$ established above and applying the classical Schauder fixed point theorem we infer that the operator F has at least one fixed point $x=x(t)$ in Y. Obviously, the function $x(t)$ is a solution of (1.1).

Moreover, keeping in mind that $Y\in\ker\mu$, we obtain that $x(t)=o(\exp(ML(t))$ as $t\to\infty$. □

4. Example

Consider the functional integral equation, with deviating arguments,

$$\begin{aligned} x(t) = t^2 + \arctan\Big[&\frac{x(t/3)}{4+t^2}+\frac{1}{(1+t^3)}\int_0^t \exp(-t)s^2x(s-\exp(-s))\\ &\times\cos\big(x^2(s-\exp(-s))\big)ds\Big],\quad t\geq 0. \end{aligned}\tag{4.1}$$

Note that the above equation represents a special case of (1.1) where

$$f(t,x,y)=t^2+\arctan\Big[\frac{x}{4+t^2}+\frac{y}{(1+t^3)}\Big],$$

$\sigma_1(t)=t/3$ and $\sigma_2(t)=t-\exp(-t)$. Moreover, the functions $k(t,s)$ and $g(s,x)$ take the form $k(t,s)=s\exp(-t)$ and $g(s,x)=s\ x\cos x^2$.

It is easily seen that for (4.1), the assumption (H2) of Theorem 3.1 is satisfied with $L_0(t)=t^2$ and $L_1(t)=-\ln(1+t^3)$. In fact, observe that

$$|f(t,0,y)|\leq t^2+\Big|\arctan\Big(\frac{y}{1+t^3}\Big)\Big|\leq t^2+\frac{|y|}{1+t^3}.$$

Also, note that the function $f(t,x,y)$ is continuous on the set $\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}$ and for arbitrary $t\in\mathbb{R}_+$, $x_1,x_2,y\in\mathbb{R}$, we obtain

$$|f(t,x_1,y)-f(t,x_2,y)|\leq\frac{1}{4+t^2}|x_1-x_2|\leq\frac14|x_1-x_2|.$$

This means the function $f(t,x,y)$ satisfies the Lipschitz condition with respect to x with the constant $K=\frac14$.

Moreover the assumptions (H3) and (H4) of Theorem 3.1 are satisfied with $a(t)=\exp(-t)$, $b(t)=p(t)=t$ and hence A=1. Now, we get $(K+A/M)<1$ for $M>1$.

Similarly we can verify other details concerning the assumptions of Theorem 3.1. Taking into account the above established facts and applying Theorem 3.1, we infer that (4.1) has at least one solution $x = x(t)$ such that $x \in C_L$, where

$$L(t) = \frac{t^3}{3} + \frac{1}{3}\ln(1+t^3).$$

Apart from this, we have that

$$x(t) = o\Big(\exp\Big(M\big(\frac{t^3}{3} + \frac{1}{3}\ln(1+t^3)\big)\Big)\Big)$$

as $t \to \infty$, where $M > 1$ is a constant.

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