Electronic Journal of Differential Equations, Vol. 2008(2008), No. 88, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

POSITIVE PERIODIC SOLUTIONS FOR A PREDATOR-PREY MODEL WITH TIME DELAYS AND IMPULSIVE EFFECT

SHAN GAO, YONGKUN LI

Abstract. In this article, a two-species predator-prey model with time delays and impulsive effect is investigated. By using Mawhin's continuation theorem of coincidence degree theory, sufficient conditions are obtained for the existence of positive periodic solutions.

1. Introduction

In the past few years, predator-prey models and with many kinds of functional responses have been of great interest to both applied mathematicians and ecologists see references in this article. Recently, by using Floquet theory of linear periodic impulsive equation, Song and Li [15] considered the following T-periodic predator-prey model with modified Leslie-Gower and Holling-type II schemes and impulsive effect

$$\left.\begin{aligned} \dot{x}(t) &= x(t)\Big(r_1(t) - b_1(t)x(t) - \frac{a_1(t)y(t)}{x(t)+k_1(t)}\Big) \\ \dot{y}(t) &= y(t)\Big(r_2(t) - \frac{a_2(t)y(t)}{x(t)+k_2(t)}\Big) \end{aligned}\right\} \quad t \neq \tau_k,\ k \in \mathbb{Z}_+,$$
$$\left.\begin{aligned} x(\tau_k^+) &= (1+h_k)x(\tau_k) \\ y(\tau_k^+) &= (1+g_k)y(\tau_k) \end{aligned}\right\} \quad t = \tau_k,\ k \in \mathbb{Z}_+.$$

where $b_1(t), r_i(t), a_i(t), k_i(t) (i = 1, 2)$ are continuous ω-periodic functions such that $b_1(t) > 0, r_i(t) > 0, a_i(t) > 0, k_i(t) > 0 (i = 1, 2)$ and $\mathbb{Z}_+ = \{1, 2, \dots\}$; $h_k, g_k (k \in \mathbb{Z}_+)$ are constants and there exists an integer $q > 0$ such that $h_{k+q} = h_k, g_{k+q} = g_k, \tau_{k+q} = \tau_k + \omega$, and $1 + h_k > 0, 1 + g_k > 0$ for all $k \in \mathbb{Z}_+$. They obtain some conditions for the linear stability of trivial periodic solution and semitrivial periodic solutions.

However, as pointed out in [9], naturally, more realistic and interesting models of single or multiple species growth should take into account both the seasonality of the changing environment and the effects of time delays.

2000 *Mathematics Subject Classification.* 34K13, 34K45, 92D25.
Key words and phrases. Predator-prey model; impulse; positive periodic solution; coincidence degree.
©2008 Texas State University - San Marcos.
Submitted April 7, 2008. Published June 21, 2008.
Supported by the National Natural Sciences Foundation of China and the Natural Sciences Foundation of Yunnan Province, China.

In this paper, we consider the following ω-periodic predator-prey system with time delays and impulses:

$$\left.\begin{aligned}\dot{x}(t) &= x(t)\Big(r_1(t) - b_1(t)x(t-\tau(t)) - \frac{a_1(t)y(t-\sigma_1(t))}{x(t-\tau_1(t))+k_1(t)}\Big)\\ \dot{y}(t) &= y(t)\Big(r_2(t) - \frac{a_2(t)y(t-\sigma_2(t))}{x(t-\tau_2(t))+k_2(t)}\Big)\end{aligned}\right\} \quad t \neq t_k,\ k \in \mathbb{Z}_+,$$
$$\left.\begin{aligned}x(t_k^+) &= I_k(x(t_k)) + x(t_k^-)\\ y(t_k^+) &= J_k(y(t_k)) + y(t_k^-)\end{aligned}\right\} \quad t = t_k,\ k \in \mathbb{Z}_+. \tag{1.1}$$

where $x(t_k^+), x(t_k^-), y(t_k^+), y(t_k^-)$ represent the right and the left limit of $x(t_k)$ and $y(t_k)$, respectively, in this paper, we assume that x, y are left continuous at t_k; $b_1(t)$, $\tau(t)$, $a_i(t)$, $r_i(t)$, $k_i(t)$, $\sigma_i(t)$, $\tau_i(t)$ $(i = 1, 2)$ are all positive periodic continuous functions with period $\omega > 0$ and $\mathbb{Z}_+ = \{1, 2, \dots\}$; $I_k, J_k \in C(\mathbb{R}^+, \mathbb{R})$ satisfy that $I_k(u) > -u, J_k(v) > -v$, and there exists a positive integer p such that $t_{k+p} = t_k + \omega$, $I_{k+p} = I_k$, $J_{k+p} = J_k$, $k \in \mathbb{Z}_+$. Without loss of generality, we also assume that $[0, \omega) \cap \{t_k : k \in \mathbb{Z}_+\} = \{t_1, t_2, \dots, t_p\}$.

Our purpose of this paper is by using continuation theorem of coincidence degree theory [6] to establish criteria to guarantee the existence of positive periodic solutions of system (1.1).

2. Notation and preliminaries

To obtain our main result of this paper, we first need to make the following preparations. For any non-negative integer q, let

$$\begin{aligned}&C^{(q)}[0,\omega;\, t_1, t_2, \dots, t_p]\\ &= \Big\{x : [0,\omega] \to \mathbb{R} \text{ such that } x^{(q)}(t) \text{ exists for } t \neq t_1, \dots, t_p;\ x^{(q)}(t_k^+),\ x^{(q)}(t_k^-)\\ &\quad \text{exists at } t_1, \dots, t_p; \text{ and } x^{(j)}(t_k) = x^{(j)}(t_k^-),\ k = 1, 2, \dots, p,\ j = 0, 1, 2, \dots, q\Big\}\end{aligned}$$

with the norm

$$\|x\|_q = \max\{\sup_{t\in[0,\omega]} |x^{(j)}(t)|\}_{j=0}^q.$$

It is easy to see that $C^{(q)}[0, \omega; t_1, t_2, \dots, t_p]$ is a Banach space and the functions in $C[0, \omega;\ t_1, t_2, \dots, t_p]$ are continuous with respect to t different from $t_1, t_2, \dots, t_p$. Let

$$PC_\omega = \big\{x \in C[0, \omega; t_1, t_2, \dots, t_p] : x(0) = x(\omega)\big\}$$

with the same norm as that of $C[0, \omega; t_1, t_2, \dots, t_p]$.

Let X, Y be normed vector spaces, $L : \operatorname{Dom} L \subset X \to Y$ be a linear mapping, and $N : X \to Y$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim\ker L = \operatorname{codim} \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in Y. If L is a Fredholm mapping of index zero, and there exist continuous projectors: $P : X \to X$ and $Q : Y \to Y$ such that $\operatorname{Im} P = \ker L$, $\ker Q = \operatorname{Im} L = \operatorname{Im}(I - Q)$. It follows that mapping $L|_{\operatorname{Dom} L\cap\ker P} : (I - P)X \to \operatorname{Im} L$ is invertible. We denote the inverse of that mapping by K_P. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\ker L$, there exists an isomorphism $J : \operatorname{Im} Q \to \ker L$.

Definition 2.1. The set F is said to be quasi-equicontinuous in $[0,\omega]$ if for any $\epsilon > 0$ there exists $\delta > 0$ such that if $x \in F$, $k \in \mathbb{Z}_+$, $t_1, t_2 \in (t_{k-1}, t_k) \cap [0,\omega]$, $|t_1 - t_2| < \delta$, then $|x(t_1) - x(t_2)| < \epsilon$.

Lemma 2.2 ([2])**.** *The set $F \subset PC_\omega$ is relatively compact if and only if*

(1) *F is bounded, that is, $\|f\|_{PC_\omega} = \|f\|_0 = \sup_{t\in[0,\omega]} |f(t)| \le M$ for each $f \in F$ and some $M > 0$;*
(2) *F is quasi-equicontinuous in* $\operatorname{Dom} f$.

Now, we introduce Mawhin's continuation theorem.

Lemma 2.3 ([6])**.** *Let $\Omega \subset X$ be an open bounded set and let $N : X \to Y$ be a continuous operator which is L-compact on $\overline{\Omega}$. Assume*

(a) *for each $\lambda \in (0,1)$, $x \in \partial\Omega \cap \operatorname{Dom} L$, $Lx \ne \lambda Nx$,*
(b) *for each $x \in \partial\Omega \cap \ker L$, $QNx \ne 0$,*
(c) $\deg(JQN, \Omega \cap \ker L, 0) \ne 0$.

Then $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \operatorname{Dom} L$.

Throughout this paper, we assume that there exist $p_{1k}, p_{2k}, q_{1k}, q_{2k} \in \mathbb{R}$, $k \in Z_+$ such that

$$
\begin{gathered}
\inf_{u>0} \frac{I_k(u)}{u} \ge q_{1k} > -1, \quad \sup_{u>0} \frac{I_k(u)}{u} \le p_{1k} < +\infty, \\
\inf_{v>0} \frac{J_k(v)}{v} \ge q_{2k} > -1, \quad \sup_{v>0} \frac{J_k(v)}{v} \le p_{2k} < +\infty.
\end{gathered}
$$

For convenience, we introduce the notation

$$
\overline{f} = \frac{1}{\omega} \int_0^\omega f(t)\,\mathrm{d}t, \quad f^M = \max_{t\in[0,\,\omega]} \{f(t)\},
$$

$$
l_{1k} = \max\big\{|\ln(1+p_{1k})|, |\ln(1+q_{1k})|\big\}, \quad l_{2k} = \max\big\{|\ln(1+p_{2k})|, |\ln(1+q_{2k})|\big\}
$$

where f is a continuous ω-periodic function and $k \in Z_+$.

3. Main result

Let

$$
\begin{gathered}
H_1 = \ln\Big(\frac{\overline{r_1} + \frac{1}{\omega}\ln(\prod_{k=1}^p (1+p_{1k}))}{\overline{b_1}}\Big), \\
M_1 = H_1 + 2\omega\overline{r_1} + \ln\Big(\prod_{k=1}^p (1+p_{1k})\Big) + \sum_{k=1}^p l_{1k}; \\
H_2 = \ln\Big(\frac{\overline{r_2} + \frac{1}{\omega}\ln(\prod_{k=1}^p (1+q_{2k}))}{\overline{(\frac{a_2}{k_2})}}\Big), \\
M_2 = H_2 - 2\omega\overline{r_2} - \ln\Big(\prod_{k=1}^p (1+p_{2k})\Big) - \sum_{k=1}^p l_{2k}; \\
H_3 = \ln\Big(\frac{(k_2^M + \mathrm{e}^{M_1})(\overline{r_2} + \frac{1}{\omega}\ln(\prod_{k=1}^p (1+p_{2k})))}{\overline{a_2}}\Big), \\
M_3 = H_3 + 2\omega\overline{r_2} + \ln\Big(\prod_{k=1}^p (1+p_{2k})\Big) + \sum_{k=1}^p l_{2k};
\end{gathered}
$$

$$
\begin{gathered}
H_4=\ln\Big(\frac{\overline{r_1}+\frac{1}{\omega}\ln(\prod_{k=1}^p(1+q_{1k}))-\mathrm{e}^{M_3}\overline{(\frac{a_1}{k_1})}}{\overline{b_1}}\Big),\\
M_4=H_4-2\omega\overline{r_1}-\ln\Big(\prod_{k=1}^p(1+p_{1k})\Big)-\sum_{k=1}^p l_{1k};
\end{gathered}
$$

Our main result of this paper is as follows:

Theorem 3.1. *If*

$$
\begin{gathered}
\overline{r_1}+\frac{1}{\omega}\ln\Big(\prod_{k=1}^p(1+p_{1k})\Big)>0,\\
\overline{r_2}+\frac{1}{\omega}\ln\Big(\prod_{k=1}^p(1+q_{2k})\Big)>0,\\
\overline{r_1}+\frac{1}{\omega}\ln\Big(\prod_{k=1}^p(1+q_{1k})\Big)-\mathrm{e}^{M_3}\overline{(\frac{a_1}{k_1})}>0,
\end{gathered}
$$

then (1.1) *has at least one* ω*-periodic positive solution.*

Proof. Let $x(t)=\mathrm{e}^{u(t)},\ y(t)=\mathrm{e}^{v(t)}$ then (1.1) is reformulated as

$$
\begin{gathered}
\left.\begin{aligned}
\dot u(t)&=r_1(t)-b_1(t)\exp\{u(t-\tau(t))\}-\frac{a_1(t)\exp\{v(t-\sigma_1(t))\}}{\exp\{u(t-\tau_1(t))\}+k_1(t)}\\
\dot v(t)&=r_2(t)-\frac{a_2(t)\exp\{v(t-\sigma_2(t))\}}{\exp\{u(t-\tau_2(t))\}+k_2(t)}
\end{aligned}\right\}\\
\text{for } t\neq t_k,\ k\in\mathbb{Z}_+,\text{ and}\\
\left.\begin{aligned}
u(t_k^+)&=e_k(u(t_k))+u(t_k^-)\\
v(t_k^+)&=f_k(v(t_k))+v(t_k^-)
\end{aligned}\right\}\quad t=t_k,k\in\mathbb{Z}_+,
\end{gathered}
\tag{3.1}
$$

where

$$
\begin{gathered}
e_k(u(t_k))=\ln\Big(\frac{I_k(\exp\{u(t_k)\})+\exp\{u(t_k)\}}{\exp\{u(t_k)\}}\Big),\\
f_k(v(t_k))=\ln\Big(\frac{J_k(\exp\{v(t_k)\})+\exp\{v(t_k)\}}{\exp\{v(t_k)\}}\Big).
\end{gathered}
$$

It is easy to see that

$$
\begin{gathered}
\ln(1+q_{1k})\le e_k(u(t_k))\le\ln(1+p_{1k}),\\
\ln(1+q_{2k})\le f_k(v(t_k))\le\ln(1+p_{2k}).
\end{gathered}
$$

If system (3.1) has an ω-periodic solution $(u(t),v(t))$, then

$$
(\mathrm{e}^{u(t)},\ \mathrm{e}^{v(t)})=(x^*(t),\ y^*(t))
$$

is a positive ω-periodic solution to system (1.1). So, in the following, we discuss the existence of ω-periodic solution to system (3.1). Here, we denote

$$
\begin{gathered}
A(t)=r_1(t)-b_1(t)\exp\{u(t-\tau(t))\}-\frac{a_1(t)\exp\{v(t-\sigma_1(t))\}}{\exp\{u(t-\tau_1(t))\}+k_1(t)},\\
B(t)=r_2(t)-\frac{a_2(t)\exp\{v(t-\sigma_2(t))\}}{\exp\{u(t-\tau_2(t))\}+k_2(t)}.
\end{gathered}
$$

To use the continuation theorem of coincidence degree theory to establish the existence of an ω-periodic solution of (3.1), we take

$$X = PC_\omega \times PC_\omega, \quad Y = X \times \mathbb{R}^{2p}.$$

Then X is a Banach space with the norm

$$\|x\|_X = \|(u,v)\|_X = \|u\|_0 + \|v\|_0 = \sup_{t\in[0,\omega]} |u(t)| + \sup_{t\in[0,\omega]} |v(t)|,$$

and Y is also a Banach space with the norm

$$\|z\|_Y = \|x\|_X + \|y\|_2, \quad x \in X,\ y \in \mathbb{R}^{2p},$$

where $\|\cdot\|_2$ in $\mathbb{R}^n$ is defined as

$$\|\xi\|_2 = \|(\xi_1, \xi_2, \dots, \xi_n)\|_2 = \sum_{i=1}^n |\xi_i|.$$

So if $x = (u,v) \in X \cap \mathbb{R}^2$, then $\|x\|_X = \|x\|_2$. Let

$$\operatorname{Dom} L = \Big\{(u,v) : u, v \in C^{(1)}[0,\omega; t_1, t_2, \dots, t_p]; u(0) = u(\omega), v(0) = v(\omega)\Big\},$$

$$L : \operatorname{Dom} L \cap X \to Y$$

$$(u,v) \to (\dot{u}, \dot{v}, \Delta u(t_1), \dots, \Delta u(t_p), \Delta v(t_1), \dots, \Delta v(t_p)),$$

and let $N : X \to Y$ with

$$N(x) = N(u,v) = \big(A(t), B(t), \Delta u(t_1), \dots, \Delta u(t_p), \Delta v(t_1), \dots, \Delta v(t_p)\big).$$

Obviously, $\ker L = \Big\{(u,v) : u, v \in \mathbb{R}, t \in [0,\omega]\Big\} = \mathbb{R}^2$,

$$\operatorname{Im} L = \Big\{z = (f, g, c_1, \dots, c_p, d_1, \dots, d_p) \in Y : \int_0^\omega f(s)\,\mathrm{d}s + \sum_{k=1}^p c_k = 0,$$
$$\int_0^\omega g(s)\,\mathrm{d}s + \sum_{k=1}^p d_k = 0\Big\}$$

and $\dim\ker L = \operatorname{codim} \operatorname{Im} L = 2$. So that, $\operatorname{Im} L$ is closed in Y, L is a Fredholm mapping of index zero. Define the two projectors

$$Px = \frac{1}{\omega}, \int_0^\omega x(t)\,\mathrm{d}t,$$

$$Qz = \Big(\frac{1}{\omega}\Big[\int_0^\omega f(s)\,\mathrm{d}s + \sum_{k=1}^p c_k\Big], \frac{1}{\omega}\Big[\int_0^\omega g(s)\,\mathrm{d}s + \sum_{k=1}^p d_k\Big], \underbrace{0, \dots, 0}_{2p}\Big).$$

It is easy to show that P and Q are continuous and satisfy

$$\operatorname{Im} P = \ker L = \mathbb{R}^2, \quad \operatorname{Im} L = \ker Q = \operatorname{Im}(I - Q).$$

Further, let $L_P = L\big|_{\operatorname{Dom} L \cap \ker P}$ and the generalized inverse $K_P = L_P^{-1}$ is given by

$$K_P z = \Big(\int_0^t f(s)\,\mathrm{d}s + \sum_{t>t_k} c_k - \frac{1}{\omega}\int_0^\omega \int_0^t f(s)\,\mathrm{d}s\,\mathrm{d}t - \sum_{k=1}^p c_k + \frac{1}{\omega}\sum_{k=1}^p t_k c_k,$$
$$\int_0^t g(s)\,\mathrm{d}s + \sum_{t>t_k} d_k - \frac{1}{\omega}\int_0^\omega \int_0^t g(s)\,\mathrm{d}s\,\mathrm{d}t - \sum_{k=1}^p d_k + \frac{1}{\omega}\sum_{k=1}^p t_k d_k\Big).$$

Thus, the expression of QNx is

$$\Big(\frac{1}{\omega}\big[\int_0^\omega A(s)\,\mathrm{d}s\ +\sum_{k=1}^p e_k(u(t_k))\big],\ \frac{1}{\omega}\ \big[\int_0^\omega B(s)\,\mathrm{d}s\ \ +\sum_{k=1}^p f_k(v(t_k))\big],\ \underbrace{0,\dots,0}_{2p}\Big),$$

and then

$$\begin{aligned}
&K_P(I-Q)Nx\\
&=\Big(\int_0^t A(s)\,\mathrm{d}s+\sum_{t>t_k} e_k(u(t_k))-\frac{1}{\omega}\int_0^\omega\int_0^t A(s)\,\mathrm{d}s\,\mathrm{d}t+\frac{1}{\omega}\sum_{k=1}^p t_k e_k(u(t_k))\\
&\quad+(\frac{1}{2}-\frac{t}{\omega})\int_0^\omega A(s)\,\mathrm{d}s-(\frac{1}{2}+\frac{t}{\omega})\sum_{k=1}^p e_k(u(t_k))\ ,\\
&\quad\int_0^t B(s)\,\mathrm{d}s+\sum_{t>t_k} f_k(v(t_k))-\frac{1}{\omega}\int_0^\omega\int_0^t B(s)\,\mathrm{d}s\,\mathrm{d}t+\frac{1}{\omega}\sum_{k=1}^p t_k f_k(v(t_k))\\
&\quad+(\frac{1}{2}-\frac{t}{\omega})\int_0^\omega B(s)\,\mathrm{d}s-(\frac{1}{2}+\frac{t}{\omega})\sum_{k=1}^p f_k(v(t_k))\Big).
\end{aligned}$$

Hence, QN and $K_P(I-Q)N$ are both continuous. Using Lemma 2.2, it is easy to show that $K_P(I-Q)N(\overline{\Omega})$ is compact for any open bounded set $\Omega\subset X$. Moreover, $QN(\overline{\Omega})$ is bounded. Therefore, N is L-compact on $\overline{\Omega}$ for any open bounded set $\Omega\subset X$.

Now, it needs to show that there exists a domain Ω that satisfies all the requirements given in Lemma 2.3. Corresponding to operator equation $Lx=\lambda Nx$, $\lambda\in(0,1)$, $x=(u,v)$, we get

$$\begin{gathered}
\left.\begin{aligned}
\dot u(t)&=\lambda\Big[r_1(t)-b_1(t)\exp\{u(t-\tau(t))\}-\frac{a_1(t)\exp\{v(t-\sigma_1(t))\}}{\exp\{u(t-\tau_1(t))\}+k_1(t)}\Big]\\
\dot v(t)&=\lambda\Big[r_2(t)-\frac{a_2(t)\exp\{v(t-\sigma_2(t))\}}{\exp\{u(t-\tau_2(t))\}+k_2(t)}\Big]
\end{aligned}\right\}\\
\text{for } t\neq t_k,\ k\in\mathbb{Z}_+,\text{ and}\\
\left.\begin{aligned}
u(t_k^+)&=\lambda e_k(u(t_k))+u(t_k^-)\\
v(t_k^+)&=\lambda f_k(v(t_k))+v(t_k^-)
\end{aligned}\right\}\quad t=t_k,\ k\in\mathbb{Z}_+.
\end{gathered}\tag{3.2}$$

Suppose $x=(u,v)$ is an ω-periodic solution to system (3.2). By integrating (3.2) over $[0,\omega]$ we obtain

$$\begin{gathered}
\begin{aligned}
&\int_0^\omega r_1(t)\,\mathrm{d}t+\sum_{k=1}^p e_k(u(t_k))\\
&=\int_0^\omega\Big[b_1(t)\exp\{u(t-\tau(t))\}+\frac{a_1(t)\exp\{v(t-\sigma_1(t))\}}{\exp\{u(t-\tau_1(t))\}+k_1(t)}\Big]\,\mathrm{d}t,
\end{aligned}\\
\int_0^\omega r_2(t)\,\mathrm{d}t+\sum_{k=1}^p f_k(v(t_k))=\int_0^\omega\frac{a_2(t)\exp\{v(t-\sigma_2(t))\}}{\exp\{u(t-\tau_2(t))\}+k_2(t)}\,\mathrm{d}t.
\end{gathered}\tag{3.3}$$

From (3.2) and (3.3), we have

$$\int_0^\omega |\dot u(t)|\,\mathrm{d}t < 2\int_0^\omega r_1(t)\,\mathrm{d}t + \sum_{k=1}^p e_k(u(t_k)), \tag{3.4}$$

$$\int_0^\omega |\dot v(t)|\,\mathrm{d}t < 2\int_0^\omega r_2(t)\,\mathrm{d}t + \sum_{k=1}^p f_k(v(t_k)). \tag{3.5}$$

Since $u(t), v(t) \in PC_\omega$, there exist $\xi_1, \xi_2, \eta_1, \eta_2 \in [0,\omega]$ such that

$$\begin{aligned} u(\xi_1) &= \min_{t\in[0,\omega]} u(t), \quad u(\eta_1) = \max_{t\in[0,\omega]} u(t),\\ v(\xi_2) &= \min_{t\in[0,\omega]} v(t), \quad v(\eta_2) = \max_{t\in[0,\omega]} v(t). \end{aligned} \tag{3.6}$$

Then by (3.3) and (3.6), we obtain

$$\begin{gathered} \overline{r_1} + \frac{1}{\omega}\sum_{k=1}^p e_k(u(t_k)) \ge \frac{1}{\omega}\int_0^\omega b_1(t)\mathrm{e}^{u(\xi_1)}\,\mathrm{d}t\,,\\ \overline{r_2} + \frac{1}{\omega}\sum_{k=1}^p f_k(v(t_k)) \le \frac{1}{\omega}\int_0^\omega \frac{a_2(t)\mathrm{e}^{v(\eta_2)}}{k_2(t)}\,\mathrm{d}t\,; \end{gathered}$$

that is,

$$u(\xi_1) \le \ln\big[\frac{\overline{r_1} + \frac{1}{\omega}\sum_{k=1}^p e_k(u(t_k))}{\overline{b_1}}\big] \le \ln\big[\frac{\overline{r_1} + \frac{1}{\omega}\ln\big(\prod_{k=1}^p(1+p_{1k})\big)}{\overline{b_1}}\big] =: H_1$$

and

$$v(\eta_2) \ge \ln\big[\frac{\overline{r_2} + \frac{1}{\omega}\sum_{k=1}^p f_k(v(t_k))}{\overline{\big(\frac{a_2}{k_2}\big)}}\big] \ge \ln\big[\frac{\overline{r_2} + \frac{1}{\omega}\ln\big(\prod_{k=1}^p(1+q_{2k})\big)}{\overline{\big(\frac{a_2}{k_2}\big)}}\big] =: H_2.$$

Hence

$$\begin{aligned} u(t) &\le u(\xi_1)\ + \int_0^\omega |\dot u(t)|\,\mathrm{d}t + \sum_{k=1}^p |e_k(u(t_k))|\\ &\le H_1 + 2\omega\overline{r_1} + \ln\big(\prod_{k=1}^p(1+p_{1k})\big) + \sum_{k=1}^p l_{1k} =: M_1 \end{aligned}$$

and

$$\begin{aligned} v(t) &\ge v(\eta_2) - \int_0^\omega |\dot v(t)|\,\mathrm{d}t - \sum_{k=1}^p |f_k(v(t_k))|\\ &\ge H_2 - 2\omega\overline{r_2} - \ln\big(\prod_{k=1}^p(1+p_{2k})\big) - \sum_{k=1}^p l_{2k} =: M_2. \end{aligned}$$

So we have

$$\overline{r_2} + \frac{1}{\omega}\sum_{k=1}^p f_k(v(t_k)) \ge \frac{1}{\omega}\int_0^\omega \frac{a_2(t)\mathrm{e}^{v(\xi_2)}}{k_2(t)+\mathrm{e}^{M_1}}\,\mathrm{d}t \ge \frac{1}{\omega}\int_0^\omega \frac{a_2(t)\mathrm{e}^{v(\xi_2)}}{k_2^M+\mathrm{e}^{M_1}}\,\mathrm{d}t\,;$$

that is,

$$v(\xi_2) \le \ln\Big(\frac{\big(k_2^M + \mathrm{e}^{M_1}\big)\big(\overline{r_2} + \frac{1}{\omega}\sum_{k=1}^p f_k(v(t_k))\big)}{\overline{a_2}}\Big)$$

$$\le \ln\Big(\frac{\big(k_2^M+\mathrm{e}^{M_1}\big)\big(\overline{r_2}+\frac{1}{\omega}\ln(\prod_{k=1}^p(1+p_{2k}))\big)}{\overline{a_2}}\Big)=: H_3.$$

Thus

$$\begin{aligned}
v(t)&\le v(\xi_2)+\int_0^\omega |\dot v(t)|\,\mathrm{d}t+\sum_{k=1}^p |f_k(v(t_k))|\\
&\le H_3+2\omega\overline{r_2}+\ln\big(\prod_{k=1}^p(1+p_{2k})\big)+\sum_{k=1}^p l_{2k}=: M_3.
\end{aligned}$$

Similarly, we have

$$\overline{r_1}+\frac{1}{\omega}\sum_{k=1}^p e_k(u(t_k))\le \frac{1}{\omega}\int_0^\omega b_1(t)\mathrm{e}^{u(\eta_1)}\,\mathrm{d}t+\frac{1}{\omega}\int_0^\omega \frac{a_1(t)\mathrm{e}^{M_3}}{k_1(t)}\,\mathrm{d}t;$$

that is,

$$\begin{aligned}
u(\eta_1)&\ge \ln\Big[\frac{\overline{r_1}+\frac{1}{\omega}\sum_{k=1}^p e_k(v(t_k))-\mathrm{e}^{M_3}\overline{\big(\frac{a_1}{k_1}\big)}}{\overline{b_1}}\Big]\\
&\ge \ln\Big[\frac{\overline{r_1}+\frac{1}{\omega}\ln\big(\prod_{k=1}^p(1+q_{1k})\big)-\mathrm{e}^{M_3}\overline{\big(\frac{a_1}{k_1}\big)}}{\overline{b_1}}\Big]=: H_4.
\end{aligned}$$

Then

$$\begin{aligned}
u(t)&\ge u(\eta_1)-\int_0^\omega |\dot u(t)|\,\mathrm{d}t-\sum_{k=1}^p |e_k(u(t_k))|\\
&\ge H_4-2\omega\overline{r_1}-\ln\big(\prod_{k=1}^p(1+p_{1k})\big)-\sum_{k=1}^p l_{1k}=: M_4.
\end{aligned}$$

Now, we have

$$M_4\le u(t)\le M_1,\quad M_2\le v(t)\le M_3.$$

Let $D=|M_1|+|M_2|+|M_3|+|M_4|$. We have

$$\|x\|_X=\|u\|_0+\|v\|_0\le D.$$

Clearly, D is independent of λ. Denote $M=D+D_0$, where D_0 is taken sufficiently large such that each solution (u^*,v^*) of

$$\begin{gathered}
\int_0^\omega r_1(t)\,\mathrm{d}t+\sum_{k=1}^p e_k(u(t_k))=\int_0^\omega \Big[b_1(t)\mathrm{e}^u+\frac{a_1(t)\mathrm{e}^v}{\mathrm{e}^u+k_1(t)}\Big]\,\mathrm{d}t,\\
\int_0^\omega r_2(t)\,\mathrm{d}t+\sum_{k=1}^p f_k(v(t_k))=\int_0^\omega \frac{a_2(t)\mathrm{e}^v}{\mathrm{e}^u+k_2(t)}\,\mathrm{d}t
\end{gathered}\tag{3.7}$$

satisfies $\|(u^*,v^*)\|_X<D_0$, and we can obtain D_0 by repeating the above arguments. Then $\|(u^*,v^*)\|_X<M$.

Let $\Omega=\big\{x=(u,v)\in X,\ \|x\|_X<M\big\}$, which satisfies condition (a) of Lemma 2.3.

When $x\in\partial\Omega\cap\ker L=\partial\Omega\cap R^2$, x is a constant vector in R^2 with $\|x\|_X=M$. Then

$$QNx=\Big(\frac{1}{\omega}\Big[\int_0^\omega\Big(r_1(t)-b_1(t)\exp\{u\}-\frac{a_1(t)\exp\{v\}}{\exp\{u\}+k_1(t)}\Big)\,\mathrm{d}t+\sum_{k=1}^p e_k(u(t_k))\Big],$$

$$\frac{1}{\omega}\Big[\int_0^\omega\Big(r_2(t)-\frac{a_2(t)\exp\{v\}}{\exp\{u\}+k_2(t)}\Big)\,\mathrm{d}t+\sum_{k=1}^{p}f_k(v(t_k))\Big],\underbrace{0,\dots,0}_{2p}\Big)$$
$$\neq 0,$$

which shows that condition (b) in Lemma 2.3 holds.

Finally, we prove that condition (c) in Lemma 2.3 is satisfied. The isomorphism J of $\operatorname{Im} Q$ onto $\ker L$ can be defined by

$$J:\operatorname{Im} Q\to X,\quad (f,g,c_1,\dots,c_p,d_1,\dots,d_p)\to(f,g).$$

For $x\in\ker L\cap\Omega$, we have

$$\begin{aligned}
JQNx=&\Big(\frac{1}{\omega}\Big[\int_0^\omega\Big(r_1(t)-b_1(t)\exp\{u\}-\frac{a_1(t)\exp\{v\}}{\exp\{u\}+k_1(t)}\Big)\,\mathrm{d}t+\sum_{k=1}^{p}e_k(u(t_k))\Big],\\
&\frac{1}{\omega}\Big[\int_0^\omega\Big(r_2(t)-\frac{a_2(t)\exp\{v\}}{\exp\{u\}+k_2(t)}\Big)\,\mathrm{d}t+\sum_{k=1}^{p}f_k(v(t_k))\Big]\Big)\\
=&\Big(\overline{r_1}-\overline{b_1}\mathrm{e}^u-\frac{\mathrm{e}^v}{\omega}\int_0^\omega\frac{a_1(t)}{\mathrm{e}^u+k_1(t)}\,\mathrm{d}t+\frac{1}{\omega}\sum_{k=1}^{p}e_k(u(t_k)),\\
&\overline{r_2}-\frac{\mathrm{e}^v}{\omega}\int_0^\omega\frac{a_2(t)}{\mathrm{e}^u+k_2(t)}\,\mathrm{d}t+\frac{1}{\omega}\sum_{k=1}^{p}f_k(v(t_k))\Big).
\end{aligned}$$

Denote $\varphi:\operatorname{Dom} L\times[0,1]\to X$ as the form

$$\begin{aligned}
\varphi(u,v,\mu)=&\Big(\overline{r_1}-\overline{b_1}e^u,\ \overline{r_2}-\frac{\mathrm{e}^v}{\omega}\int_0^\omega\frac{a_2(t)}{\mathrm{e}^u+k_2(t)}\,\mathrm{d}t\Big)\\
&+\mu\Big(-\frac{\mathrm{e}^v}{\omega}\int_0^\omega\frac{a_1(t)}{\mathrm{e}^u+k_1(t)}\,\mathrm{d}t+\frac{1}{\omega}\sum_{k=1}^{p}e_k(u(t_k)),\ \frac{1}{\omega}\sum_{k=1}^{p}f_k(v(t_k))\Big),
\end{aligned}$$

where $\mu\in[0,1]$ is a parameter. With the mapping φ, we have $\varphi(u,v,\mu)\neq 0$ for $(u,v)\in\partial\Omega\cap\ker L$. Otherwise, there exists a constant vector (u,v) with $\|(u,v)\|_X=M$ implies $\varphi(u,v,\mu)=0$; i.e.,

$$\overline{r_1}-\overline{b_1}\mathrm{e}^u-\frac{\mu\mathrm{e}^v}{\omega}\int_0^\omega\frac{a_1(t)}{\mathrm{e}^u+k_1(t)}\,\mathrm{d}t+\frac{\mu}{\omega}\sum_{k=1}^{p}e_k(u(t_k))=0$$

and

$$\overline{r_2}-\frac{\mathrm{e}^v}{\omega}\int_0^\omega\frac{a_2(t)}{\mathrm{e}^u+k_2(t)}\,\mathrm{d}t+\frac{\mu}{\omega}\sum_{k=1}^{p}f_k(v(t_k))=0.$$

Similar to the above discussion, we know that $\|(u,v)\|_X<M$, which contradicts $\|(u,v)\|_X=M$. From the property of coincidence degree theory, we can obtain

$$\begin{aligned}
\deg(JQNx,\Omega\cap\ker L,0)&=\deg(\varphi(u,v,1),\Omega\cap\ker L,0)\\
&=\deg(\varphi(u,v,0),\Omega\cap\ker L,0).
\end{aligned}$$

Obviously, the following algebraic equation has a unique solution (u^*,v^*)

$$\begin{gathered}
\overline{r_1}-\overline{b_1}\mathrm{e}^u=0,\\
\overline{r_2}-\frac{\mathrm{e}^v}{\omega}\int_0^\omega\frac{a_2(t)}{\mathrm{e}^u+k_2(t)}\,\mathrm{d}t=0.
\end{gathered}$$

So

$$\deg(JQNx, \Omega \cap \ker L, 0) = \deg(\varphi(u, v, 0), \Omega \cap \ker L, 0) = 1 \neq 0.$$

By Lemma 2.3, the system (1.1) has at least one ω-periodic solution in Ω. The proof is complete. $\square$

4. An Example

Consider the system

$$\left.\begin{aligned} \dot{x}(t) = x(t)\Big(r_1 + \sin 2\pi t - b_1 x(t-\tau(t)) - \frac{a_1(1+\theta\cos 2\pi t)y(t-\sigma_1(t))}{x(t-\tau_1(t))+k_1}\Big) \\ \dot{y}(t) = y(t)\Big(r_2 - \frac{a_2(1+\theta\cos 2\pi t)y(t-\sigma_2(t))}{x(t-\tau_2(t))+k_2}\Big) \\ \text{for } t \neq t_k, k \in \mathbb{Z}_+, \text{ and} \\ \left.\begin{aligned} x(t_k^+) &= (1-h)x(t_k^-) \\ y(t_k^+) &= (1-g)y(t_k^-) \end{aligned}\right\} \quad t = t_k, k \in \mathbb{Z}_+, \end{aligned}\right\} \tag{4.1}$$

where $r_1 = 1.1$, $r_2 = 1.25$, $b_1 = 0.6$, $a_1 = 0.05$, $a_2 = 0.7$, $k_1 = k_2 = 1$, $h = 0.5$, $g = 0.7$, $\theta = 0.5$. Obviously, in this case, $\omega = 1, p = 1$ and

$$H_1 < -0.38, \quad M_1 < 1.82, \quad H_3 < -0.66, \quad M_3 < 1.84,$$
$$0.40 < r_1 + \ln(1-h) < 0.41, \quad 0.04 < r_2 + \ln(1-g) < 0.05,$$
$$r_1 + \ln(1-h) - \mathrm{e}^{M_3}\frac{a_1}{k_1} > 0.40 - 0.32 = 0.08 > 0.$$

So that, all conditions of Theorem 3.1 are satisfied. Therefore, (4.1) has at least one ω-periodic positive solution.

References

[1] Z. Amine, R. Ortega, A periodic prey-predator system, J. Math. Anal. Appl. 185 (1994) 477-489.

[2] D. D. Bainov, P. S. Simeonov; *Impulsive Differential Equations: Periodic Solutions and Applications*, Longman Scientific and Technical, New York, 1993.

[3] F. D. Chen, J. L. Shi; *On a delayed nonautonomous ratio-dependent predator-prey model with Holling type functional response and diffusion*, Appl. Math. Comput. 192(2)(15) (2007) 358-369.

[4] A. D'Onofrio, Pulse vaccination strategy in the SIR epidemic model: global asymptotic stable eradication in presence of vaccine failures, Math. Comput. Modell. 36 (2002) 473-489.

[5] Y. H. Fan, W. T. Li; *Permanence in delayed ratio-dependent predator-prey models with monotonic functional responses*, Nonlinear Anal.: Real World Appl. 8(2)(2007) 424-434.

[6] R. E. Gaines, J. L. Mawhin; *Coincidence Degree and Nonlinear Differential Equations*, Springer-Verlag, Berlin, 1977.

[7] H. F. Huo, W. T. Li, J. J. Nieto; *Periodic solutions of delayed predator-prey model with the Beddington-DeAngelis functional response*, Chaos, Solitons & Fractals, 33(2)(2007) 505-512.

[8] A. Lakmeche, O. Arino, Nonlinear mathematical model of pulsed therapy of heterogeneous tumor, Nonlinear Anal. 2 (2001) 455-465.

[9] Y. K. Li, Y. Kuang; *Periodic Solutions of Periodic Delay Lotka-Volterra Equations and Systems*, J. Math. Anal. Appl. 255(2001) 260-280.

[10] Y. K. Li; *Periodic solutions of a periodic delay predator-prey system*, Proc. Amer. Math. Soc. 127(1999)1331-1335.

[11] B. Liu, Y. J. Zhang, L. S. Chen; *The dynamical behaviors of a Lotka-Volterra predator-prey model concerning integrated pest management*, Nonlinear Analysis: Real World Applications 6(2)(2005) 227-243.

[12] J. López-Gómez, R. Ortega, A. Tineo, The periodic predator-prey Lotka-Volterra model, Adv. Diff. Equations 1 (1996) 403-423.
[13] Y. Saito, T. Hara and W. Ma, Necessary and sufficient conditions for permanence and global stability of a Lotka-Volterra system with two delays, J. Math. Anal. Appl. 236 (1999) 534-556.
[14] B. Shulgin, L. Stone, Z. Agur, Theoretical examination of pulse vaccination policy in the SIR epidemic model, Math. Comput. Model. 31 (4/5) (2000) 207-215.
[15] X. Song, Y. Li, Dynamic behaviors of the periodic predator-prey model with modified Leslie-Gower Holling-type II schemes and impulsive effect Nonlinear Anal.: Real World Appl. 9(1)(2008) 64-79.
[16] S.Y. Tang, L.S. Chen, The periodic predator-prey Lotka-Volterra model with impulsive effect, J. Mech. Med. Biol. 2 (2002) 1-30.
[17] X. Y. Zhou, X. Y. Shi, X. Y. Song; *Analysis of nonautonomous predator-prey model with nonlinear diffusion and time delay*, Appl. Math. Comput. 196(1)(15) (2008) 129-136.

Shan Gao
Department of Mathematics, Yunnan University, Kunming, Yunnan, 650091, China
E-mail address: 2002711036@163.com

Yongkun Li
Department of Mathematics, Yunnan University, Kunming, Yunnan, 650091, China
E-mail address: yklie@ynu.edu.cn