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A CHARACTERIZATION OF DICHOTOMY IN TERMS OF BOUNDEDNESS OF SOLUTIONS FOR SOME CAUCHY PROBLEMS

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Abstract. We prove that a quadratic matrix of order n having complex entries is dichotomic (i.e. its spectrum does not intersect the imaginary axis) if and only if there exists a projection P on $\mathbb{C}^n$ such that $Pe^{tA} = e^{tA}P$ for all $t \geq 0$ and for each real number μ and each vector $b \in \mathbb{C}^n$ the solutions of the following two Cauchy problems are bounded:

$$\begin{gathered} \dot{x}(t) = Ax(t) + e^{i\mu t}Pb, \quad t \geq 0, \\ x(0) = 0 \end{gathered}$$

and

$$\begin{gathered} \dot{y}(t) = -Ay(t) + e^{i\mu t}(I-P)b, \quad t \geq 0, \\ y(0) = 0\,. \end{gathered}$$

1. Notation and Preliminary Results

It is well-known that if a nonzero solution of the scalar differential equation $\dot{x}(t) = ax(t)$, $t \geq 0$ is asymptotically stable then each another solution has the same property and this it happens if and only if the real part of the complex number a is negative or if and only if for each real number μ and each complex number b the solution of the Cauchy Problem

$$\begin{gathered} \dot{z}(t) = az(t) + e^{i\mu t}b, \quad t \geq 0, \\ z(0) = 0 \end{gathered}$$

is bounded.

This result can be extended with approximatively the same formulation for the case of bounded linear operators acting on a Banach space X. See [1]. The result can be also extended for strongly continuous bounded semigroups, see [3, 4, 5, 8]. For discrete systems see [6].

Under a slightly different assumption the result on stability is also preserved for any strongly continuous semigroups acting on complex Hilbert spaces, see for example [9, 10] and references therein. See also, [7, 2], for counter-examples. In

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this paper the above result is extended for the case of dichotomic matrices. Our proof uses the Spectral Decomposition Theorem which is reminded below.

It is well known that the solution of the linear Cauchy Problem

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0, \quad t \in \mathbb{R} \tag{1.1}$$

is given by $x(t) = e^{tA}x_0$. Here $x_0 \in \mathbb{C}^n$ and A is a quadratic matrix of order n having complex numbers as entries. Let p_A be the characteristic polynomial associated with the matrix A and let $\sigma(A) = \{\lambda_1, \lambda_2, \dots, \lambda_k\}$, $k \le n$ be its spectrum. There exist the integer numbers $n_1, n_2, \dots, n_k \ge 1$ such that

$$p_A(\lambda) = (\lambda - \lambda_1)^{n_1}(\lambda - \lambda_2)^{n_2} \dots (\lambda - \lambda_k)^{n_k}, \quad n_1 + n_2 + \dots + n_k = n.$$

Let $j \in \{1, 2, \dots, k\}$ and $Y_j := \ker(A - \lambda_j I)^{n_j}$. Clearly Y_j is an e^{tA}-invariant subspace of $\mathbb{C}^n$ and $\dim(Y_j) \ge 1$. Moreover, the following Spectral Decomposition Theorem holds.

Theorem 1.1. *For each $x \in \mathbb{C}^n$ there exist $y_j \in Y_j$ $(j \in \{1, 2, \dots, k\})$ such that*

$$e^{tA}x = e^{tA}y_1 + e^{tA}y_2 + \dots + e^{tA}y_k, \quad t \in \mathbb{R}.$$

Moreover, if $y_j(t) := e^{tA}y_j$ then $y_j(t) \in Y_j$ for all $t \in \mathbb{R}$ and there exist a $\mathbb{C}^n$-valued polynomials $p_j(t)$ with $\deg(p_j) \le n_j - 1$ such that

$$y_j(t) = e^{\lambda_j t}p_j(t), \quad t \in \mathbb{R}, \ j \in \{1, 2, \dots, k\}. \tag{1.2}$$

To make this article self contained, we give here a sketch of the proof. From the Hamilton-Cayley theorem and using the fact that

$$\ker[pq(A)] = \ker[p(A)] \oplus \ker[q(A)]$$

whenever the complex valued polynomials p and q are relative prime follows that

$$\mathbb{C}^n = Y_1 \oplus Y_2 \oplus \dots \oplus Y_k. \tag{1.3}$$

Let $x \in \mathbb{C}^n$. For each $j \in \{1, 2, \dots, k\}$ there exists a unique $y_j \in Y_j$ such that

$$x = y_1 + y_2 + \dots + y_k$$

and then

$$e^{tA}x = e^{tA}y_1 + e^{tA}y_2 + \dots + e^{tA}y_k, \quad t \in \mathbb{R}. \tag{1.4}$$

Let $z_j(t) = e^{-\lambda_j t}y_j(t)$. A simple calculus shows that

$$\frac{d^{n_j}z_j(t)}{dt} = e^{-\lambda_j t}(A - \lambda_j)^{n_j}y_j(t) = 0.$$

The last equality follows because $y_j(t)$ belongs to Y_j for each $t \in \mathbb{R}$. Then z_j is a $\mathbb{C}^n$-valued polynomial having degree less then n_j.

2. Dichotomy and Boundedness

Let us denote $\mathbb{C}_+ := \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$, $\mathbb{C}_- := \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$ and $i\mathbb{R} = \{i\eta : \eta \in \mathbb{R}\}$. Clearly $\mathbb{C} = \mathbb{C}_+ \cup \mathbb{C}_- \cup i\mathbb{R}$.

A matrix A is called:

(i) *stable* if $\sigma(A)$ belongs to $\mathbb{C}_-$ or, equivalently, if there exist two positive constants N and ν such that $\|e^{tA}\| \le Ne^{-\nu t}$ for all $t \ge 0$,

(ii) *expansive* if $\sigma(A)$ belongs to $\mathbb{C}_+$ and

(iii) *dichotomic* if $\sigma(A)$ does not intersect the set $i\mathbb{R}$.

Our first result reads as follows.

Theorem 2.1. *The matrix A is stable if and only if for each $\mu \in \mathbb{R}$ and each $b \in \mathbb{C}^n$ the solution of the Cauchy Problem*

$$\begin{gathered} \dot{y}(t) = Ay(t) + e^{i\mu t}b, \quad t \geq 0, \\ y(0) = 0 \end{gathered} \tag{2.1}$$

is bounded.

Proof. **Necessity:** Let $\mu \in \mathbb{R}$ and $b \in \mathbb{C}^n$ be given. The solution of (2.1) is

$$\psi_{\mu,b}(t) = \int_0^t e^{(t-s)A}e^{i\mu s}b\,ds \quad t \geq 0.$$

Then

$$\begin{aligned} \|\psi_{\mu,b}(t)\| &\leq \int_0^t \|e^{(t-s)A}e^{i\mu s}b\|ds \\ &= \int_0^t \|e^{(t-s)A}\|\|b\|ds \\ &\leq \int_0^t Ne^{-\nu(t-s)}\|b\|ds \\ &\leq \frac{N}{\nu}\|b\|. \end{aligned}$$

Thus $\psi_{\mu,b}$ is bounded.

Sufficiency: The solution of (2.1) can be written as

$$\psi_{\mu,b}(t) = e^{i\mu t}\int_0^t e^{(-i\mu I + A)r}b\,dr, \quad t \geq 0.$$

It is clear that

$$\sigma(-i\mu I + A) = \{-i\mu + \lambda_1, -i\mu + \lambda_2, \dots, -i\mu + \lambda_k\}.$$

Suppose for the contrary that the matrix A is not stable i.e. there exists $\nu \in \{1, 2, \dots, k\}$ such that $\mathrm{Re}(\lambda_\nu) \geq 0$. Then $\mathrm{Re}(-i\mu + \lambda_\nu) \geq 0$. Denoting $B_\mu := (-i\mu I + A)$ may write

$$e^{B_\mu s}b = e^{B_\mu s}z_1 + e^{B_\mu s}z_2 + \dots + e^{B_\mu s}z_k.$$

Let choose $b = z_\nu$ be a nonzero vector. Then $e^{(-i\mu I + A)s}b = e^{B_\mu s}z_\nu$ and by Spectral Decomposition Theorem follows that $e^{B_\mu s}z_\nu = e^{\mu_\nu s}p(s)$, where $\mu_\nu := -i\mu + \lambda_\nu$ and p is a $\mathbb{C}^n$-valued polynomial with $\deg(p) \leq n_\nu - 1$.

We are going to consider two cases.

Case 1: When $\mathrm{Re}(\lambda_\nu) > 0$. Applying again Theorem 1.1 above, obtain

$$\psi_{\mu,b}(t) = e^{i\mu t}\int_0^t e^{\mu_\nu s}p(s)ds = e^{i\mu t}e^{\mu_\nu t}q(t), \quad t \geq 0.$$

Here $q(t)$ is a $\mathbb{C}^n$-valued, nonzero polynomial. Thus $\psi_{\mu,b}$ is an unbounded function and we arrived at a contradiction.

Case 2: When $\mathrm{Re}(\lambda_\nu) = 0$. Let $\mu = \frac{\lambda_\nu}{i}$. Then $\mu_\nu t = 0$ and

$$\psi_{\mu,b}(t) = e^{i\mu t}\int_0^t p(s)ds = \begin{cases} e^{i\mu t}q(t) \text{if } \deg(p) \geq 1 \\ te^{i\mu t} & \text{if } \deg(p) = 0. \end{cases}$$

Here q is a polynomial of degree greater than 1. In this case $\psi_{\mu,b}$ is also unbounded, which is a contradiction. □

Having in mind that A is expansive if and only if $(-A)$ is stable we obtain the following result.

Corollary 2.2. *The matrix A is expansive if and only if for each $\mu \in \mathbb{R}$ and each $b \in \mathbb{C}^n$ the solution of (2.1), with $-A$ instead of A, is bounded.*

A linear map P acting on $\mathbb{C}^n$ is called *projection* if $P^2 = P$.

Theorem 2.3. *The matrix A is dichotomic if and only if there exist a projection P having the property $e^{tA}P = Pe^{tA}$ for all $t \geq 0$ such that for each μ and each $b \in \mathbb{C}^n$ the following two inequalities hold*

$$\sup_{t\geq 0} \| \int_0^t e^{(-i\mu+A)s} Pbds\| < \infty, \tag{2.2}$$

$$\sup_{t\geq 0} \| \int_0^t e^{(-i\mu-A)s} (I-P)bds\| < \infty\,. \tag{2.3}$$

Proof. **Necessity:** Working under the assumption that A ia a dichotomic matrix we may suppose that there exists $\nu \in \{1, 2, \dots, k\}$ such that

$$\mathrm{Re}(\lambda_1) \geq \mathrm{Re}(\lambda_1) \geq \dots \Re(\lambda_\nu) > 0 > \Re(\lambda_{\nu+1}) \geq \dots \geq \Re(\lambda_k).$$

Having in mind the decomposition of $\mathbb{C}^n$ given by (1.3) let us consider

$$X_0 = Y_1 \oplus Y_2 \oplus \dots \oplus Y_\nu, \quad X_1 = Y_{\nu+1} \oplus Y_{\nu+2} \oplus \dots \oplus Y_k.$$

Then $\mathbb{C}^n = X_0 \oplus X_1$. Let us define $P : \mathbb{C}^n \to \mathbb{C}^n$ by $Px = x_0$, where $x = x_0 + x_1$, $x_0 \in X_0$ and $x_1 \in X_1$. It is clear that P is a projection. Moreover for all $x \in \mathbb{C}^n$ and all $t \geq 0$, this yields

$$Pe^{tA}x = P(e^{tA}(x_0 + x_1)) = P(e^{tA}(x_0) + e^{tA}(x_1)) = e^{tA}(x_0) = e^{tA}Px,$$

where the fact that X_0 is an $e^{tA}-$ invariant subspace, was used. Then $Pe^{tA} = e^{tA}P$. Now, we have

$$\begin{aligned} e^{s(-i\mu I+A)}Pb &= e^{-i\mu s}Pe^{sA}b \\ &= e^{-i\mu s}P\big(e^{\lambda_1 s}p_1(s) + \dots + e^{\lambda_\nu s}p_\nu(s) + \dots + e^{\lambda_k s}p_k(s)\big) \\ &= e^{-i\mu s}\big(e^{\lambda_1 s}p_1(s) + e^{\lambda_2 s}p_2(s) + \dots + e^{\lambda_\nu s}p_k(s)\big), \end{aligned}$$

where $p_1, p_2, \dots, p_\nu$ are polynomials as in Theorem 1.1. Now it is clear that the map $t \mapsto \int_0^t e^{(-i\mu+A)s}Pbds$ is bounded. The condition (2.3) can be verified in a similar manner.

Sufficiency: Suppose for a contradiction that A is not dichotomic. Then there exists $j \in \{1, 2, \dots, k\}$ such that $\lambda_j = i\eta$ with $\eta \in \mathbb{R}$. Let us take $b = x_j \in Y_j$, $x_j \neq 0$ and consider $x_{j0} =: Px_j$ and $x_{j1} =: (I-P)x_j$. We have

$$\psi_{\mu,Px_j}(t) = \int_0^t e^{-i\mu s}e^{sA}Px_jds = \int_0^t e^{i(-\mu+\eta)s}p_j(s)ds.$$

First consider the case when $\deg(p_j) \geq 1$. If $x_{j0} \neq 0$ the map $t \mapsto \psi_{\mu,Px_j}(t)$ is clearly unbounded and if $x_{j1} \neq 0$ we may repeat the above argument in order to arrive at the contradiction that the map $t \mapsto \int_0^t e^{-i\mu s}e^{-sA}(I-P)x_jds$ is unbounded. Next we consider the case when $\deg(p_j) = 0$. If $x_{j0} \neq 0$ choose $\mu = \eta$ and then there exists $p_j \in Y_j$, $p_j \neq 0$ such that $\psi_{\mu,Px_j}(t) = tp_j$ which is unbounded. If $x_{j0} = 0$ choose $\mu = -\eta$ and then

$$\int_0^t e^{-i\mu s} e^{-sA}(I-P)x_j ds = \int_0^t e^{-i\mu s} e^{-sA} x_{j1} ds = \int_0^t q_j ds = q_j t, \quad q_j \in Y_j.$$

Here $q_j \neq 0$ because $x_{j1} \neq 0$. Then $\psi_{\mu,(I-P)x_j}$ is an unbounded map. This is a contradiction and the proof is complete. □

It is clear that the Theorem 2.1 is a particular case ($P = 1$) of Theorem 2.3. We present its proof because it is different from that in [1].

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