Electronic Journal of Differential Equations, Vol. 2008(2008), No. 91, pp. 1–19.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

STRONG SOLUTIONS FOR SOME NONLINEAR PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

MOHAMED ALIA, KHALIL EZZINBI

ABSTRACT. In this work, we use the Kato approximation to prove the existence of strong solutions for partial functional differential equations with infinite delay. We assume that the undelayed part is m-accretive in Banach space and the delayed part is Lipschitz continuous. The phase space is axiomatically defined. Firstly, we show the existence of the mild solution in the sense of Evans. Secondly, when the Banach space has the Radon-Nikodym property, we prove the existence of strong solutions. Some applications are given for parabolic and hyperbolic equations with delay. The results of this work are extensions of the Kato-approximation results of Kartsatos and Parrot [8, 9].

1. Introduction

In this work, we study the existence and the regularity of solutions for the following partial functional differential equation with infinite delay

$$\begin{gathered} u'(t) + Au(t) \ni F(u_t) \quad \text{for } t \geq 0 \\ u_0 = \phi \in \mathcal{B}, \end{gathered} \tag{1.1}$$

where A is a nonlinear multivalued operator with domain $D(A)$ in a Banach space X, $\mathcal{B}$ is the space of functions defined on $]-\infty,0]$ with values in X, satisfying the Hale and Kato's assumptions [6]. For $t \geq 0$, the history function $u_t \in \mathcal{B}$ is defined by

$$u_t(\theta) = u(t+\theta) \quad \text{for } \theta \in]-\infty, 0],$$

$F : \mathcal{B} \to X$ is a continuous function. Note that the difference between the finite and infinite delay lies in the fact that in general the function

$$t \to u_t \tag{1.2}$$

is not continuous from $[0,T]$ into $\mathcal{B}$. In finite delay, usually the phase space is $C([-r,0];X)$ the space of continuous functions from $[-r,0]$ to X, consequently the history function (1.2) is continuous. The main problem of differential equations involving infinite delay is the choice of the phase space for which the history function

2000 *Mathematics Subject Classification.* 34K30, 37L05, 47H06, 47H20.
Key words and phrases. Partial functional differential equations; infinite delay; m-accretive operator; Kato approximation; mild solution in the sense of Evans; strong solution; Radon-Nikodym property.

Submitted October 25, 2007. Published June 21, 2008.

(1.2) is continuous. For more details, about this topics we refer to Hale and Kato [6] and Hino, Murakami and Naito [7]. In [10], Kato proposed a new approach to prove the existence of solution for the evolution equation

$$\begin{gathered} x'(t) + Ax(t) = 0 \\ x(0) = x_0 \end{gathered} \tag{1.3}$$

where A is m-accretive in a Banach space X such that the dual X^* is uniformly convex. The author proposed the following approximation which called the Kato approximation

$$\begin{gathered} x_n'(t) + A_n x_n(t) = 0 \\ x_n(0) = x_0 \end{gathered} \tag{1.4}$$

where A_n is the Yosida approximation of A to show the existence of solutions for equation (1.3).

Kartsatos and Parrott [8] employed the Kato approximation to prove the existence of strong solutions for the partial functional differential equation

$$\begin{gathered} u'(t) + B(t)u(t) = F(u_t) \quad \text{for } t \geq 0 \\ u_0 = \varphi \in C([-r, 0]; X), \end{gathered} \tag{1.5}$$

where $B(t)$ is m-accretive on X, the authors proved, the existence of strong solution if the dual space X^* is uniformly convex. In [9], Kartsatos and Parrott considered equation (1.5) in general Banach space and proved the existence of a Lipschitz mild solution which becomes a strong solution when the phase space X is reflexive. In [11], Ruess studied the existence of solutions for the following multivalued partial functional differential equation

$$\begin{gathered} u'(t) + B(t)u(t) \ni G(t, x_t) \quad \text{for } t \geq 0 \\ u_0 = \varphi \in C([-r, 0]; X) \quad \text{or} \quad \varphi \in BUC((-\infty, 0]; X), \end{gathered}$$

where $BUC((-\infty, 0]; X)$ is the space of bounded uniformly continuous functions from $(-\infty, 0]$ to X, for every $t \geq 0$, the operator $B(t)$ is m-accretive in a Banach space X, the authors proved the existence of strong solutions when X is reflexive and its norm is differentiable at any $x \neq 0$. In [12], Ruess studied also the existence of solution for the following equation

$$\begin{gathered} u'(t) + \alpha u(t) + Bu(t) \ni G(x_t) \quad \text{for } t \geq 0 \\ u_0 = \varphi \in \mathcal{M}, \end{gathered} \tag{1.6}$$

where the phase space $\mathcal{M} = C([-r, 0]; X)$ or $\in \mathcal{B}$, $\alpha \in \mathbb{R}$ and B is m-accretive operator, $G : \mathcal{M} \to X$ is Lipschitz continuous, the authors proved the existence of strong solution of equation (1.6) if one of the following conditions holds:

(a) X is reflexive and its norm is differentiable at any $x \neq 0$ and $\varphi \in \hat{D}(\mathcal{A})$, where $\hat{D}(\mathcal{A})$ denotes the generalized domain of the operator

$$\begin{gathered} D(\mathcal{A}) = \{\varphi \in \mathcal{M} : \varphi' \in \mathcal{M},\ \varphi(0) \in D(B),\ \varphi'(0) \in G(\varphi) - \alpha\varphi(0) - B\varphi(0)\} \\ \mathcal{A}\varphi = -\varphi'. \end{gathered}$$

(b) X has the Radon-Nikodym property, $D(B)$ is closed, B is single valued with $B : D(B) \to X$ norm weakly continuous and $\varphi \in \hat{D}(\mathcal{A})$.

(c) X is any Banach space, $D(B)$ is closed, B is single valued with $B : D(B) \to X$ is continuous and either:
(c1) $\varphi \in \hat{D}(\mathcal{A})$
(c2) $\varphi \in \overline{D(\mathcal{A})}$ and B maps bounded sets into bounded sets.

(d) X is reflexive, $B : D(B) \to X$ is single valued and demiclosed, namely, the graph of B is norm-weakly closed in $X \times X$ and $\varphi \in \hat{D}(\mathcal{A})$.

More details can be found in the book K. S. Ha [13] where an overview on nonlinear theory of partial functional differential equations is given.

Travis and Webb [14] gave the basic theory on the existence and stability of equation (1.1) when $-A$ is linear, densely defined and satisfies the Hille-Yosida condition, more results and applications can be found in the book Wu [15]. Adimy, Bouzahir and Ezzinbi [1] gave the basic theory of the existence, regularity and stability of solution of equation (1.1) when $-A$ is a linear operator, not necessarily densely defined and satisfies the well known Hille-Yosida condition, by renorming the space X, the Hille-Yosida condition is equivalent to say that A is m-accretive, in this work, the authors investigated several results on the existence of solutions and stability by using the integrated semi-group theory. Here we propose to extend the works of Kartsatos and Parrott [8], [9] and Ruess [12]. To simplify our analysis, we consider the case where A is time-independent, but the same approach still works in general context. Here we use the Kato approximation to show the existence of strong solutions in Banach spaces that have the Radon-Nikodym property. The study of the existence of strong solutions requires some hypotheses about regularity of the space X and the initial data φ. More precisely, we propose the Kato approximation

$$\begin{gathered} u_n'(t) + A_n u_n(t) = F(u_{nt}) \quad \text{for } t \geq 0, \\ u_{n0} = \varphi \in \mathcal{B}, \end{gathered} \tag{1.7}$$

where A_n is the Yosida approximation of A. Our aim is to prove that the solution u_n converges uniformly on $[0, T]$ to the mild solution of equation (1.1). The advantage of this approximation is the fact that the right hand side of equation (1.7) is a Lipschitz continuous, consequently the solutions of equation (1.7) are C^1-functions on $[0, T]$.

This work is organized as follows: In section 2, we recall some results on the existence of strong solution for evolution problem involving m-accretive operators. In section 3, we prove the existence of mild and strong solutions for equation (1.1). Finally, for illustration, we propose to show the existence of solutions for some partial differential equation with delay.

2. Preliminary results

In this section we recall some preliminary results on m-accretive operators and some results on the phase space that will be used in the whole of this work. Let X be a Banach space and $A : X \to 2^X$ be an operator on X with domain defined by

$$D(A) = \{x \in X : Ax \text{ is non empty in } X\}.$$

We say that $(x, y) \in A$ if $x \in D(A)$ and $y \in Ax$.

Definition 2.1. A is said to be accretive if for $\lambda > 0$, $(x_1, y_1) \in A$ and $(x_2, y_2) \in A$ we have

$$|x_1 - x_2| \leq |x_1 - x_2 + \lambda(y_1 - y_2)|.$$

Proposition 2.2 ([5]). *If A is an accretive operator, then for all $\lambda > 0$, $I + \lambda A$ is a bijection from $D(A)$ into $R(I + \lambda A)$. Moreover, $(I + \lambda A)^{-1}$ is nonexpansive on $R(I + \lambda A)$.*

Definition 2.3. Let $A : D(A) \subset X \to 2^X$. Then A is said to be m-accretive if A is accretive and for some $\lambda > 0$, we have

$$R(I + \lambda A) = X.$$

Remark 2.4. If A is m-accretive, then for all $\lambda > 0$, we have $R(I + \lambda A) = X$.

Definition 2.5. The duality mapping $J : X \to 2^{X^*}$ is defined by

$$J(x) = \{x^* \in X^* :< x^*, x >= |x^*|^2 = |x|^2\}.$$

By the Hahn-Banach Theorem, $J(x)$ is a non empty set for all $x \in X$. For a general Banach space, the duality mapping J is multi-valued. If the dual X^* is strictly convex, J is single-valued. Moreover, if X^* is uniformly convex, then J is uniformly continuous on bounded sets.

Definition 2.6. For every $(x, y) \in X$, we define the bracket $[.,.]$ by

$$[x, y] = \lim_{h \to 0} \frac{|x + hy| - |x|}{h}.$$

The following results are well known.

Proposition 2.7 ([5]). *Let $x, y, z \in X$ and $\alpha, \beta \in \mathbb{R}$. Then the following statements hold:*

- (i) $[\alpha x, \beta y] = |\beta|[x, y]$ *for* $\alpha\beta > 0$.
- (ii) $[x, \alpha x + y] = \alpha|x| + [x, y]$.
- (iii) $[x, y] \geq 0$ *if and only if* $|x + hy| \geq |x|$ *for* $h \geq 0$.
- (iv) $|[x, y]| \leq |y|$.
- (v) $[x, y + z] \leq [x, y] + [x, z]$.
- (vi) $[x, y] \geq -[x, -y]$.
- (vii) $[x, y] = \max_{x^* \in \frac{1}{|x|} J(x)} \langle x^*, y\rangle$ *for* $x \neq 0$.
- (viii) *Let u be a function from a real interval J to X such that $u'(t_0)$ exists for an interior point t_0 of J. Then $D_+|u(t_0)|$ exists and*
$$D_+|u(t_0)| = [u(t_0), u'(t_0)],$$
where $D_+|u(t_0)|$ denotes the right derivative of $|u(t)|$ at t_0.

Proposition 2.8 ([8]). *Let $A : X \to 2^X$ be an operator in X. Then the following statements are equivalent*

- (i) *A is accretive,*
- (ii) *$(I + \lambda A)^{-1}$ is nonexpansive on $R(I + \lambda A)$,*
- (iii) $[x_1 - x_2, y_1 - y_2] \geq 0$ *for any* $(x_1, y_1), (x_2, y_2) \in A$,
- (iv) *for all $(x_1, y_1), (x_2, y_2) \in A$, there exists $x^* \in J(x_1 - x_2)$ such that*
$$< x^*, y_1 - y_2 >\geq 0.$$

Consider the Cauchy problem

$$\begin{gathered} u'(t) + Au(t) \ni f(t) \quad \text{for } t \in [0, T] \\ u(0) = u_0. \end{gathered} \tag{2.1}$$

Definition 2.9. A function $u : [0,T] \to X$ is said to be a strong solution of (2.1) if

(i) u is absolutely continuous on $[0,T]$.
(ii) u is differentiable on $[0,T]$ almost everywhere.
(iii) $u'(t) + Au(t) \ni f(t)$ for a.e. $t \in [0,T]$.
(iv) $u(0) = u_0$.

Definition 2.10. [5] For a given $\varepsilon > 0$, a partition $t_0 < t_1 < \dots < t_n$ of $[0, t_n]$, and a finite sequence $f_0, f_1, \dots, f_n$ in X, the equation

$$\frac{u_k - u_{k-1}}{t_k - t_{k-1}} + Au_k \ni f_k \quad \text{for } k = 1, 2, \dots, n.$$

is called a ε-discretization of $u'(t) + Au(t) \ni f(t)$, on $[0,T]$ if,

$$0 \le t_0 \le \varepsilon, \quad 0 \le T - t_n < \varepsilon, \quad t_k - t_{k-1} < \varepsilon, \quad \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \|f(\tau) - f_k\| d\tau < \varepsilon.$$

Moreover, the step function

$$u_\varepsilon(t) = \begin{cases} u_0 & \text{for } t = 0 \\ u_k & \text{for } t \in]t_{k-1}, t_k] \end{cases}$$

is called ε-solution of this discretization.

Definition 2.11 ([5]). A continuous function $u : [0,T] \to X$ satisfying $u(0) = u_0$ is called a mild solution (in the sense of Evans) of equation (2.1), if, for all $\varepsilon > 0$ there exists u_ε an $\varepsilon-$solution of an ε-discretization on $[0,T]$ such that

$$|u(t) - u_\varepsilon(t)| < \varepsilon \quad \text{for } t \in [0,T].$$

Proposition 2.12 ([5]). *If A is accretive, then the following results hold*

(i) *the mild solution of equation* (2.1) *if it exists, is unique.*
(ii) *If u is a strong solution of equation* (2.1)*, then u is a mild solution.*

Theorem 2.13 ([5]). *Let A be a m-accretive operator and $f \in L^1(0,T;X)$. Suppose that $u_0 \in \overline{D(A)}$, then equation* (2.1) *has a unique mild solution.*

Theorem 2.14 ([3, p.102]). *Let A be an m-accretive operator on X and take $f \in L^1(0,T;X)$, then the function u is a strong solution of equation* (2.1) *if and only if u is a mild solution which is absolutely continuous and almost everywhere differentiable on* $[0,T]$.

Definition 2.15. A Banach space X is said to have the Radon-Nikodym property if and only if every absolutely continuous function $g : [a,b] \to X$ is almost everywhere differentiable.

Definition 2.16 ([5, p.194]). The generalized domain $\hat{D}(A)$ of A is defined by

$$\hat{D}(A) = \{x \in X : |x|_A = \lim_{\lambda \to 0} |A_\lambda x| < \infty\}.$$

Proposition 2.17. *Let $A : X \to 2^X$ be m-accretive operator in X. Then $D(A) \subset \hat{D}(A) \subset \overline{D(A)}$.*

As a consequence of [5, Theorem 5], we deduce the following result.

Theorem 2.18 ([5]). *Assume that A is m-accretive, $f \in C([0,T];X)$ and $u_0 \in \overline{D(A)}$. If X has the Radon-Nikodym, then every absolutely continuous mild solution of* (2.1) *becomes a strong solution of* (2.1).

Definition 2.19 ([3, p.32]). Let $A_n : D(A_n) \subset X \to 2^X$ be a sequence of multi-valued operators on X. We define the $\liminf\limits_{n\to+\infty} A_n$ by the operator $A_\infty : D(A_\infty) \subset X \to 2^X$ such that

$y_\infty \in A_\infty x_\infty$ if and only if there exist $x_n \in D(A_n)$ and $y_n \in A_n x_n$ such that $x_n \to x_\infty$ and $y_n \to y_\infty$ as $n \to +\infty$.

For $\lambda > 0$, we define the resolvent of A by

$$J_\lambda = (I + \lambda A)^{-1}.$$

The Yosida approximation of A is defined for $\lambda > 0$ by

$$A_\lambda = \frac{1}{\lambda}(I - J_\lambda).$$

Proposition 2.20 ([10]). *If A is an accretive operator, then for $\lambda > 0$, the following statements hold*

(i) *A_λ is accretive and if A is m-accretive, so is A_λ.*
(ii) *A_λ is a Lipschitz mapping on $R(I + \lambda A)$ with coefficient $\frac{2}{\lambda}$.*

Theorem 2.21 ([3, p.164]). *Let A be a m-accretive operator on X, then*

$$A = \liminf_{\lambda \to 0^+} A_\lambda.$$

where A_λ is the Yosida approximation of A.

Theorem 2.22 ([3, p.159]). *Let $T > 0$, $\omega \in \mathbb{R}$, $(A_n + \omega I)_{n\ge 1}$ be a sequence of m-accretive operators, $x_n \in \overline{D(A_n)}$ and $f_n \in L^1(0,T;X)$ for $n \ge 1$. Let u_n be the mild solution of*

$$\begin{gathered} u_n'(t) + A_n u_n(t) \ni f_n(t) \quad \text{for } t \in [0,T] \\ u_n(0) = x_n. \end{gathered} \tag{2.2}$$

If $f_n \to f_\infty$ in $L^1(0,T;X)$, $x_n \to x_\infty$ and $A_\infty = \liminf_{n\to+\infty} A_n$, then

$$\lim_{n\to+\infty} u_n(t) = u_\infty(t) \quad \text{uniformly on } [0,T],$$

where u_∞ is the mild solution of the equation

$$\begin{gathered} u_\infty'(t) + A_\infty u_\infty(t) \ni f_\infty(t) \quad \text{for } t \in [0,T] \\ u_\infty(0) = x_\infty. \end{gathered}$$

Proposition 2.23 ([3, p.90]). *Let A be such that $A + \omega I$ is m-accretive for some $\omega \in \mathbb{R}$. Let f, g be two functions in $L^1(0,T;X)$. If u_1 and u_2 are respectively mild solutions of $u'(t) + Au(t) \ni f(t)$ and $v'(t) + Av(t) \ni g(t)$ for $t \in [0,T]$. Then for $0 \le s \le t \le T$, the following estimate holds*

$$|u_1(t) - u_2(t)| \le e^{\omega(t-s)}|u_1(s) - u_2(s)| + \int_s^t e^{\omega(t-\tau)}|f(\tau) - g(\tau)|d\tau.$$

In the following, we assume that the phase space $\mathcal{B}$ satisfies the the following assumptions which were introduced by Hale and Kato [6]:

(A1) There exist constant $H > 0$ and functions $K, M : \mathbb{R}^+ \to \mathbb{R}^+$ with K continuous and $M \in L^\infty_{\rm loc}(\mathbb{R}^+)$ such that for all $\sigma \in \mathbb{R}$ and for any $a > 0$ if $x : (-\infty, \sigma + a] \to X$ is such that $x_\sigma \in \mathcal{B}$ and $x :[\sigma, \sigma + a] \to X$ is continuous, then for all $t \in [\sigma, \sigma + a]$ we have
  (i) $x_t \in \mathcal{B}$
  (ii) $|x(t)| \le H|x_t|_{\mathcal{B}}$ (in other words $|\varphi(0)| \le H|\varphi|_{\mathcal{B}}$, for any $\varphi \in \mathcal{B}$),
  (iii) $|x_t|_{\mathcal{B}} \le K(t - \sigma) \sup_{\sigma \le s \le t} |x(t)| + M(t - \sigma)|x_\sigma|_{\mathcal{B}}$.

(A2) The function $t \to x_t$ is continuous from $[\sigma, \sigma + a]$ to $\mathcal{B}$.

(B) $\mathcal{B}$ is complete.

Let C_{00} be the space of continuous functions from $(-\infty, 0]$ into X with compact supports. In the sequel we suppose that $\mathcal{B}$ satisfies

(C) If a uniformly bounded sequence $(\varphi_n)_{n\ge0}$ in C_{00} converges compactly to a function φ in $(-\infty, 0]$, then $\varphi \in \mathcal{B}$ and $|\varphi_n - \varphi|_{\mathcal{B}} \to 0$ as $n \to +\infty$.

Let $B_0 = \{\varphi \in \mathcal{B} : \varphi(0) = 0\}$. Consider the family of the linear operators defined on B_0 by

$$(S_0(t)\varphi)(\theta) = \begin{cases} 0 & \text{if } -t \le \theta \le 0. \\ \varphi(t + \theta) & \text{if } \theta < -t. \end{cases}$$

Then $(S_0(t))_{t\ge0}$ defines a strongly continuous semigroup on B_0.

Definition 2.24 ([7])**.** We say that $\mathcal{B}$ is a fading memory space if
(i) $\mathcal{B}$ satisfies assumption (C),
(ii) $|S_0(t)\varphi|_{\mathcal{B}} \to 0$ as $t \to +\infty$ for all $\varphi \in \mathcal{B}$.

Let $BC(]-\infty, 0]; X)$ be the space of bounded continuous functions with values in X endowed with the supremum norm. Then we have the following interesting result.

Proposition 2.25 ([7])**.** *If $\mathcal{B}$ is a fading memory space, then $BC(-\infty, 0]; X)$ is continuously embedded in $\mathcal{B}$; namely, there exists a constant $c > 0$ such that*

$$|\varphi|_{\mathcal{B}} \le c|\varphi|_{\rm BC} \quad \textit{for all } \varphi \in BC((-\infty, 0]; X).$$

3. Mild and strong solution of (1.1)

Definition 3.1 (In the sense of Evans)**.** A function $u : (-\infty, +\infty) \to X$ is said to be a mild solution of equation (1.1) if:
(i) $u_0 = \phi$
(ii) u is mild solution in the sense of Evans of the equation

$$u'(t) + Au(t) \ni f(t) \quad \text{for } t \ge 0$$

where $f(t) = F(u_t)$ for $t \ge 0$.

Definition 3.2. A function $u : (-\infty, T] \to X$ is said to be a strong solution of equation (1.1) if:
(i) $u_0 = \phi$
(ii) u is absolutely continuous
(iii) u is almost everywhere differentiable on $[0, T]$ and

$$u'(t) + Au(t) \ni F(u_t) \quad \text{for a.e. } t \in [0, T].$$

Firstly, we prove the existence of the mild solution. For this goal, we assume:

(H1) $(A + \omega I)$ is m-accretive for some $\omega \in \mathbb{R}$.

(H2) There exists a constant $L > 0$ such that
$$|F(\phi) - F(\psi)| \leq L|\phi - \psi|_{\mathcal{B}} \quad \text{for } \phi, \psi \in \mathcal{B}.$$

Theorem 3.3. *Assume that* (H1), (H2) *hold. Let $\phi \in \mathcal{B}$ be such that $\phi(0) \in \overline{D(A)}$. Then equation* (1.1) *has a unique mild solution defined on* $[0, +\infty)$.

Proof. Without loss of generality we assume that $\omega = 0$. Let $T > 0$. Consider the set
$$Y = \{v : [0, T] \to X \text{ is continuous and } v(0) = \phi(0)\}.$$
For $v \in Y$, we consider the equation
$$\begin{gathered} u'(t) + Au(t) \ni F(\tilde{v}_t) \quad \text{for } t \in [0, T] \\ u(0) = \phi(0) \end{gathered} \tag{3.1}$$
where
$$\tilde{v} = \begin{cases} \phi & \text{on } (-\infty, 0] \\ v & \text{on } [0, T] \end{cases}$$
From assumption (A2) the mapping $t \mapsto \tilde{v}_t$ is continuous. Consequently, the mapping $t \mapsto F(\tilde{v}_t)$ is continuous.

In virtue of Theorem 2.13, equation (3.1) has a unique mild solution $u(v)$ on $[0, T]$. Let us now define the operator
$$\begin{gathered} \mathbb{K} : Y \to Y \\ v \to u(v) \end{gathered}$$
and show that $\mathbb{K}$ has an unique fixed point on Y. Notice that $\mathbb{K}$ is well defined and $\mathbb{K}(Y) \subset Y$.

Let v_1 and v_2 be in Y. Set $u_1 = \mathbb{K}(v_1)$ and $u_2 = \mathbb{K}(v_2)$. Then
$$\begin{gathered} u_1'(t) + Au_1(t) \ni F(\tilde{v}_{1_t}) \\ u_2'(t) + Au_2(t) \ni F(\tilde{v}_{2_t}). \end{gathered}$$
By Proposition 2.23, we deduce that
$$|u_1(t) - u_2(t)| \leq L \int_0^t |\tilde{v}_{1_s} - \tilde{v}_{2_s}|_{\mathcal{B}} ds.$$
From assumption (A1)(iii) and using the fact that $\tilde{v}_{1_0} = \tilde{v}_{2_0} = \phi$, we deduce that
$$\begin{aligned} |\tilde{v}_{1_s} - \tilde{v}_{2_s}|_{\mathcal{B}} &\leq K(s) \sup_{0 \leq \tau \leq s} |v_1(\tau) - v_2(\tau)| \\ &\leq K(s) \sup_{0 \leq \tau \leq T} |v_1(\tau) - v_2(\tau)|. \end{aligned}$$
Set
$$K_T = \sup_{t \in [0, T]} K(t).$$
Hence
$$|u_1(t) - u_2(t)| \leq K_T T \sup_{\tau \in [0, T]} |v_1(\tau) - v_2(\tau)|.$$
Thus
$$\sup_{t \in [0, T]} |u_1(t) - u_2(t)| \leq L K_T T \sup_{t \in [0, T]} |v_1(t) - v_2(t)|.$$
Finally for T appropriately small, $\mathbb{K}$ is strictly contractive. By the Banach fixed point theorem we have the existence and uniqueness of u which is a mild solution

of equation (1.1) on $[0, T]$. We proceed by steep and we can extend continuously the solution on $[0, T]$ for every $T > 0$. $\square$

As a consequence of Theorem 2.18, we deduce the following result.

Theorem 3.4. *Assume that X has the Radon-Nikodym property and u is a mild solution of equation equation* (1.1). *If u is lipschitz continuous on $[0, T]$, then u becomes a strong solution.*

For the regularity of the mild solution we suppose the following hypotheses:

(H3) X has Radon-Nikodym property.
(H4) $\mathcal{B}$ is a fading memory space.
(H5) $\phi \in C^1((-\infty, 0]; X) \cap \mathcal{B}$, $\phi' \in \mathcal{B}$ such that ϕ' is bounded and $\phi(0) \in \hat{D}(A)$.

Consider the Kato approximation

$$\begin{gathered} u_n'(t) + A_n u_n(t) = F(u_{n_t}) \quad \text{for } t \geq 0 \\ u_{n_0} = \phi \end{gathered} \tag{3.2}$$

where for $n \geq 1$,

$$J_n = (I + (\frac{1}{n})A)^{-1}$$

is the resolvent of A and $A_n = n(I - J_n)$ is the Yosida approximation of A.

Now, We state our main result of this work on the existence of strong solutions.

Theorem 3.5. *Assume that* (H1)–(H5) *hold. Then there exists a unique strong solution u of equation* (1.1) *on* $[0, +\infty)$ *such that*

$$u(t) = \lim_{n \to +\infty} u_n(t)$$

uniformly on each compact subset of $[0, +\infty)$*, where u_n is the solution of equation* (3.2). *Moreover, $u(t) \in \hat{D}(A)$ for $t \geq 0$.*

Let $T > 0$. The proof will be done in the following steps:

(i) The approximate equation (3.2) with second term $-A_n u_n(t) + F(u_{n_t})$ is Lipschitz with respect to the second variable. Hence by a fixed point argument we show that equation (3.2) has a unique solution u_n on $[0, T]$ which is of class C^1 on $[0, T]$.
(ii) We prove that u_n and u_n' are uniformly bounded on $[0, T]$.
(iii) We prove that the strong limit of u_n exists uniformly in $[0, T]$ as $n \to +\infty$ which is denoted by u .
(iv) We prove that u is a strong solution of equation (1.1).

Lemma 3.6. *Suppose that* (H1), (H2) *are satisfied and $\phi \in \mathcal{B}$ is such that $\phi(0) \in \hat{D}(A)$. Then for every $T > 0$, there exists $\varrho > 0$ such that $|u_n(t)| \leq \varrho$ for all n, and for $t \in [0, T]$.*

Proof. Let $a = \phi(0)$. Then

$$\begin{aligned} D_+|u_n(t) - a| &= [u_n(t) - a, u_n'(t)] \\ &= [u_n(t) - a, -A_n u_n(t) + F(u_{n_t})] \\ &= [u_n(t) - a, -A_n u_n(t) + A_n a - A_n a + F(u_{n_t}) - F(\phi) + F(\phi)] \\ &\leq [u_n(t) - a, -A_n u_n(t) + A_n a] + |A_n a| + |F(\phi)| + L|u_{n_t} - \phi|. \end{aligned}$$

Since A is m-accretive, it follows that $[u_n(t) - a, -A_n u_n(t) + A_n a] \leq 0$. Consequently,

$$D_+|u_n(t) - a| \leq |A_n a| + |F(\phi)| + L|u_{n_t} - \phi|.$$

Since $\phi(0) \in \hat{D}(A)$, $\sup_{n\geq 1} |A_n a| < \infty$; and consequently

$$D_+|u_n(t) - a| \leq k_1 + L|u_{n_t} - \phi|_{\mathcal{B}}, \tag{3.3}$$

where $k_1 = \sup_{n\geq 1} |A_n a| + |F(\phi)|$. By solving the differential inequality (3.3), we deduce

$$|u_n(t) - a| \leq k_1 T + L \int_0^t |u_{n_s} - \phi|_{\mathcal{B}} ds \quad \text{for } t \in [0, T],$$

consequently,

$$\sup_{s\in[0,t]} |u_n(s) - a| \leq k_1 T + L \int_0^t |u_{n_s} - \phi|_{\mathcal{B}} ds.$$

It follows that

$$K(t) \sup_{s\in[0,t]} |u_n(s) - a| \leq K(t) k_1 T + LK(t) \int_0^t |u_{n_s} - \phi|_{\mathcal{B}} ds;$$

moreover,

$$\begin{aligned}
&K(t) \sup_{s\in[0,t]} |u_n(s) - a| + M(t)|\phi - a|_{\mathcal{B}} \\
&\leq K(t) k_1 T + LK(t) \int_0^t |u_{n_s} - \phi|_{\mathcal{B}} ds + M(t)|\phi - a|_{\mathcal{B}} \\
&\leq K_T k_1 T + LK_T \int_0^t |u_{n_s} - \phi|_{\mathcal{B}} ds + M_T|\phi - a|_{\mathcal{B}},
\end{aligned}$$

where $M_T = \sup_{t\in[0,T]} M(t)$. Let $k_2 = K_T k_1 T + m|\phi - a|_{\mathcal{B}}$. We obtain

$$K(t) \sup_{s\in[0,t]} |u_n(s) - a| + M(t)|\phi - a|_{\mathcal{B}} \leq k_2 + LK_T \int_0^t |u_{n_s} - \phi|_{\mathcal{B}} ds.$$

Applying assumption (A1)(iii), we have

$$|u_{n_t} - a|_{\mathcal{B}} K(t) \sup_{s\in[0,t]} |u_n(s) - a| + M(t)|\phi - a|_{\mathcal{B}} \leq k_2 + LK_T \int_0^t |u_{n_s} - \phi|_{\mathcal{B}} ds.$$

Consequently

$$\begin{aligned}
|u_{n_t} - \phi|_{\mathcal{B}} &\leq |u_{n_t} - a|_{\mathcal{B}} + |\phi - a|_{\mathcal{B}} \\
&\leq |\phi - a|_{\mathcal{B}} + k_2 + LK_T \int_0^t |u_{n_s} - \phi|_{\mathcal{B}} ds.
\end{aligned}$$

we set $k_3 = |\phi - a|_{\mathcal{B}} + k_2$, we then have

$$|u_{n_t} - \phi|_{\mathcal{B}} \leq k_3 + LK_T \int_0^t |u_{n_s} - \phi|_{\mathcal{B}} ds.$$

Gronwall's Lemma implies

$$|u_{n_t} - \phi|_{\mathcal{B}} \leq k_3 e^{LK_T T}.$$

Since for all $\psi \in \mathcal{B}$, we have $|\psi(0)| \leq H|\psi|_{\mathcal{B}}$, it follows that

$$|u_n(t) - \phi(0)| \leq H|u_{n_t} - \phi|_{\mathcal{B}},$$
$$|u_n(t) - \phi(0)| \leq Hk_3 e^{LK_T T} = N.$$

Finally, we arrive at

$$|u_n(t)| \leq |\phi(0)| + N,$$

which implies that $(u_n)_{n\geq 1}$ is uniformly bounded in $C([0, T]; X)$. □

To prove that $(u_n')_{n\geq 1}$ is uniformly bounded, we need the following two lemmas.

Lemma 3.7 ([8])**.** *Let $w \in C^1([0,T];X)$. Then for any $s \in [0,T)$ one has*

$$\lim_{h\to 0^+} \sup_{\theta\in[-s,0]} \frac{|w(s+\theta+h)-w(s+\theta)|}{h} = \sup_{\theta\in[-s,0]} |w'(s+\theta)|.$$

Lemma 3.8 ([8])**.** *Let $w \in C^1([-h_0,0];X)\cap C^1([0,h_0];X)$. Then*

$$\limsup_{h\to 0^+} \sup_{\theta\in[-(s+h),-s]} \frac{|w(s+\theta+h)-w(s+\theta)|}{h} \leq |w_+'(0)| + |w_-^{'}(0)|$$

for $s \geq 0$ where $w_+'(0)$ and $w_-'(0)$ denote respectively the right and left derivative at 0.

Lemma 3.9. *There exists a constant $\beta > 0$ such that $|u_n'(t)| \leq \beta$ for all $n \geq 1$ and $t \in [0,T]$.*

Proof. Let $z_n(t) = u_n(t+h) - u_n(t)$. Then

$$D_+|z_n(t)| = [z_n(t), z_n'(t)] = [z_n(t), -A_n u_n(t+h) + A_n u_n(t) + F(u_{n_{t+h}}) - F(u_{n_t})].$$

Since A_n, is accretive,

$$[z_n(t), -A_n u_n(t+h) + A_n u_n(t)] \leq 0.$$

Consequently

$$D_+|z_n(t)| \leq L|u_{n_{t+h}} - u_{n_t}|_{\mathcal{B}},$$

which implies that

$$|z_n(t)| \leq |z_n(0)| + L\int_0^t |u_{n_{s+h}} - u_{n_s}|_{\mathcal{B}} ds,$$

$$\frac{|u_n(t+h)-u_n(t)|}{h} \leq \frac{|u_n(h)-u_n(0)|}{h} + L\int_0^t \frac{|u_{n_{s+h}} - u_{n_s}|_{\mathcal{B}}}{h} ds.$$

It remains to estimate

$$\int_0^t \frac{|u_{n_{s+h}} - u_{n_s}|_{\mathcal{B}}}{h} ds.$$

Using Proposition 2.25, we deduce that

$$|u_{n_{s+h}} - u_{n_s}|_{\mathcal{B}} \leq c|u_{n_{s+h}} - u_{n_s}|_{_{BC}} = c\sup_{\theta\leq 0} |u_n(s+\theta+h) - u_n(s+\theta)|.$$

We have to estimate

$$\sup_{\theta\leq 0} \frac{|u_n(s+\theta+h)-u_n(s+\theta)|}{h}.$$

In fact one has,

$$\begin{aligned}\sup_{\theta\leq 0} \sup |u_n(s+\theta+h) - u_n(s+\theta)| \leq &\sup_{\theta\leq -(s+h)} |u_n(s+\theta+h) - u_n(s+\theta)| \\ &+ \sup_{\theta\in[-(s+h),-s]} |u_n(s+\theta+h) - u_n(s+\theta)| \\ &+ \sup_{\theta\in[-s,0]} |u_n(s+\theta+h) - u_n(s+\theta)|\end{aligned}$$

For $s+\theta+h \leq 0$ and $s+\theta \leq 0$, one has

$$\begin{aligned}\sup_{\theta \leq -(s+h)} \frac{|u_n(s+\theta+h)-u_n(s+\theta)|}{h} &= \sup_{\theta \leq -(s+h)} \frac{|\phi(s+\theta+h)-\phi(s+\theta)|}{h} \\ &\leq \sup_{\theta \leq 0} |\phi'(\theta)| = N_1.\end{aligned}$$

If $\theta \in [-(s+h),-s]$, then $s+\theta+h \geq 0$ and $s+\theta \leq 0$. Since $u_n \in C^1([0,T];X)$ and $\phi \in C^1(-\infty,0];X)$, hence Lemma 3.8 yields

$$\limsup_{h\to 0^+} \sup_{\theta\in[-(s+h),-s]} \frac{|u_n(s+\theta+h)-u_n(s+\theta)|}{h} \leq |u_n'(0)|+|\phi'(0)|$$

with $u_n'(0)$ denotes the right derivative of u_n at 0, and $\phi'(0)$ denotes the left derivative of ϕ at 0. If $\theta \in [-s,0]$ then $s+\theta \geq 0$, and Lemma 3.7 yields

$$\begin{aligned}&\limsup_{h\to 0^+} \sup_{\theta\in[-s,0]} \frac{|u_n(s+\theta+h)-u_n(s+\theta)|}{h}. \\ &= \sup_{\theta\in[-s,0]} \sup_{h\to 0^+} \frac{|u_n(s+\theta+h)-u_n(s+\theta)|}{h}. \\ &= \sup_{\theta\in[-s,0]} |u_n'(s+\theta)|.\end{aligned}$$

$$\begin{aligned}\int_0^t \frac{|u_{n_{s+h}}-u_{n_s}|_{BC}}{h} ds &= \int_0^t \sup_{\theta\leq 0} \frac{|u_n(s+\theta+h)-u_n(s+\theta)|}{h} ds \\ &\leq \int_0^t \sup_{\theta\leq -(s+h)} \frac{|u_n(s+\theta+h)-u_n(s+\theta)|}{h} ds \\ &\quad + \int_0^t \sup_{\theta\in[-(s+h),-s]} \frac{|u_n(s+\theta+h)-u_n(s+\theta)|}{h} ds \\ &\quad + \int_0^t \sup_{\theta\in[-s,0]} \frac{|u_n(s+\theta+h)-u_n(s+\theta)|}{h} ds.\end{aligned}$$

$$\begin{aligned}\limsup_{h\to 0^+} \frac{|u_n(t+h)-u_n(t)|}{h} &= \lim_{h\to 0^+} \frac{|u_n(t+h)-u_n(t)|}{h} \\ &\leq |u_n'(0)| + cN_1TL + Lc\int_0^t (|u_n'(0)|+|\phi'(0)|) ds \\ &\quad + Lc\int_0^t \sup_{\theta\in[-s,0]} |u_n'(s+\theta)| ds.\end{aligned}$$

Consequently,

$$\begin{aligned}|u_n'(t)| &= \lim_{h\to 0^+} \frac{|u_n(t+h)-u_n(t)|}{h} \\ &\leq (1+cLT)|u_n'(0)| + cL(N_1+|\phi'(0)|)T \\ &\quad + Lc\int_0^t \sup_{\theta\in[-s,0]} |u_n'(s+\theta)| ds.\end{aligned}$$

Furthermore,

$$|u_n'(0)| \leq |A_n\phi(0)| + |F(\phi)| \leq k_0 + |F(\phi)|,$$

where $k_0 = \sup_{n\geq 1} |A_n a|$. Hence

$$
\begin{aligned}
|u_n'(t)| &= \lim_{h\to 0^+} \frac{|u_n(t+h) - u_n(t)|}{h} \\
&\leq (1 + cLT)(k_0 + |F(\phi)|) + Lc(N_1 + |\phi'(0)|)T \\
&\quad + Lc\int_0^t \sup_{\theta\in[-s,0]} |u_n'(s+\theta)|ds.
\end{aligned}
$$

Let

$$
k_3 = (1 + cLT)(k_0 + |F(\phi)|) + Lc(N_1 + |\phi'(0)|)T.
$$

Hence for $\theta \leq 0$ such that $-t \leq \theta$, we get

$$
\sup_{\theta\in[-t,0]} |u_n'(t+\theta)| \leq k_3 + Lc\int_0^t \sup_{\theta\in[-s,0]} |u_n'(s+\theta)|ds.
$$

Gronwall's Lemma implies

$$
\sup_{\theta\in[-t,0]} |u_n'(t+\theta)| \leq k_3 e^{LcT} = \beta.
$$

Finally for $\theta = 0$ we conclude $|u_n'(t)| \leq \beta$ which proves $(u_n'(t))_n$ is uniformly bounded. $\square$

Lemma 3.10. *Suppose that* (H1)–(H5) *hold. Then the sequence* $(u_n)_{n\geq 1}$ *converges uniformly to the mild solution* u *of* (1.1) *on* $[0, T]$.

Proof. Let u be the mild solution of (1.1) and v_n be the mild solution of the equation

$$
\begin{gathered}
v_n'(t) + A_n v_n(t) = F(u_t) \quad \text{for } t \in [0, T] \\
v_n(0) = \phi(0).
\end{gathered} \tag{3.4}
$$

From Theorem 2.22, we deduce that $v_n \to u$ as $n \to \infty$ uniformly on $[0, T]$. Setting

$$
z_n(t) = u_n(t) - v_n(t) \quad \text{for } t \in [0, T],
$$

we have

$$
D_+|z_n(t)| = [z_n(t), z_n'(t)] = [z_n(t), -A_n u_n(t) + A_n v_n(t) + F(u_{n_t}) - F(u_t)].
$$

Thus

$$
D_+|z_n(t)| \leq L|u_{n_t} - u_t|_{\mathcal{B}}.
$$

Hence

$$
\begin{aligned}
|u_n(t) - v_n(t)| &\leq L\int_0^t |u_{n_s} - u_s|_{\mathcal{B}} ds. \\
&\leq Lc\int_0^t |u_{n_s} - u_s|_{BC} ds. \\
&\leq Lc\int_0^t \sup_{\theta\leq 0} |u_n(s+\theta) - u(s+\theta)|ds. \\
&\leq LcT \sup_{\tau\in[0,T]} |u_n(\tau) - u(\tau)|.
\end{aligned}
$$

It follows that

$$\begin{aligned}\sup_{t\in[0,T]}|u_n(t)-v_n(t)| &\leq LcT \sup_{\tau\in[0,T]}|u_n(\tau)-u(\tau)|\\ &\leq LcT \sup_{t\in[0,T]}(|u_n(t)-v_n(t)|+|v_n(t)-u(t)|).\end{aligned}$$

Let T_0 be such that For $LcT_0<1$, we deduce that

$$\sup_{t\in[0,T_0]}|u_n(t)-v_n(t)| \leq \frac{LcT_0}{1-LcT_0}\sup_{t\in[0,T_0]}|v_n(t)-u(t)|,$$

and $v_n\to u$ uniformly on $[0,T_0]$, which implies

$$|u_n(t)-v_n(t)|\to 0 \text{ as } n\to\infty \text{ uniformly on } [0,T_0].$$

Consequently, for T_0 small enough, we have

$$u_n\to u \quad \text{uniformly on } [0,T_0].$$

Since the derivation of u_n' are uniformly bounded , which implies that u is lipschitz continuous on $[0,T_0]$. Since X has the Radon-Nikodym property, it follows that u is almost everywhere differentiable, by Theorem 2.18, we deduce that u is a strong solution of equation (1.1) on $[0,T_0]$, for T_0 small enough. The strong solution can be extended on $[0,+\infty)$, in fact, consider the equation

$$\begin{gathered} w'(t)+Aw(t)\ni F(w_t) \quad \text{for } t\in[T_0,T_1]\\ w_{T_0}=u_{T_0}, \end{gathered}\tag{3.5}$$

Arguing as above, we prove for T_1-T_0 small enough that (3.5) has a strong solution on $[T_0,T_1]$ which extends the strong solution of (1.1) on the entire interval $[T_0,T_1]$, we use the same argument to extend continuously the strong solution in the whole interval $[0,+\infty)$. To show that $u(t)\in\hat{D}(A)$ for $t\geq 0$. we use the following Lemma.

Lemma 3.11 ([5]). *Assume A is m-accretive and $u_0\in\hat{D}(A)$. If f is measurable and of essentially bounded variation on $[0,T]$. Let u be the mild solution solution of equation (2.1). Then $u(t)\in\hat{D}(A)$ for $t\geq 0$.*

In our case, $f(t)=F(u_t)$ for $t\geq 0$. Since the initial value φ is a Lipschitz continuous function on $(-\infty,0]$ and the mild solution of equation (1.1) is Lipschitz on $[0,T]$, using the fact that $\mathcal{B}$ is a fading memory space, we deduce that the function $t\to u_t$ is Lipschitz and consequently, we deduce that the function $t\to F(u_t)$ is Lipschitz and of course is of essentially bounded variation on $[0,T]$, by Lemma, we conclude that $u(t)\in\hat{D}(A)$ for $t\geq 0$. □

4. Applications

Example 1: Parabolic case. Let β be a maximal monotone subset of $\mathbb{R}\times\mathbb{R}$ such that $0\in D(\beta)$ and $\beta_p\subset L^p(0,1)\times L^p(0,1)$, $1<p<+\infty$, be the operator defined by

$$\begin{aligned} D(\beta_p)=\{u\in L^p(0,1) &: \text{there exists } v\in L^p(0,1) \text{ such that}\\ & v(x)\in\beta(u(x)) \text{ a.e. in } [0,1]\}\\ \beta_p(u)=\{v\in L^p(0,1) &: v(x)\in\beta(u(x)) \text{ a.e. in } [0,1]\}. \end{aligned}$$

Lemma 4.1 ([2]). *β_p is m-accretive on $L^p(0,1)$.*

Proposition 4.2 ([2]). *The operator $A : L^p(0,1) \to L^p(0,1)$ defined by*

$$D(A) = W_0^{1,p} \cap W_0^{2,p} \cap D(\beta_p)$$
$$A(u) = -\Delta u + \beta_p(u)$$

is m-accretive in $L^p(0,1)$.

To apply the previous abstract results, we consider the following multivalued parabolic partial functional differential equation

$$\begin{gathered}
\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} + \beta(u(t,x)) \ni \int_{-\infty}^0 G(\theta, u(t+\theta,x))d\theta \quad \text{for } t \in [0,1],\ x \in]0,1[\\
u(t,0) = u(t,1) = 0 \quad \text{for } t \in [0,1], \\
u(\theta,x) = \varphi(\theta,x) \quad \text{for } \theta \in \mathbb{R}^-,\ x \in]0,1[\,.
\end{gathered} \tag{4.1}$$

The phase space is

$$B = C_\gamma = \big\{\varphi \in C(]-\infty,0]; L^p(0,1) : \sup_{\theta\le 0} e^{\gamma\theta}|\varphi(\theta)|_p < +\infty\big\},$$

where $\gamma > 0$, endowed with the norm

$$|\varphi|_{C_\gamma} = \sup_{\theta \le 0} e^{\gamma\theta}|\varphi(\theta)|_p,$$

where

$$|\varphi(\theta)|_p = \Big(\int_0^1 |\varphi(\theta)(x)|^p dx\Big)^{1/p}.$$

Let $X = L^p(0,1)$, with $1 < p < +\infty$. $G :]-\infty,0] \times \mathbb{R} \to \mathbb{R}$ is such that

(i) the mapping $\theta \mapsto G(\theta,0)$ belongs to $L^1(-\infty,0)$.
(ii) $|G(\theta,x_1) - G(\theta,x_2)| \le \vartheta(\theta)|x_1 - x_2|$ for all $\theta \in]-\infty,0]$ and $x_1,x_2 \in \mathbb{R}$.

We assume that $\vartheta e^{-(\gamma+\varepsilon)} \in L^q(]-\infty,0])$ for some $\varepsilon > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 4.3 ([7]). *C_γ satisfies assumptions* (A1), (A2) *and* (B)*; moreover C_γ is a fading memory space.*

We introduce the function $F : C_\gamma \to L^p(0,1)$ defined by

$$(F\varphi)(x) = \int_{-\infty}^0 G(\theta,\varphi(\theta)(x))d\theta \quad \text{for a.e. } x \in [0,1].$$

Lemma 4.4. *Under the above conditions, the function $F : C_\gamma \to L^p(0,1)$ is Lipschitz continuous.*

Proof. Let $\varphi \in C_\gamma$ and $x \in [0,1]$. Then

$$\begin{aligned}
|(F(\varphi))(x) - (F(0))(x)| &= \Big|\int_{-\infty}^0 G(\theta,\varphi(\theta)(x))d\theta - \int_{-\infty}^0 G(\theta,0)d\theta\Big| \\
&\le \int_{-\infty}^0 |G(\theta,\varphi(\theta)(x)) - G(\theta,0)|d\theta \\
&\le \int_{-\infty}^0 \vartheta(\theta)|\varphi(\theta)(x)|d\theta \\
&\le \int_{-\infty}^0 \vartheta(\theta)e^{-(\gamma+\varepsilon)\theta}e^{(\gamma+\varepsilon)\theta}|\varphi(\theta)(x)|d\theta.
\end{aligned}$$

Hence

$$|(F\varphi)(x)-(F(0))(x)|^p \le \Big(\int_{-\infty}^0 \vartheta(\theta)e^{-(\gamma+\varepsilon)\theta}e^{(\gamma+\varepsilon)\theta}|\varphi(\theta)(x)|d\theta\Big)^p.$$

Using Hypothesis (ii) and Hölder's inequality, we obtain

$$|(F\varphi)(x)-(F(0))(x)|^p \le \Big(\int_{-\infty}^0 (\vartheta(\theta))^q e^{-q(\gamma+\varepsilon)\theta}d\theta\Big)^{p/q}\int_{-\infty}^0 e^{p(\gamma+\varepsilon)\theta}|\varphi(\theta)(x)|^p d\theta)$$

and

$$\begin{aligned}&\int_0^1 |(F\varphi)(x)-(F(0))(x)|^p dx\\ &\le \int_{-\infty}^0 \Big((\vartheta(\theta))^q e^{-q(\gamma+\varepsilon)\theta}d\theta\Big)^{p/q}\int_0^1\int_{-\infty}^0 e^{p(\gamma+\varepsilon)\theta}|\varphi(\theta)(x)|^p d\theta dx.\end{aligned}$$

Let

$$\lambda = (\int_{-\infty}^0 (\vartheta(\theta))^q e^{-q(\gamma+\varepsilon)\theta}d\theta)^{p/q} < +\infty.$$

By hypothesis (ii),

$$\begin{aligned}\int_0^1 |(F\varphi)(x)-(F(0))(x)|^p dx &\le \lambda\int_{-\infty}^0 e^{p\varepsilon\theta}\int_0^1 e^{p\gamma\theta}|\varphi(\theta)(x)|^p dx\,d\theta\\ &\le \lambda\Big(\sup_{\theta\le 0} e^{p\gamma\theta}\int_0^1 |\varphi(\theta)(x)|^p dx\Big)\int_{-\infty}^0 e^{p\varepsilon\theta}d\theta\\ &\le \frac{1}{p\varepsilon}\lambda|\varphi|^p_{C_\gamma}.\end{aligned}$$

Hence

$$|F(\varphi)-F(0)|_p \le \big(\frac{1}{p\varepsilon}\lambda\big)^{1/p}|\varphi|_{C_\gamma}.$$

Since$|F(0)|_p<\infty$, $F(\varphi)\in L^p(0,1)$. Now, let $\varphi,\psi\in C_\gamma$ and $x\in[0,1]$. Then

$$\begin{aligned}|(F\varphi)(x)-(F\psi)(x)| &= \int_{-\infty}^0 G(\theta,\varphi(\theta)(x))d\theta - \int_{-\infty}^0 G(\theta,\psi(\theta)(x))d\theta\Big|\\ &\le \int_{-\infty}^0 |G(\theta,\varphi(\theta)(x))-G(\theta,\psi(\theta)(x))|d\theta\\ &\le \int_{-\infty}^0 \vartheta(\theta)|\varphi(\theta)(x)-\psi(\theta)(x)|d\theta\\ &\le \int_{-\infty}^0 \vartheta(\theta)e^{-(\gamma+\varepsilon)\theta}e^{(\gamma+\varepsilon)\theta}|\varphi(\theta)(x)-\psi(\theta)(x)|d\theta.\end{aligned}$$

Hence

$$|(F\varphi)(x)-(F\psi)(x)|^p \le \Big(\int_{-\infty}^0 \vartheta(\theta)e^{-(\gamma+\varepsilon)\theta}e^{(\gamma+\varepsilon)\theta}|\varphi(\theta)(x)-\psi(\theta)(x)|d\theta\Big)^p.$$

By Hölder's inequality,

$$\begin{aligned}&|(F\varphi)(x)-(F\psi)(x)|^p\\ &\le \Big(\int_{-\infty}^0 (\vartheta(\theta))^q e^{-q(\gamma+\varepsilon)\theta}d\theta\Big)^{p/q}\int_{-\infty}^0 e^{p(\gamma+\varepsilon)\theta}|\varphi(\theta)(x)-\psi(\theta)(x)|^p d\theta.\end{aligned}$$

Thus

$$\begin{aligned}
&\int_0^1 |(F\varphi)(x) - (F\psi)(x)|^p dx \\
&\leq \Big(\int_{-\infty}^0 (\vartheta(\theta))^q e^{-q(\gamma+\varepsilon)\theta} d\theta \Big)^{p/q} \int_0^1 \int_{-\infty}^0 e^{p(\gamma+\varepsilon)\theta} |\varphi(\theta)(x) - \psi(\theta)(x)|^p d\theta \, dx.
\end{aligned}$$

Then

$$\begin{aligned}
\int_0^1 |(F\varphi)(x) - (F\psi)(x)|^p dx &\leq \lambda \int_{-\infty}^0 e^{p\varepsilon\theta} \int_0^1 e^{p\gamma\theta} |\varphi(\theta)(x) - \psi(\theta)(x)|^p dx \, d\theta \\
&\leq \lambda (\sup_{\theta\leq 0} e^{p\gamma\theta} \int_0^1 |\varphi(\theta)(x) - \psi(\theta)(x)|^p dx) \int_{-\infty}^0 e^{p\varepsilon\theta} d\theta \\
&\leq \frac{1}{p\varepsilon} \lambda |\varphi - \psi|_{C_\gamma}^p.
\end{aligned}$$

Therefore,

$$|F(\varphi) - F(\psi)|_p \leq (\frac{1}{p\varepsilon}\lambda)^{1/p} |\varphi - \psi|_{C_\gamma}.$$

□

Let function ϕ defined by

$$\phi(\theta)(x) = \varphi(\theta, x) \quad \text{for } \theta \leq 0, \ x \in [0,1].$$

Then (4.1) takes the abstract form

$$\begin{gathered}
u'(t) + Au(t) \ni F(u_t) \quad \text{for } t \geq 0 \\
u_0 = \phi \in C_\gamma.
\end{gathered} \tag{4.2}$$

Consequently, by Theorem 3.5, we deduce the following result.

Proposition 4.5. *Under the above assumption, let $\phi \in C_\gamma \cap C^1(]-\infty,0];X)$ be such that $\phi' \in C_\gamma$, ϕ' bounded and $\phi(0) \in \hat{D}(A)$. Then (4.2) has a unique strong solution u and the function v defined by*

$$v(t,x) = u(t)(x) \quad \textit{for a.e. } (t,x) \in [0,1]\times]0,1[$$

satisfies (4.1) for almost everywhere $(t,x) \in [0,1]\times]0,1[$.

Example 2: Hyperbolic case. We consider the hyperbolic equation

$$\begin{gathered}
\frac{\partial}{\partial t} u(t,x) + \frac{\partial}{\partial x}(g(u(t,x))) = \int_{-\infty}^0 H(\theta, x, u(t+\theta, x)) d\theta \quad \text{for } t \geq 0, \ x \in \mathbb{R} \\
u(\theta, x) = \varphi_0(\theta, x) \quad \text{for } \theta \leq 0, \ x \in \mathbb{R}
\end{gathered} \tag{4.3}$$

where $g : \mathbb{R} \to \mathbb{R}$ is continuous and strictly monotone with $g(\mathbb{R}) = \mathbb{R}$. $H :]-\infty, 0] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and the initial value function $\varphi_0 :]-\infty, 0] \times \mathbb{R} \to \mathbb{R}$ will be defined in the sequel.

Let $X = L^1(\mathbb{R})$ and define the operator

$$\begin{gathered}
D(A) = \big\{ v \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) : \frac{d}{dx}(g(v(x)) \in L^1(\mathbb{R}) \big\} \\
Av = \frac{d}{dx}(g(v(x)).
\end{gathered}$$

Lemma 4.6 ([9]). *A is m-accretive operator in $L^1(\mathbb{R})$.*

As above, we choose the phase space

$$\mathcal{B} = C_\gamma = \{\varphi \in C(]-\infty,0]; L^1(\mathbb{R})) : \sup_{\theta\le 0} e^{\gamma\theta}|\varphi(\theta)|_1 < +\infty\},$$

where $\gamma > 0$, we provide C_γ with the norm

$$|\varphi|_{C_\gamma} = \sup_{\theta\le 0} e^{\gamma\theta}|\varphi(\theta)|_1,$$

where

$$|\varphi(\theta)|_1 = \int_{-\infty}^{\infty} |\varphi(\theta)(x)|dx.$$

Let F be defined on C_γ by

$$F(\varphi)(x) = \int_{-\infty}^{0} H(\theta, x, \varphi(\theta,x))d\theta \quad \text{for } t \ge 0,\ x \in \mathbb{R}.$$

And the function ϕ defined by

$$\phi(\theta)(x) = \varphi_0(\theta, x) \quad \text{for for } \theta \le 0,\ x \in \mathbb{R}$$

Then equation 4.3 takes the abstract form

$$\begin{gathered} u'(t) + Au(t) = F(u_t) \quad \text{for } t \ge 0 \\ u_0 = \phi \in C_\gamma \end{gathered}$$

We assume that H satisfies

$$|H(\theta,x,y_1) - H(\theta,x,y_2)| \le \kappa(\theta)|y_1 - y_2| \quad \text{for } \theta \in]-\infty,0]\ x, y_1, y_2 \in \mathbb{R}$$

with

$$\int_{-\infty}^{0} e^{-\gamma\theta}\kappa(\theta)d\theta < \infty.$$

Moreover, we assume that

$$H(.,.,0) \in L^1(]-\infty,0] \times \mathbb{R}).$$

Under the above condition, $F : C_\gamma \to L^1(\mathbb{R})$ is Lipschitz continuous. Let $\varphi \in C_\gamma$. Then $F(\varphi) \in L^1(\mathbb{R})$ due to the fact, that

$$F(0) \in L^1(\mathbb{R}).$$

For the Lipschitz condition, take $\varphi, \psi \in C_\gamma$ and $x \in \mathbb{R}$. Then

$$|(F(\varphi) - F(\psi))(x)| \le \int_{-\infty}^{0} \kappa(\theta)|\varphi(\theta,x) - \psi(\theta,x)|d\theta \quad \text{for } x \in \mathbb{R}.$$

It follows that

$$\int_{-\infty}^{\infty} |(F(\varphi) - F(\psi))(x)|dx \le \int_{-\infty}^{0} e^{-\gamma\theta}\kappa(\theta)e^{-\gamma\theta}\int_{-\infty}^{\infty} |\varphi(\theta,x) - \psi(\theta,x)|dx\, d\theta.$$

Consequently,

$$|F(\varphi) - F(\psi)|_1 \le \int_{-\infty}^{0} e^{-\gamma\theta}\kappa(\theta)d\theta|\varphi - \psi|_{C_\gamma}.$$

By theorem 3.3, we deduce the following result.

Proposition 4.7. *Let the initial data function φ_0 be such that $\phi \in C_\gamma$ and $\phi(0) \in \overline{D(A)}$. Then* (1.1) *has a unique mild solution defined on* $[0,+\infty)$.

Acknowledgments. The authors would like to thank the anonymous referees for their careful reading of the original manuscript. Their valuable suggestions made numerous improvements.

References

[1] M. Adimy, H. Bouzahir and K. Ezzinbi; *Local Existence and Stability for Some Partial Functional Differential Equation with Infinite Delay*, Nonlinear Analysis, Theory Methods and Applications, 48, 323-348, (2002).

[2] V. Barbu; *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff Leyden, (1976).

[3] P. Benilan M. G. Crandall and A. Pazy; *Nonlinear Evolution Equations in Banach Spaces*, (Unpublished book).

[4] B. Burch and J. Goldstein; *Nonlinear semigroups and a problem in heat conduction*, Houston J. Math.4, 311-328, (1978).

[5] L. C. Evans; *Nonlinear evolution equations in arbitrary Banach spaces*, Israel J. Math., Vol. 26, No. 1, 1-42, (1977).

[6] J. K. Hale and T. Kato; *Phase space for retarded equations with infinite delay*, Funkcial Ekvac, 21, 11-41, (1978).

[7] Y. Hino ,S. Murakam, Naito; *Functional Differental Equations with Infinite Delay*, Springer-Verlag, (1991).

[8] A. G. Kartsatos and M. E. Parrott; *Convergence of the Kato approximants for evolution equations involving functional perturbations*, Journal of Differential Equations, 47, 358-377, (1983).

[9] A. G. Kartsatos and M. E. Parrott; *The weak solution of functional differential equation in a Banach Space*, Journal of Differential Equation, 75, 290-302, (1988).

[10] T. Kato; *Nonlinear semigroups and evolution equations, Journal of the Mathematical Society of Japan*, Vol. 19, 508-520, (1967).

[11] W. M. Ruess; *The evolution operator approach to functional differential equations with delay*, Proceedings of the American Mathematical Society, Vo. 119, 783-791, (1993).

[12] W. M. Ruess and W. H. Summers; *Operator semigroups for functional differential equations with delay*, Transactions of the American Mathematical Society, Vol. 341, No. 2, 695-719, (1994).

[13] K. Sik Ha; *Nonlinear Functional Evolutions in Banach Spaces*, Kluwer Academic Publishers, (2003).

[14] C. C. Travis and G. F. Webb; *Existence and stability for partial functional differential equations*, Transactions of the American Mathematical Society, Vol. 200, 395-418, (1974).

[15] J. Wu; *Theory and Applications of Partial Functional Differential Equations*, Graduate Texts in Applied Mathematics, Springer-Verlag, Vol. 119, (1996).

Mohamed Alia
Université Cadi Ayyad, Faculté des Sciences Semlalia, Département de Mathématiques, BP. 2390, Marrakech, Morocco
E-mail address: monsieuralia@yahoo.fr

Khalil Ezzinbi
Université Cadi Ayyad, Faculté des Sciences Semlalia, Département de Mathématiques, BP. 2390, Marrakech, Morocco
E-mail address: ezzinbi@ucam.ac.ma