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ANALYTIC SOLUTION OF AN INITIAL-VALUE PROBLEM FROM STOKES FLOW WITH FREE BOUNDARY

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Abstract. We study an initial-value problem arising from Stokes flow with free boundary. If the initial data is analytic in disk $\mathcal{R}_r$ containing the unit disk, it is proved that unique solution, which is analytic in $\mathcal{R}_s$ for $s \in (1, r)$, exists locally in time.

1. Introduction

The study of the deformation and breakup of bubbles in a slow viscous flow is of importance in many practical applications such as the rheology and mixing in multiphase viscous system. There has been a lot of research on this subject, the review article by Stone [10] summarizes the state of affair in the early nineties. This problem has been recently studied by some investigators. Tanveer and Vasconcelos [11, 12] obtained polynomial exact solutions; Cummings et al [5] also obtained explicit solutions and some conserved quantities. Crowdy and Siegel [2] obtained new conserved quantities and exact solution based on Cauchy transform approach. Nie et al [6] numerically studied the singularity formation of the Stokes flow. Prokert [9] obtained existence result of solutions in Sobolev space for a similar problem.

In this paper, we are going to establish a local existence result for an initial-value problem arising from free evolving bubble in Stokes flow. We first derive the initial-value problem using complex variables theory [1], then obtain the local existence result based on a Nirenberg theorem [7, 8] on abstract Cauchy-Kowalewski problem in properly chosen Banach spaces. The same techniques have been used for other problems [13, 14].

2. Stokes flow with free boundary

We consider the quasi-steady evolution of a bubble in an ambient Stokes flow [11, 12]. The fluid inside the bubble has a negligible viscosity and is at a constant pressure, which is set to be zero. The fluid outside the bubble has a viscosity μ and is incompressible. Under the assumption of no inertial effects, gravitational

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or other body forces, the fluid motion is governed by Stokes equation and the incompressibility condition

$$\mu\Delta u = \nabla p, \tag{2.1}$$

$$\nabla\cdot u = 0 \tag{2.2}$$

The above equations hold in the fluid region outside of the bubble.

On the bubble boundary, we have stress condition

$$-pn_j + 2\mu e_{jk}n_k = \tau\kappa n_j \tag{2.3}$$

where $n = (n_1, n_2)$ is a unit normal vector pointing outward from the bubble, τ is the surface tension coefficient, κ is the curvature and

$$e_{jk} = \frac{1}{2}\Big(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\Big) \tag{2.4}$$

is the rate of strain tensor.

The kinematic condition on the free boundary is

$$u\cdot n = V_n, \tag{2.5}$$

where V_n is the normal component of the free surface motion. Introducing streaming function $\psi(x,y)$ such that

$$u = \nabla^{\perp}\psi \tag{2.6}$$

then $\psi(x,y)$ satisfies the biharmonic equation

$$\nabla^4\psi = 0\,. \tag{2.7}$$

Here $\psi(x,y)$ can be expressed as

$$\psi = \operatorname{Im}[z^* f(z,t) + g(z,t)] \tag{2.8}$$

where $z = x + iy$ and $*$ denotes complex conjugate. Here Goursat functions $f(z,t)$ and $g(z,t)$ are analytic functions in the fluid region.

In terms of Goursat functions, the physical quantities are established

$$\begin{gathered} \frac{p}{\mu} - i\omega = 4f'(z,t), \\ u = u_1 + iu_2 = -f(z,t) + z[f'(z)]^* + [g(z)]^*, \\ e_{11} + ie_{12} = z[f''(z)]^* + [g''(z)]^*. \end{gathered} \tag{2.9}$$

where $*$ denotes the complex conjugate and ω is the vorticity.

Defining s to be the arc-length traversed in a counterclockwise direction around the bubble boundary, then the stress condition can be written as

$$f(z,t) + z[f'(z,t)]^* + [g'(z,t)]^* = -i\frac{z_s}{2}, \tag{2.10}$$

the kinematic equation can be written as

$$\operatorname{Im}[(z_t + 2f)z_s^*] = -\frac{1}{2}. \tag{2.11}$$

Equations (2.10) and (2.1) will be supplemented by far field conditions on f and g at infinity.

3. An initial-value problem

We consider the conformal mapping $z(\xi,t)$ that maps the interior of the unit circle $|\xi| < 1$ in the ξ plane to the fluid region in z-plane such that the $\xi = 0$ is mapped to the point $z = \infty$. So $\xi z(\xi,t)$ is analytic in $|\xi| < 1$. The kinematic condition can be written as

$$\mathrm{Re}\,\Big[\frac{z_t + 2f(z,t)}{\xi z_\xi}\Big] = \frac{\tau}{2|z_\xi|}. \tag{3.1}$$

as in Tanveer and Vasconcelos [11], we use the far field condition

$$f(z) \sim az + b + O(1/z) \quad \text{as } |z| \to \infty \tag{3.2}$$

where a and b are functions of t only. In particular, we choose $f(z) = az + b$ where a and b are constants. From Poisson's formula, (3.1) becomes

$$z_t + 2(az+b) = \xi z_\xi I_-(\xi,t) \text{ for } |\xi| < 1; \tag{3.3}$$

where $I_-(\xi,t)$ is defined by

$$I_-(\xi,t) = \frac{\tau}{4\pi i}\int_{|\xi|=1} \frac{d\xi'}{\xi'}\Big[\frac{\xi'+\xi}{\xi'-\xi}\Big]\frac{1}{|z_\xi|}. \tag{3.4}$$

Let $h(\xi,t) = \xi z(\xi,t)$, then $h(\xi,t)$ is analytic in $|\xi| < 1$. If $h(\xi,t)$ can be analytically extended to some region where $|\xi| > 1$, then

$$h_t + 2ah + 2b\xi = [\xi h_\xi - h]I_+[h](\xi,t) + \frac{\tau(\xi h_\xi - h)^{1/2}}{(\xi\bar{h}_\xi - \bar{h})^{1/2}}, \tag{3.5}$$

where $\bar{h}(\xi,t)$ is defined as

$$\bar{h}(\xi,t) = [h(\frac{1}{\xi^*})]^*, \tag{3.6}$$

and

$$I_+[h](\xi,t) = \frac{\tau}{4\pi i}\int_{|\xi|=1} \frac{d\xi'}{\xi'}\Big[\frac{\xi'+\xi}{\xi'-\xi}\Big]\frac{1}{|\xi h_\xi - h|} \quad \text{for } |\xi| > 1. \tag{3.7}$$

Making the change of variable,

$$v(\xi,t) = \frac{1}{(\xi h_\xi - h)^{1/2}}, \tag{3.8}$$

and using (3.5), we obtain

$$v_t = \xi v_\xi I_+[v] - \frac{1}{2}\xi v[I_+[v]]_\xi + \frac{1}{2}\tau\xi v\bar{v}v_\xi - \frac{1}{2}\tau\xi v^2\bar{v}_\xi + \frac{1}{2}vI_+[v] + \tau v^2\bar{v} + av, \tag{3.9}$$

where

$$I_+[v](\xi,t) = \frac{\tau}{4\pi i}\int_{|\xi|=1} \frac{d\xi'}{\xi'}\Big[\frac{\xi'+\xi}{\xi'-\xi}\Big]v(\xi',t)\bar{v}(\xi',t) \quad \text{for } |\xi| > 1. \tag{3.10}$$

The analytic continuation of (3.9) to $|\xi| < 1$ is

$$v_t = \xi v_\xi I_-[v] - \frac{1}{2}\xi v[I_-[v]]_\xi + \frac{1}{2}vI_-[v] + av, \tag{3.11}$$

where

$$I_-[v](\xi,t) = \frac{\tau}{4\pi i}\int_{|\xi|=1} \frac{d\xi'}{\xi'}\Big[\frac{\xi'+\xi}{\xi'-\xi}\Big]v(\xi',t)\bar{v}(\xi',t) \quad \text{for } |\xi| < 1. \tag{3.12}$$

We will consider equation (3.9) (3.11) with the initial condition

$$v(\xi,0) = v_0(\xi). \tag{3.13}$$

Let us first introduce a scale of Banach spaces which are spaces of bounded analytic functions in disks.

Definition 3.1. Let $\mathcal{R}_s$ be the disk in complex ξ plane with radius s; i.e., $\mathcal{R}_s = \{\xi, |\xi| < s\}$; we define function space $\mathbf{B}_s$ consisting of functions $f(\xi)$ is analytic in $\mathcal{R}_s$ and continuous on $\overline{\mathcal{R}_s}$ with norm $\|f\|_s = \sup_{\mathcal{R}_s} |f(\xi)|$.

Also we define the constant

$$M = \|v_0\|_r \,. \tag{3.14}$$

We will obtain the following local existence result.

Theorem 3.2. *If $v_0 \in \mathbf{B}_r$ with $r > 1$, then there exists one and only one solution $v \in C^1([0,T), \mathcal{B}_s)$, $1 < s < r, \|v\| \le 2M$ to (3.9) and $v|_{t=0} = v_0$, where $T = a_0(r-s)$, a_0 is a suitable positive constant independent of s.*

The proof of above theorem will be based on Nirenberg-Nishida theorem [7, 8].

Theorem 3.3 (Nirenberg-Nishida)**.** *Let $\{\mathbf{B_s}\}_{r_1 \le s \le r}$ be a scale of Banach spaces satisfying that $\mathbf{B}_s \subset \mathbf{B}_{s'}$, $\|\cdot\|_{s'} \le \|\cdot\|_s$ for any $r_1 < s' < s < r$. Consider the abstract Cauchy-Kowalewski problem*

$$\frac{du}{dt} = \mathcal{L}(u(t), t), \quad u(0) = 0 \tag{3.15}$$

Assume the following conditions on $\mathcal{L}$:

(i) *For some constants $M > 0, \delta > 0$ and every pair of numbers s, s' such that $r_1 < s' < s < r$, $(u,t) \to \mathcal{L}(u,t)$ is a continuous mapping of*
$$\{u \in \mathbf{B}_s : \|u\|_s < M\} \times \{t; |t| < \delta\} \text{ into } \mathbf{B}_{s'} \tag{3.16}$$

(ii) *For any $r_1 \le s' < s \le r$ and all $u, v \in \mathbf{B}_s$ with $\|u\|_s < M, \|v\|_s < M$ and for any t, $|t| < \delta$, $\mathcal{L}$ satisfies*
$$\|\mathcal{L}(u,t) - \mathcal{L}(v,t)\|_{s'} \le C\frac{\|u-v\|_s}{s-s'} \tag{3.17}$$
where C is some positive constant independent of t, u, v, s, s'.

(iii) *$\mathcal{L}(0,t)$ is a continuous function of t, $|t| < \delta$ with values in $\mathbf{B}_s$ for every $r_1 < s < r$ and satisfies, with some positive constant K,*
$$\|\mathcal{L}(0,t)\|_s \le \frac{K}{(r-s)} \tag{3.18}$$

Under the preceding assumptions there is a positive constant a_0 such that there exists a unique function $u(t)$ which, for every $r_1 < s < r$ and $|t| < a_0(r-s)$, is a continuously differentiable function of t with values in $\mathbf{B}_s, \|u\|_s < M$, and satisfies (3.15).

4. Properties of Banach space $\mathbf{B}_s$

Let r_0 and r_1 be two fixed numbers so that $r > r_1 > r_0 > 1$, then $\mathcal{R}_{r_0} \subset \mathcal{R}_{r_1} \subset \mathcal{R}_r$ and $\mathbf{B}_r \subset \mathbf{B}_{r_1} \subset \mathbf{B}_{r_0}$.

In this and the following sections, $C > 0$ represents a generic constant, it may vary from line to line. C may depend on r_0, r_1 and r; but it is always independent of s and s'.

Lemma 4.1. *If $f \in \mathbf{B}_s, r_1 < s < r$, then $\|f_\xi\|_{r_0} \le K_1\|f\|_s$ where K_1 is a positive constant independent of s and f.*

Proof. For $\xi \in \mathcal{R}_{r_0}$ and $t \in \{|t| = r_1\}$, we have $|t-\xi| \geq |r_1 - r_0|$. From Cauchy Integral Formula, we have

$$f_\xi(\xi) = \frac{1}{2\pi i}\int_{|t|=r_1} \frac{f(t)}{(t-\xi)^2}dt$$

so

$$|f(\xi)| \leq \frac{1}{2\pi}\int_{|t|=r_1} \frac{|f(t)|}{|t-\xi|^2}|dt| \leq \frac{r_1\|f\|_s}{2\pi(r_1-r_0)^2},$$

which completes the lemma. □

Definition 4.2. We define the function

$$\bar{f}(\xi) = [f(\frac{1}{\xi^*})]^* \tag{4.1}$$

where * denotes complex conjugate.

Remark 4.3. For $s > 0$, if $f \in \mathbf{B}_s$, then $\bar{f}$ is analytic in $|\xi| > \frac{1}{s}$ and $|\bar{f}| \leq \|f\|_s$ for $|\xi| > \frac{1}{s}$.

Lemma 4.4. *If $f \in \mathbf{B}_s, r_1 < s < r$, then $|\bar{f}_\xi(\xi)| \leq K_2\|f\|_s$ for $r \geq |\xi| \geq \frac{1}{r_0}$, where K_2 is a positive constant independent of s and f.*

Proof. Due to Remark 4.3, $\bar{f}$ is analytic in $|\xi| \geq \frac{1}{r_0}$, by Cauchy Integral Formula, we have for $1 \leq |\xi| \leq s$

$$\bar{f}_\xi(\xi,t) = \frac{1}{2\pi i}\int_{|\xi'|=2r} \frac{\bar{f}(\xi',t)}{(\xi'-\xi)^2}d\xi' - \frac{1}{2\pi i}\int_{|\xi'|=\frac{1}{r_1}} \frac{\bar{f}(\xi',t)}{(\xi'-\xi)^2}d\xi'$$

For $\frac{1}{r_0} \leq |\xi| \leq r, \xi' \in \{|\xi'| = \frac{1}{r_1}\} \cup \{|\xi'| = 2r\}$, we have $|\xi - \xi| \geq C$, where C depends only on r_0, r_1 and r; hence each integral in the above equation can be bounded by $C\|f\|_s$. This completes the proof. □

The following lemma is essential for applying the Nirenberg-Nishida Theorem.

Lemma 4.5. *If $f \in \mathbf{B}_s$, $r_1 < s' < s \leq r$; then $f_\xi \in \mathbf{B}_{s'}$ and*

$$\|f_\xi\|_{s'} \leq \frac{K_3}{s-s'}\|f\|_s, \tag{4.2}$$

where $K_3 > 0$ is independent of s, s' and f.

Proof. Since $\operatorname{dist}(\partial\mathbf{B}_{s'}, \partial\mathbf{B}_s) = s - s'$, for $\xi \in \mathbf{B}_{s'}$, we are able to find a disk $D(\xi)$ centered at ξ with radius $s - s'$ such that $D(\xi)$ is contained in $\mathcal{R}_s$. Using Cauchy integral formula, we have

$$f_\xi(\xi) = \frac{1}{2\pi i}\int_{|t-\xi|=s-s'} \frac{f(t)}{(t-\xi)^2}dt$$

so

$$\begin{aligned}
|f_\xi(\xi)| &\leq \frac{1}{2\pi}\int_{|t-\xi|=s-s'} \frac{|f(t)|}{|t-\xi|^2}|dt| \\
&\leq \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(\xi+|s-s'|e^{i\theta})|}{s-s'}d\theta \\
&\leq \frac{K_3\|f\|_s}{s-s'},
\end{aligned}$$

which proves the lemma. □

Definition 4.6. We define the function

$$G_1[v](\xi,t) = v(\xi,t)\bar{v}(\xi,t). \tag{4.3}$$

Remark 4.7. If $v \in \mathbf{B}_s, s > 1$, then $G_1[v]$ are analytic in $\frac{1}{s} \le |\xi| \le s$.

Lemma 4.8. *If $v \in \mathbf{B}_s$ and $1 < s < r$, then $|G_1[v](\xi,t)| \le \|v\|_s^2$ for $\frac{1}{s} \le |\xi| \le s$.*

The above lemma follows from equation (4.3).

Lemma 4.9. *If $v \in \mathbf{B}_s$ and $r_1 < s < r$, then $I_+[v](\xi,t) \le C\|v\|^2$ for $|\xi| \ge 1$, where $C > 0$ is a constant independent of s, g and h.*

Proof. Due to Remark 4.7, the integrands of I_+ are analytic in $\frac{1}{r_0} \le |\xi| \le 1$, changing the contour of integration in the definitions of I^+ from $|\xi| = 1$ to $|\xi| = \frac{1}{r_0}$ gives

$$I_+[v](\xi,t) = \frac{1}{4\pi i}\int_{|\xi'|=\frac{1}{r_0}} \frac{d\xi'}{\xi'}[\frac{\xi+\xi'}{\xi'-\xi}]G_1[v](\xi',t) \tag{4.4}$$

For $|\xi| > 1$ and $|\xi'| = \frac{1}{r_0}$, from simple geometry, we have

$$|\frac{\xi+\xi'}{\xi'-\xi}| \le C \tag{4.5}$$

where C depends on only r_0. The lemma now follows from (4.4) and Lemma 4.8. ☐

Similarly we have

Lemma 4.10. *If $v \in \mathbf{B}_s$ and $1 < s < r$, then $(I^+[v])_\xi(\xi,t) \le C\|v\|^2$ for $|\xi| \ge 1$.*

Proof. Taking derivative in (4.4),

$$(I_+[v])_\xi(\xi,t) = \frac{1}{4\pi i}\int_{|\xi'|=\frac{1}{r_0}} \frac{d\xi'}{\xi'}[\frac{\xi+\xi'}{(\xi'-\xi)^2}]G_1[v](\xi',t) \tag{4.6}$$

For $|\xi| > 1$ and $|\xi'| = \frac{1}{r_0}$, from simple geometry, we have

$$|\frac{\xi+\xi'}{(\xi'-\xi)^2}| \le C \tag{4.7}$$

where C depends on only r_0. The lemma now follows from (4.7) and Lemma 4.8. ☐

Lemma 4.11. *If $v \in \mathbf{B}_s$ and $1 < s < r$, then $I_-[v](\xi,t) \le C\|v\|^2$ for $|\xi| \le 1$.*

Proof. Due to Remark 4.7, the integrands of I_- is analytic in $r_0 \ge |\xi| \ge 1$, changing the contour of integration in the definitions of I_- from $|\xi| = 1$ to $|\xi| = r_0$ gives

$$I_-[v](\xi,t) = \frac{1}{4\pi i}\int_{|\xi'|=r_0} \frac{d\xi'}{\xi'}[\frac{\xi+\xi'}{\xi'-\xi}]G_1[v](\xi',t) \tag{4.8}$$

For $|\xi| < 1$ and $|\xi'| = r_0$, from simple geometry, we have

$$|\frac{\xi+\xi'}{\xi'-\xi}| \le C \tag{4.9}$$

where C depends on only r_0. The lemma now follows from(4.8), (4.9) and Lemma 4.8. ☐

Lemma 4.12. *If $v \in \mathbf{B}_s$ and $1 < s < r$, then $(I_-)_\xi[v](\xi,t) \le C\|v\|^2$ for $|\xi| \le 1$.*

Proof. Taking derivatives in (4.8),

$$(I_-[v])_\xi(\xi,t) = \frac{1}{4\pi i}\int_{|\xi'|=r_0} \frac{d\xi'}{\xi'}[\frac{\xi+\xi'}{(\xi'-\xi)^2}]G_1[v](\xi,t) \tag{4.10}$$

For $|\xi| < 1$ and $|\xi'| = r_0$, from simple geometry, we have

$$|\frac{\xi+\xi'}{(\xi'-\xi)^2}| \le C \tag{4.11}$$

where C depends on only r_0. The lemma now follows from (4.10), (4.11) and Lemma 4.8. □

5. Proof of the main theorem

In this section, to prove the main theorem, we apply Nirenberg's theorem to initial problem (3.9), (3.11) and (3.13). To this end, we need more estimates of the type as in (3.17).

Lemma 5.1. *If $u \in \mathbf{B}_s, v \in \mathbf{B}_s, r_1 < s' < s < r$, then $\|u_\xi - v_\xi\|_{s'} \le \frac{C}{s-s'}\|u-v\|_s$, where $C > 0$ is independent of s, s' and u and v.*

The above lemma follows from applying Lemma 4.5 with $f = u - v$.

Lemma 5.2. *If $u \in \mathbf{B}_s, v \in \mathbf{B}_s, r_1 < s < r$, then $|\bar{u}_\xi - \bar{v}_\xi| \le C\|u-v\|_s$ for $\frac{1}{r_1} \le |\xi|$, where $C > 0$ is independent of s, u and v.*

The above lemma follows from applying Lemma 4.4 with $f = u - v$.

Lemma 5.3. *If $u \in \mathbf{B}_s, v \in \mathbf{B}_s, \|u\|_s \le M, \|v\|_s \le M$, $r_1 < s < r$, then for $|\xi| > \frac{1}{r_0}$,*

$$|G_1[v](\xi,t) - G_1[u](\xi,t)| \le C\|v-u\|_s.$$

Proof. From (4.3), we have

$$G_1[v] - G_1[u] = (v-u)\bar{v} + u(\bar{v} - \bar{u})$$

which proves the lemma by using Remark 4.3. □

Lemma 5.4. *If $u \in \mathbf{B}_s, v \in \mathbf{B}_s, \|u\|_s \le M, \|v\|_s \le M$, $r_1 < s < r$, then for $|\xi| \le 1$,*

$$|I_-[v](\xi,t) - I_-[u](\xi,t)| \le C\|v-u\|_s;$$
$$|(I_-[v])_\xi(\xi,t) - (I_-[u])_\xi(\xi,t)| \le C\|v-u\|_s.$$

Proof. From (4.8), we have

$$I_-[v](\xi,t) - I_-[u](\xi,t) = \frac{1}{4\pi i}\int_{|\xi'|=r_0} \frac{d\xi'}{\xi'}[\frac{\xi+\xi'}{\xi'-\xi}](G_2[v](\xi',t) - G_2[u](\xi',t))$$

Now the lemma can be proved in the same fashion as Lemma 4.11 and Lemma 4.8 in light of Lemma 5.3. □

Lemma 5.5. *If $u \in \mathbf{B}_s, v \in \mathbf{B}_s, \|u\|_s \le M, \|v\|_s \le M$, then for $|\xi| \ge 1$,*

$$|I_+[v](\xi,t) - I_+[u](\xi,t)| \le C\|v-u\|_s,$$
$$|(I_+[v])_\xi(\xi,t) - (I_+[u])_\xi(\xi,t)| \le C\|v-u\|_s.$$

The above lemma can be proved in the same fashion as Lemma 4.9 and Lemma 4.7, in light of Lemma 5.3.

Let $v \in \mathcal{B}_s$. We define the following operator; for $|\xi| > 1$, $L[v]$ is defined by

$$L[v](\xi,t) = \xi v_\xi I_+[v] - \frac{1}{2}\xi v[I_+[v]]_\xi + \frac{1}{2}\tau\xi v\bar{v}v_\xi - \frac{1}{2}\tau\xi v^2\bar{v}_\xi + \frac{1}{2}vI_+[v] + \tau v^2\bar{v} + av, \quad (5.1)$$

The analytic continuation of $L[v]$ to $|\xi| < 1$ is

$$L[v](\xi,t) = \xi v_\xi I_-[v] - \frac{1}{2}\xi v[I_-[v]]_\xi + \frac{1}{2}vI_-[v] + av\,. \quad (5.2)$$

Lemma 5.6. *If $u \in \mathbf{B}_s$, $v \in \mathbf{B}_s$, $\|u\|_s \le M$, $\|v\|_s \le M$, then for $|\xi| \ge 1$ and $r_1 < s' < s < r$, we have*

$$\|L[u] - L[v]\|_{s'} \le \frac{C}{s-s'}\|v-u\|_s,$$

where $C > 0$ is independent of s and s'.

Proof. By (5.2), for $|\xi| < 1$,

$$\begin{aligned} L[v](\xi,t) - L[u](\xi,t) &= \xi(v_\xi - u_\xi)I_-[v] + \xi u_\xi\{I_-[v] - I_-[u]\} + a(v-u) \\ &\quad - \frac{1}{2}\xi(v-u)(I_-[v])_\xi - \frac{1}{2}\xi u\{(I_-[v])_\xi - (I_-[u])_\xi\}, \end{aligned} \quad (5.3)$$

and for $|\xi| > 1$,

$$\begin{aligned} L[v](\xi,t) - L[u](\xi,t) &= \xi(v_\xi - u_\xi)I_+[v] + \xi u_\xi\{I_+[v] - I_+[u]\} \\ &\quad + a(v-u) - \frac{1}{2}\xi(v-u)(I_+[v])_\xi - \frac{1}{2}\xi u\{(I_+[v])_\xi - (I_+[u])_\xi\} \\ &\quad + \frac{1}{2}\tau\xi\{(v-u)\bar{v}v_\xi + u(\bar{v} - \bar{u}v_\xi + u\bar{u}(v_\xi - u_\xi)\} \\ &\quad - \frac{1}{2}\tau\xi\{(v-u)(v+u)\bar{v}_\xi + u^2(\bar{v}_\xi + \bar{u}_\xi)(\bar{v}_\xi - \bar{u}_\xi)\} \\ &\quad + \tau(v-u)(v+u)\bar{v} + \tau u^2(\bar{v} - \bar{u}), \end{aligned} \quad (5.4)$$

By Lemmas 4.4–4.12 and 5.1– 5.5, each term in equations (5.3) and (5.4) can bounded by $\frac{C}{s-s'}\|v-u\|_s$; hence the proof is complete. □

Let $p = v - v_0$, then v is a solution of initial problem (3.9), (3.11) and (3.13) if and only if p solves the following initial problem

$$p_t = \mathcal{L}[p], p|_{t=0} = 0. \quad (5.5)$$

where the operator $\mathcal{L}$ is defined by

$$\mathcal{L}[p] = L[p + v_0] \quad (5.6)$$

Lemma 5.7. *If $p \in B_s, u \in B_s, \|p\|_s \le M$ and $\|u\|_s \le M, r_1 < s' < s < r$, then*

$$\|\mathcal{L}[p] - \mathcal{L}[u]\|_{s'} \le \frac{C}{s-s'}\|p-u\|_s$$

The proof of the above lemma follows from Lemma 5.6 and (5.6).

Lemma 5.8. *If $r_1 < s' < r$, then $\|\mathcal{L}[0]\|_{s'} \le \frac{K}{r-s'}$.*

Proof. From (5.6), we have $\mathcal{L}[0] = L[v_0]) = L[v_0] - L[0]$, for any s such that $s' < s < r$, using Lemma 5.6 with $v = v_0, u = 0$, we obtain

$$\|\mathcal{L}[0]\|_{s'} \leq \frac{C}{s - s'}\|v_0\|_s,$$

Letting $s \to r$ in the above equation, we complete the proof. □

Proof of Theorem 3.2. We first apply Nirenberg theorem to system (5.5). For $p \in B_s$, by Lemma 5.7 with $p = p, u = 0$, we have $\mathcal{L}[p] \in B_{s'}$. Since the system (5.5) is autonomous, the continuity of the operator $\mathcal{L}$ is implied by Lemma 5.7; hence (3.16) holds. (3.17) and (3.18) are given by Lemma 5.7 and Lemma 5.8 respectively. Therefore, there exists unique solution $p \in B_s, \|p\|_s \leq M$, so $v = p + v_0$ is the unique solution of the problem (3.9), (3.11) and (3.13). □

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