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STABILIZED QUASI-REVERSIBILITY METHOD FOR A CLASS OF NONLINEAR ILL-POSED PROBLEMS

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Abstract. In this paper, we study a final value problem for the nonlinear parabolic equation

$$\begin{gathered} u_t + Au = h(u(t), t), \quad 0 < t < T \\ u(T) = \varphi, \end{gathered}$$

where A is a non-negative, self-adjoint operator and h is a Lipchitz function. Using the stabilized quasi-reversibility method presented by Miller, we find optimal perturbations, of the operator A, depending on a small parameter ϵ to setup an approximate nonlocal problem. We show that the approximate problems are well-posed under certain conditions and that their solutions converges if and only if the original problem has a classical solution. We also obtain estimates for the solutions of the approximate problems, and show a convergence result. This paper extends the work by Hetrick and Hughes [11] to nonlinear ill-posed problems.

1. Introduction

Let A be a self-adjoint operator on a Hilbert space H such that $-A$ generates a compact contraction semi-group on H. We shall consider the final value problem of finding a function $u : [0, T] \to H$ satisfying

$$u_t + Au = h(u(t), t), \quad 0 < t < T, \tag{1.1}$$

$$u(T) = \varphi, \tag{1.2}$$

for some prescribed final value φ in a Hilbert space H. Such problem are not well posed, that is, even if a unique solution exists on $[0, T]$ it need not depend continuously on the final value φ . Hence, a regularization is in order. We note that this type of problems has been considered by many authors, using different approaches. In their pioneering work Lattes and Lions [17] presented, in a heuristic approach, the quasi-reversibility method. In this method the main ideas are replacing A by an operator $A_\epsilon = f_\epsilon(A)$. Originally, $f_\epsilon(A) = A - \epsilon A^2$ which yields the well-posed

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problem, in the backward direction,

$$\begin{gathered} u_t + Au - \epsilon A^2 u = 0, \quad t \in [0,T], \\ u(T) = \varphi. \end{gathered} \tag{1.3}$$

The stability of this method is of order $e^{c\epsilon^{-1}}$. In [31], the problem is approximated by

$$\begin{gathered} u_t + Au + \epsilon A u_t = 0, \quad t \in [0,T], \\ u(T) = \varphi. \end{gathered} \tag{1.4}$$

In [22], using the method of stabilized quasi reversibility, the author studied the general approximated problem

$$\begin{gathered} u_t + f(A)u = 0, \quad t \in [0,T], \\ u(T) = \varphi. \end{gathered} \tag{1.5}$$

It is clear that (1.3) and (1.4) are special case of (1.5) where $f(x) = x - \epsilon x^2$ and $f(x) = x/(1+\epsilon x)$ respectively. Note that the solution of (1.5) has the form $e^{(T-t)f(A)}\varphi$. And since these functions f are bounded by c/ϵ, we know that their stability is of order $e^{c/\epsilon}$. Hence, the stability in this case are quite large as in the original quasi-reversibility methods. To improve the stability result of this problem (1.5), Miller gave some appropriate conditions on the "corrector" $f(A)$ and obtain the stability of order $c\epsilon^{-1}$.

In 1983, Showalter presented a method called the quasiboundary value (QBV) method, to regularize that linear homogeneous problem, which gave a stability estimate better than the one of discussed methods. The main idea of this method is adding an appropriate "corrector" into the final data. Using this method, Clark-Oppenheimer [5], and Denche-Bessila [7], recently, regularized the backward problem by replacing the final condition with

$$u(T) + \epsilon u(0) = \varphi$$

and

$$u(T) - \epsilon u'(0) = \varphi,$$

respectively.

In 2005, Ames and Hughes [3] applied semigroup theory and other operator-theoretic methods to prove Holder continuous dependence for homogeneous ill-posed Cauchy problems. The authors consider the above problem in Banach space and give the conditions of the function f, to obtained the stability estimate

$$\|u(t) - v(t)\| \le C\beta^{1-w(t)} M^{w(t)}$$

where $u(t)$ is the solution of (1.1)–(1.2) and $v(t)$ the solution of (1.5).

Although there are many works on the linear homogeneous case of the backward problem (ill-posed problem), the literature on the linear nonhomogeneous case and the nonlinear case are quite scarce. A conditional stability result for the Ginzburg-Landau equation was given in [A]. In [27], the authors used the QR method and the eigenvalue-expansion method to regularize a 1-D linear nonhomogeneous backward problem. In [32], the authors used an improved version of QBV method to regularize the latter problem. Recently, Hetrick and Hughes [11] extended the earlier work of Ames and Hughes [3], by considered nonhomogeneous ill-posed problems and proving the continuous dependence in Banach spaces. However, the nonlinear case of the problem in [3] in Banach space is not given here and will be presented in future work.

Most of the above articles give better results than the quasi-reversibility method given by Miller. So, it is difficult to consider the backward problem using quasi-reversibility method. Up to the present, we can find only a few papers which study (1.1)-(1.2) using quasi-reversibility, such as [18]. In fact, Long and Dinh [18] approximated (1.1)-(1.2) by the problem

$$\begin{gathered} v_\beta'(t) + A_\beta v_\beta(t) = e^{-(1-t)\beta A A_\beta} h(v_\beta) \\ v_\beta(1.1) = \varphi \end{gathered}$$

where $f_\beta(A) = A_\beta = A(I+\beta A)^{-1}$ is the approximate operator for A. Although v^β is a good approximation of u, the authors can not prove that v^β is a regularized solution of u. So, the quasi-reversibility method given in [18], is not effective to regularize the backward problem with the large time.

This paper is a generalization of Miller's paper for the nonlinear right hand side. We prove that our method gives the same stability order as previous method in [27, 33]. By replacing the operator A by $f_\epsilon(A)$, chosen latter under some better conditions, we approximate the problem (1.1)-(1.2) as the follows:

$$u_t^\epsilon + f_\epsilon(A)u^\epsilon = h(u^\epsilon(t), t), \quad 0 < t < T, \tag{1.6}$$

$$u^\epsilon(T) = \varphi, \tag{1.7}$$

with $0 < \epsilon < 1$.

This paper is organized as follows. In the section 2, we derive conditions on the perturbation $f_\epsilon(A)$ and show that (1.6)-(1.7) is well-posed. Moreover, the stability of this method is of order $c\epsilon^{\frac{t}{T}-1}$. Also, we find some conditions on f_ϵ so that we can get error estimate

$$\|u^\epsilon(t) - u(t)\| \le C\beta(\epsilon)^{t/T}, \tag{1.8}$$

where $\|\cdot\|$ is the norm in H, $\beta(\epsilon) \to 0$ when $\epsilon \to 0$ and C depends on $u(t)$. Finally, we consider the example and numerical experiment will be given in Section 4, which show that the efficient of our method.

2. Approximation of the non-linear problem

We assume that H is a separable Hilbert space and A is self-adjoint and that 0 is in the resolvent set of A. We also assume that A^{-1} is compact. Let $\{\phi_n\}$ be an orthonormal eigenbasic on H corresponding to the eigenvalues $\{\lambda_n\}$ of A; i.e., $A\phi_n = \lambda_n \phi_n$. Without loss of generality, we shall assume that

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \dots, \quad \lim_{n\to\infty} \lambda_n = \infty.$$

For every v in H having the expansion $v = \sum_{n=1}^\infty v_n\phi_n$, $v_n \in \mathbb{R}$, $n = 1, 2, \dots$ and $g : \mathbb{R} \to \mathbb{R}$, we define $g(A)v = \sum_{n=1}^\infty g(\lambda_n)v_n\phi_n$. If $v \in H$, we define

$$\operatorname{Dom}(g(A)) = \{v \in H : \|g(A)v\|^2 = \sum_{n=1}^\infty g^2(\lambda_n)v_n^2 < \infty\}$$

Definition. Let fixed $\epsilon \in (0,1)$. Let $f_\epsilon : [0,\infty) \to \mathbb{R}$ be a bounded Borel function, and assume that there exists $\beta(\epsilon) > 0$ satisfies $\beta(\epsilon) \to 0$ when $\epsilon \to 0$, and $|f_\epsilon(\alpha)| \le -\frac{1}{T}\ln(\beta(\epsilon))$ for all $\alpha \in [0,\infty)$.

1. f is said to satisfy Condition (A) if

$$\|(-A + f_\epsilon(A))u\| \le \beta(\epsilon)\|e^{TA}u\|$$

for all $u \in \mathrm{Dom}(e^{TA}) = \{u \in H : \|e^{TA}u\| = \sqrt{\sum_{n=1}^\infty e^{2T\lambda_n}u_n^2} < \infty\}$.

Yongzhong Huang [13, p.759] gave the approximate operator

$$A_\epsilon = -\frac{1}{pT}\ln(\epsilon + e^{-pT\epsilon})$$

In the case $p = 1$, we have $f_\epsilon(x) = -\frac{1}{T}\ln(\epsilon + e^{-Tx})$, where $x \in (0,\infty)$. Then, it is easy to see that $|f_\epsilon(\alpha)| \le -\frac{1}{T}\ln(\epsilon)$. Also we have

$$\begin{aligned}
\|(-A + f_\epsilon(A))u\|^2 &= \sum_{n=1}^\infty (-\lambda_n - \frac{1}{T}\ln(\epsilon + e^{-T\lambda_n}))^2 u_n^2 \\
&\le \sum_{n=1}^\infty \frac{1}{T^2}\ln^2(1 + \epsilon e^{T\lambda_n})u_n^2 \\
&\le \sum_{n=1}^\infty \frac{\epsilon^2}{T^2} e^{2T\lambda_n}u_n^2 = \frac{\epsilon^2}{T^2}\|e^{TA}u\|^2
\end{aligned}$$

Hence, f_ϵ satisfies condition (A).

2. Let $0 \le s \le t \le T$ and $u \in H$. Then we define the operator

$$e^{(s-t)f_\epsilon(A)}u = \sum_{n=1}^\infty e^{(s-t)f_\epsilon(\lambda_n)}u_n\phi_n$$

Lemma 2.1. *Let $\epsilon > 0$ be such that $0 < \beta(\epsilon) < 1$ and $u \in H$ has the eigen-function expansion $u = \sum_{n=1}^\infty u_n\phi_n$ where $u_n =< u, \phi_n >$. Then*

$$\|f_\epsilon(A)u\| \le -\frac{1}{T}\ln(\beta(\epsilon))\|u\|$$

Proof. Suppose that $u \in H$ has the eigen-function expansion $u = \sum_{n=1}^\infty u_n\phi_n$ where $u_n = \langle u, \phi_n\rangle$. Then, using the expansion of $f_\epsilon(A)u$, and that f_ϵ is bounded, we obtain

$$\|f_\epsilon(A)u\|^2 = \sum_{n=1}^\infty f_\epsilon^2(\lambda_n)u_n^2 \le \frac{1}{T^2}\ln^2\frac{1}{(\beta(\epsilon))}\sum_{n=1}^\infty u_n^2 = \ln^2\frac{1}{(\beta(\epsilon))}\|u\|^2$$

This completes the proof. □

Lemma 2.2. *Let ϵ, s, t be as in Lemma 2.1. Then for $u \in H$, we have*

$$\|e^{(s-t)f_\epsilon(A)}u\| \le (\beta(\epsilon))^{\frac{t-s}{T}}\|u\|$$

Proof. Using that f_ϵ is bounded, we have

$$\|e^{(s-t)f_\epsilon(A)}u\|^2 = \sum_{n=1}^\infty e^{2(s-t)f_\epsilon(\lambda_n)}u_n^2 \le \exp(\frac{s-t}{T}\ln\frac{1}{(\beta(\epsilon))})\sum_{n=1}^\infty u_n^2 = (\beta(\epsilon))^{\frac{t-s}{T}}\|u\|^2$$

□

Theorem 2.3. *Let ϵ be as in Lemma 2.1, $\varphi \in H$ and let $h : H \times \mathbb{R} \to H$ be a continuous operator satisfying $\|h(w(t),t) - h(v(t),t)\| \le k\|w - v\|$ for a $k > 0$ independent of $w(t), v(t) \in H$, $t \in \mathbb{R}$ and f_ϵ satisfies Condition (A). Then the approximate problem* (1.6)-(1.7) *has a unique solution $u^\epsilon \in C([0,T];H)$.*

First, we consider two following propositions which are useful to the proof of Theorem 2.3.

Proposition 2.4. *The integral equation*

$$u^\epsilon(t) = e^{(T-t)f_\epsilon(A)}\varphi - \int_t^T e^{(s-t)f_\epsilon(A)} h(u^\epsilon(s),s)ds \tag{2.1}$$

has a unique solution and this solution satisfies the approximate problem (1.6)-(1.7).

Proof. We put

$$F(w)(t) = e^{(T-t)f_\epsilon(A)}\varphi - \int_t^T e^{(s-t)f_\epsilon(A)} h(w(s),s)ds$$

We claim that, for every $w, v \in C([0,T];H)$ we have

$$\|F^m(w)(.,t) - F^m(v)(.,t)\| \le \Big(\frac{k(T-t)}{\beta(\epsilon)}\Big)^m |||w - v||| \tag{2.2}$$

where $C = \max\{T, 1\}$ and $|||\cdot|||$ is sup norm in $C([0,T];H)$. We shall prove the latter inequality by induction.

For $m = 1$, we have

$$\begin{aligned}
\|F(w)(.,t) - F(v)(.,t)\| &= \|\int_t^T e^{(s-t)f_\epsilon(A)}(h(w(s),s) - h(v(s),s))ds\| \\
&\le \int_t^T \|e^{(s-t)f_\epsilon(A)}\|\|h(w(s),s) - h(v(s),s)\|ds \\
&\le \int_t^T \frac{k}{\beta(\epsilon)^{\frac{s-t}{T}}}\|w(s) - v(s)\|ds \\
&\le \frac{k}{\beta(\epsilon)}\int_t^T \|w(s) - v(s)\|ds \\
&\le \frac{k}{\beta(\epsilon)}(T-t)|||w - v||||
\end{aligned}$$

(We can choose ϵ such that $0 < \beta(\epsilon) < 1$)

Suppose that (2.2) holds for $m = j$. We prove that (2.2) holds for $m = j + 1$. We have

$$\begin{aligned}
\|F^{j+1}(w)(.,t) - F^{j+1}(v)(.,t)\| &= \|\int_t^T e^{(s-t)f_\epsilon(A)}(h(F^j w)(s) - h(F^j v)(s))ds\| \\
&\le \int_t^T \|e^{(s-t)f_\epsilon(A)}\|\|h(F^j w)(s) - h(F^j v)(s)\|ds \\
&\le \int_t^T \frac{k(T-t)}{\beta(\epsilon)^{\frac{s-t}{T}}}\|h(F^j w)(s) - h(F^j v)(s)\|ds \\
&\le \frac{k(T-t)}{\beta(\epsilon)}\int_t^T k\|(F^j w)(s) - (F^j v)(s)\|ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\beta(\epsilon)}(T-t)k\int_t^T \|G^j(w)(.,s)-G^j(v)(.,s)\|^2 ds\\
&\leq \frac{1}{\beta(\epsilon)}(T-t)k\int_t^T \frac{k^j}{\beta(\epsilon)^j}(T-s)^j ds|||w-v|||\\
&\leq \big(\frac{k}{\beta(\epsilon)}\big)^{(j+1)}(T-t)^{j+1}|||w-v|||.
\end{aligned}
$$

Therefore, by the induction principle, we have (2.2).

We consider $F : C([0,T];H) \to C([0,T];H)$. Since $\lim_{m\to\infty}\left(\frac{kT}{\beta(\epsilon)}\right)^m = 0$, there exists a positive integer number m_0 such that F^{m_0} is a contraction. It follows that the equation $F^{m_0}(w) = w$ has a unique solution $u_\epsilon \in C([0,T];H)$.

We claim that $F(u^\epsilon) = u^\epsilon$. In fact, one has $F(F^{m_0}(u^\epsilon)) = F(u^\epsilon)$. Hence $F^{m_0}(F(u^\epsilon)) = F(u^\epsilon)$. By the uniqueness of the fixed point of F^{m_0}, one has $F(u^\epsilon) = u^\epsilon$, i.e., the equation $F(w) = w$ has a unique solution $u^\epsilon \in C([0,T];H)$.

Finally, we prove the unique solution of (2.1) satisfies t (1.6)-(1.7). In fact, one has in view from (2.1), we have

$$
u^\epsilon(t) = e^{(T-t)f_\epsilon(A)}\varphi - \int_t^T e^{(s-t)f_\epsilon(A)}h(u^\epsilon(s),s)ds
$$

This also follows that $u(T) = \varphi$, hence the condition (1.7) is satisfied. The expansion formula of $u^\epsilon(t)$

$$
u^\epsilon(t) = \sum_{n=1}^\infty \Big(e^{(T-t)f_\epsilon(\lambda_n)}\varphi_n - \int_t^T e^{(s-t)f_\epsilon(\lambda_n)}h_n(u^\epsilon)(s)ds\Big)\phi_n
$$

Differentiating $u(t)$ with respect to t, we get

$$
\begin{aligned}
u^\epsilon_t(t) &= -f_\epsilon(A)e^{(T-t)f_\epsilon(A)}\varphi + f_\epsilon(A)\int_t^T e^{(s-t)f_\epsilon(A)}h(u^\epsilon(s),s)ds + h(u^\epsilon(t),t)\\
&= -f_\epsilon(A)u^\epsilon + h(u^\epsilon(t),t).
\end{aligned}
$$

This completes the proof of Proposition 2.4 □

Proposition 2.5. *Assume that f_ϵ satisfies condition A then Problem* (1.6)-(1.7) *has at most one solution in $C([0,T];H)$.*

Proof. Suppose $u(t)$ and $v(t)$ are solution in $C([0,T];H)$ of the approximate problem (1.6)-(1.7). Putting $w(t) = e^{m(t-T)}(u(t)-v(t))$ $(m>0)$, then replacing in the equation (1.6) and by direct computation, we obtain

$$
w_t + f_\epsilon(A)w(t) - mw(t) = e^{m(t-T)}h(e^{-m(t-T)}u(t),t) - h(e^{-m(t-T)}v(t),t) \quad (2.3)
$$

Multiplying two side of (2.3) with w and using global Lipchitz properties of function h we get

$$
\frac{d}{2dt}\|w(t)\|^2 + \langle f_\epsilon(A)w,w\rangle - m\|w\|^2 + k\|w\|^2 \geq 0
$$

Using the boundedness of function f_ϵ in Lemma 2.2, we have

$$
|\langle f_\epsilon(A)w,w\rangle| \leq \|f_\epsilon(A)w\|\|w\| \leq \frac{1}{T}\ln\big(\frac{1}{\beta(\epsilon)}\big)\|w\|^2
$$

It follows that

$$
\frac{d}{2ds}\|w(s)\|^2 \geq m\|w\|^2 - k\|w\|^2 - \frac{1}{T}\ln\big(\frac{1}{\beta(\epsilon)}\big)\|w\|^2 \quad (2.4)
$$

Putting the integral with s from t to T in (2.4), then choosing $m = k + \frac{1}{T}\ln(\frac{1}{\beta(\epsilon)})$, it can be rewritten as

$$\|w(T)\|^2 - \|w(t)\|^2 \geq 2(m - k - \frac{1}{T}\ln(\frac{1}{\beta(\epsilon)}) \int_t^T \|w(s)\|^2 ds = 0 \tag{2.5}$$

Using the equality $w(T) = u(T) - v(T) = 0$, one has $w(t) = 0$. This completes the proof ☐

Theorem 2.6. *The solution of* (1.6)-(1.7) *depends continuously on* φ

Proof. Let u and v be two solution of (1.6)-(1.7) corresponding with two final values φ and ω. By setting $w(t) = e^{m(t-T)}(u(t) - v(t))$ (with $m > 0$), we have $w(T) = \varphi - \omega$. In view of inequality (2.3) in Proposition 2.4, we get

$$\|w(T)\|^2 - \|w(t)\|^2 \geq 2(m - k - \frac{1}{T}\ln(\frac{1}{\beta(\epsilon)}) \int_t^T \|w(s)\|^2 ds = 0. \tag{2.6}$$

choosing $m = k + \frac{1}{T}\ln(\frac{1}{\beta(\epsilon)})$, we have

$$\|\varphi - \omega\| \geq \|w(t)\| = e^{m(t-T)}\|u(t) - v(t)\|$$

This implies

$$\|u(t) - v(t)\| \leq e^{m(T-t)}\|\varphi - \omega\| = e^{k(T-t)}\beta(\epsilon)^{\frac{t}{T}-1}\|\varphi - \omega\|$$

whihc proves continuity and that the stability of the solution is of order $E\beta(\epsilon)^{\frac{t}{T}-1}$. ☐

3. Regularization of Problem (1.1)-(1.2)

Theorem 3.1. *Let ϵ be as in Lemma 2.1. Suppose problem* (1.1)-(1.2) *has a unique solution $u(t) \in (C[0,T]; H)$ which satisfies $u(t) \in \mathrm{Dom}(e^{TA})$. Then for $0 < t \leq T$ we have the error estimate*

$$\|u(t) - u^\epsilon(t)\| \leq M\beta(\epsilon)^{t/T}$$

Moreover, there exists a $t_\epsilon \in (0, T)$ such that

$$\|u(0) - u^\epsilon(t_\epsilon)\| \leq 2C\Big(\frac{T}{\ln\frac{1}{\beta(\epsilon)}}\Big)^{1/2},$$

where

$$M = e^{k(T-t)}\int_0^T \|e^{TA}u(s)\|ds, \quad C = \max\{e^{kT}M, (\frac{1}{T} + k)M + \sup_{t\in[0,T]}\|f(0,t)\|\},$$

u^ϵ is the unique solution of (1.5), *and $u^\epsilon(t)$ is the unique solution of* (1.6)-(1.7).

Proof. We put $w^\epsilon(t) = u^\epsilon(t) - u(t)$ and $g_\epsilon(A) = -A + f_\epsilon(A)$. Then $w^\epsilon(t)$ satisfies

$$w_t^\epsilon + f_\epsilon(A)w^\epsilon = h(u^\epsilon(t), t) - h(u(t), t) + g_\epsilon(A)u(t) \tag{3.1}$$

Let $h_1 : H \times \mathbb{R} \to H$ satisfying $h_1(w(t), t)) = h(w(t) + u(t), t) - h(u(t), t)$. Using the Lipchitz property of h given in Theorem 2.6, we get $\|h_1(w(t), t))\| \leq k\|w(t)\|$. Hence, (2.6) can be written as

$$\begin{gathered} w_t^\epsilon + f_\epsilon(A)w^\epsilon = h_1(w^\epsilon(t), t) + g_\epsilon(A)u(t) \\ w^\epsilon(T) = 0 \end{gathered}$$

It is not difficult to check $w^\epsilon(t)$ satisfies

$$w^\epsilon(t) = -\int_t^T e^{(s-t)f_\epsilon(A)}[h_1(w^\epsilon(s),s) + g_\epsilon(A)u(s)]ds. \tag{3.2}$$

It follows that

$$\begin{aligned}
\|w^\epsilon(t)\| &= \|\int_t^T e^{(s-t)f_\epsilon(A)}[h_1(w^\epsilon(s),s) + g_\epsilon(A)u(s)]ds\| \\
&\le \int_t^T e^{(s-t)\|f_\epsilon(A)\|}[\|h_1(w^\epsilon(s),s)\| + \|g_\epsilon(A)u(s)\|]ds \\
&\le \beta(\epsilon)^{\frac{t}{T}} k \int_t^T \beta(\epsilon)^{\frac{-s}{T}} \|w^\epsilon(s))\|ds + \beta(\epsilon)^{\frac{t}{T}} \int_t^T \beta(\epsilon)^{\frac{-s}{T}} \|g_\epsilon(A)u\|ds \\
&\le \beta(\epsilon)^{\frac{t}{T}} k \int_t^T \beta(\epsilon)^{\frac{-s}{T}} \|w^\epsilon(s))\|ds + \beta(\epsilon)^{\frac{t}{T}} \int_0^T \beta(\epsilon)^{\frac{T-s}{T}} \|e^{TA}u(s)\|ds \\
&\le \beta(\epsilon)^{\frac{t}{T}} k \int_t^T \beta(\epsilon)^{\frac{-s}{T}} \|w^\epsilon(s))\|ds + \beta(\epsilon)^{\frac{t}{T}} T \int_0^T \|e^{TA}u(s)\|ds
\end{aligned}$$

From the above inequality, we have

$$\beta(\epsilon)^{\frac{-t}{T}} \|w^\epsilon(t)\| \le k \int_t^T \beta(\epsilon)^{\frac{-s}{T}} \|w^\epsilon(s)\|ds + \int_0^T \|e^{TA}u(s)\|ds\,.$$

Using Gronwall's inequality we obtain

$$\|w^\epsilon(t)\| \le e^{k(T-t)}\beta(\epsilon)^{\frac{t}{T}} \int_0^T \|e^{TA}u(s)\|ds,$$

or

$$\|u^\epsilon(t) - u(t)\| \le M\beta(\epsilon)^{\frac{t}{T}},$$

For $t \in (0,T)$, considering the function $h(t) = \frac{\ln t}{t} - \frac{\ln(\beta(\epsilon))}{T}$, we have $h(\beta(\epsilon)) > 0$, $\lim_{t\to 0} h(t) = -\infty$, $h'(t) > 0$ $(0 < t < \beta(\epsilon))$. It follows that the equation $h(t) = 0$ has a unique solution t_ϵ in $(0, \beta(\epsilon))$. Since $\frac{\ln t_\epsilon}{t_\epsilon} = \frac{\ln(\beta(\epsilon))}{T}$, the inequality $\ln t > -\frac{1}{t}$ gives $t_\epsilon < \sqrt{\frac{T}{\ln\frac{1}{\epsilon}}}$.

We have $u(t_\epsilon) - u(0) = \int_0^{t_\epsilon} u'(t)dt$. Hence $\|u(0) - u(t_\epsilon)\| \le t_\epsilon \sup_{t\in[0,T]} \|u'(t)\|$. On the other hand, one has

$$\begin{aligned}
\|u'(t)\| &\le \|Au(t)\| + \|f(u(t),t)\| \\
&\le \Big(\sum_{n=1}^\infty \lambda_n^2 u_n^2(t)\Big)^{1/2} + k\|u(t)\| + \|f(0,t)\| \\
&\le \frac{1}{T}\Big(\sum_{n=1}^\infty e^{2T\lambda_n} u_n^2(t)\Big)^{1/2} + k\|u(t)\| + \|f(0,t)\| \\
&\le (\frac{1}{T} + k)M + \|f(0,t)\| \le C.
\end{aligned}$$

It follows that $\|u(0) - u(t_\epsilon)\| \le Ct_\epsilon$. By the definition of t_ϵ, we get

$$\begin{aligned}
\|u(0) - u^\epsilon(t_\epsilon)\| &\le \|u(0) - u(t_\epsilon)\| + \|u(t_\epsilon) - u^\epsilon(t_\epsilon)\| \\
&\le 2Ct_\epsilon \le 2C\Big(\frac{T}{\ln(\frac{1}{\beta(\epsilon)})}\Big)^{1/2}.
\end{aligned}$$

which completes the proof. $\square$

4. Example and applications

First, we consider the model problem

$$u_t + Au(t) = h(u(t), t)$$
$$u(T) = \varphi$$

that is compared with the following well-posed problem. Taking function $f_\epsilon(x) = -\frac{1}{T}\ln(\epsilon + e^{-Tx})$ for $x \in (0, \infty)$, we have the first approximate problem

$$u_t^\epsilon - \frac{1}{T}\ln(\epsilon + e^{-TA})u^\epsilon = h(u^\epsilon(t), t) \tag{4.1}$$

$$u^\epsilon(T) = \varphi\,. \tag{4.2}$$

It is easy to check that $|f(x)| \le \frac{1}{T}\ln(\frac{1}{\epsilon})$. Then f satisfies Condition (A) with $\beta(\epsilon) = \epsilon$.

In the Hilbert space, let $H = L^2(0, \pi)$ and let $A = -\Delta$ is the Laplace operator. We take $\lambda_n = n^2$, $\phi_n = \sqrt{\frac{2}{\pi}}\sin(nx)$ are eigenvalues and orthonormal eigenfunctions, which form a basis for H. Let us consider the nonlinear backward heat problem

$$-u_{xx} + u_t = f(u) + g(x, t), \quad (x, t) \in (0, \pi) \times (0, 1) \tag{4.3}$$

$$u(0, t) = u(\pi, t) = 0, \quad t \in [0, 1], \tag{4.4}$$

$$u(x, 1) = \varphi(x), \quad x \in [0, \pi] \tag{4.5}$$

where

$$f(u) = \begin{cases} u^2 & u \in [-e^{10}, e^{10}] \\ -\frac{e^{10}}{e-1}u + \frac{e^{21}}{e-1} & u \in (e^{10}, e^{11}] \\ \frac{e^{10}}{e-1}u + \frac{e^{21}}{e-1} & u \in (-e^{11}, -e^{10}] \\ 0 & |u| > e^{11} \end{cases}$$

$$g(x, t) = 2e^t \sin x - e^{2t}\sin^2 x,$$
$$u(x, 1) = \varphi_0(x) \equiv e\sin x.$$

The exact solution of the above equation is $u(x, t) = e^t \sin x$. In particular,

$$u\big(x, \frac{999}{100}\big) \equiv u(x) = \exp\big(\frac{999}{1000}\big)\sin x \approx 2.715564905\sin x.$$

Let $\varphi_\epsilon(x) \equiv \varphi(x) = (\epsilon + 1)e\sin x$. Then

$$\|\varphi_\epsilon - \varphi\|_2 = \Big(\int_0^\pi \epsilon^2 e^2 \sin^2 x dx\Big)^{1/2} = \epsilon e\sqrt{\pi/2}.$$

Applying the method introduced in this paper, we find the regularized solution $u_\epsilon\left(x, \frac{999}{1000}\right) \equiv u_\epsilon(x)$ having the form

$$u_\epsilon(x) = v_m(x) = w_{1,m}\sin x + w_{6,m}\sin 6x\,,$$

where $v_1(x) = (\epsilon + 1)e\sin x$, $w_{1,1} = (\epsilon + 1)e$, $w_{6,1} = 0$, and $a = \frac{1}{5000}$, $t_m = 1 - am$ for $m = 1, 2, \dots, 5$, and

$$w_{i,m+1} = (\epsilon + e^{-t_m i^2})^{\frac{t_{m+1} - t_m}{t_m}} w_{i,m}$$

$$-\frac{2}{\pi}\int_{t_{m+1}}^{t_m} e^{(s-t_{m+1})i^2}\Big(\int_0^\pi \big(v_m^2(x)+g(x,s)\big)\sin(ix)dxds\Big),$$

for $i = 1, 6$. Table 1 shows the approximation error in this case.

Table 1. Error between regularized and exact solution

ϵ	u_ϵ	$\|u_\epsilon - u\|$
$\epsilon_1 = 10^{-3}$	$2.718118645\sin(x) - 0.005612885749\sin(6x)$	0.002585244486
$\epsilon_2 = 10^{-4}$	$2.715807105\sin(x) - 0.005488275207\sin(6x)$	0.0002723211648
$\epsilon_3 = 10^{-11}$	$2.715552177\sin(x) - 0.005518178192\sin(6x)$	0.00004317829056

By applying the method in [18], we have the approximate solution

$$u_\epsilon(x, \frac{999}{1000}) = v_m(x) = w_{1,m}\sin x + w_{3,m}\sin 3x,$$

where $v_1(x) = (\epsilon+1)e\sin x$, $w_{1,1} = (\epsilon+1)e$, $w_{3,1} = 0$, $a = \frac{1}{5000}$, $t_m = 1 - am$ for $m = 1, 2, \dots, 5$ and

$$\begin{aligned} w_{i,m+1} &= e^{(t_m-t_{m+1})\frac{i^2}{1+\epsilon i^2}} w_{i,m} \\ &\quad - \frac{2}{\pi}\int_{t_{m+1}}^{t_m} e^{s-t_{m+1}-\frac{(t_m-t_{m+1})\epsilon i^2}{1+\epsilon i^2}}\Big(\int_0^\pi \big(v_m^2(x)+g(x,s)\big)\sin(ix)dx\Big)ds, \end{aligned}$$

for $i = 1, 3$. Table 2 shows the approximation error in this case.

Table 2. Error between regularized and exact solution

ϵ	u_ϵ	$\|u_\epsilon - u\|$
$\epsilon_1 = 10^{-3}$	$2.718267378\sin(x) - 0.005479540370\sin(3x)$	0.006109723643
$\epsilon_2 = 10^{-4}$	$2.715832209\sin(x) - 0.005468363690\sin(3x)$	0.005474892956
$\epsilon_3 = 10^{-11}$	$2.715561633\sin(x) - 0.005467119519\sin(3x)$	0.005467120499

From the two tables, we see that the error in Table 1 is smaller and increases slower than the error in Table 2. This indicates that in this example, our our approach has a nice regularizing effect and give a better approximation that the method in [18].

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