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SOLUTIONS TO SECOND ORDER NON-HOMOGENEOUS MULTI-POINT BVPS USING A FIXED-POINT THEOREM

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Abstract. In this article, we study five non-homogeneous multi-point boundary-value problems (BVPs) of second order differential equations with the one-dimensional p-Laplacian. These problems have a common equation (in different function domains) and different boundary conditions. We find conditions that guarantee the existence of at least three positive solutions. The results obtained generalize several known ones and are illustrated by examples. It is also shown that the approach for getting three positive solutions by using multi-fixed-point theorems can be extended to nonhomogeneous BVPs. The emphasis is on the nonhomogeneous boundary conditions and the nonlinear term involving first order derivative of the unknown. Some open problems are also proposed.

1. Introduction

Multi-point boundary-value problems (BVPs) for second order differential equations without p-Laplacian have received a wide attention because of their potential applications and BVPs are fascinating and challenging fields of study, one may see the textbook by Ge [14]. There are four classes of such BVPs (including their special cases) studied in known papers:

$$\begin{gathered} x''(t)+f(t,x(t),x'(t))=0, \quad t\in(0,1),\\ x(0)=\sum_{i=1}^m \alpha_i x(\xi_i), \quad x(1)=\sum_{i=1}^m \beta_i x(\xi_i), \end{gathered} \tag{1.1}$$

$$\begin{gathered} x''(t)+f(t,x(t),x'(t))=0, \quad t\in(0,1),\\ x'(0)=\sum_{i=1}^m \alpha_i x'(\xi_i), \quad x(1)=\sum_{i=1}^m \beta_i x(\xi_i), \end{gathered} \tag{1.2}$$

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$$
\begin{gathered}
x''(t)+f(t,x(t),x'(t))=0,\quad t\in(0,1),\\
x(0)=\sum_{i=1}^m\alpha_ix(\xi_i),\quad x'(1)=\sum_{i=1}^m\beta_ix'(\xi_i),
\end{gathered}\tag{1.3}
$$

and

$$
\begin{gathered}
x''(t)+f(t,x(t),x'(t))=0,\quad t\in(0,1),\\
x'(0)=\sum_{i=1}^m\alpha_ix'(\xi_i),\quad x'(1)=\sum_{i=1}^m\beta_ix'(\xi_i),
\end{gathered}\tag{1.4}
$$

where $0<\xi_1<\dots<\xi_m<1$, $\alpha_i,\beta_i\in\mathbb{R}$ and f is a continuous function. The main methods to get solutions or multiple positive solutions of these BVPs are as follows:

(i) fixed point index theory [17, 18, 63, 64], or fixed point theorems in cones in Banach spaces [5, 7, 15, 35, 41, 42, 46], such as the Guo-Krasnoselskii's fixed-point theorem [3, 44, 48, 52, 53, 57]; Leggett-Williams theorems [33, 36, 50], the five-functional fixed point theorem [2]; the fixed point theorem of Avery and Peterson [26], etc.

(ii) Mawhin's continuation theorem [10, 19, 27, 28, 33, 34, 38, 39, 40, 42];

(iii) the shooting methods [29, 51];

(iv) upper and lower solution methods and monotone iterative techniques [1, 4, 67]; upper and lower solution methods and Leray-Schauder degree theory [23, 24, 25, 47], or the approach of a combination of nonlinear alternative of Leray-Schauder with the properties of the associated vector field at the (x,x') plane [12];

(v) the critical point theory and variational methods [30];

(vi) topological degree theory [13, 14]; the Schauder's fixed point theorem in suitable Banach space [36, 41, 42, 43, 49].

In all the above papers, the boundary conditions (BCs) are homogeneous. However, in many applications, BVPs consist of differential equations coupled with nonhomogeneous BCs, for example

$$
\begin{gathered}
y''=\frac{1}{\lambda}(1+y^2)^{\frac12},\quad t\in(a,b),\\
y(a)=a\alpha,\quad y(b)=\beta
\end{gathered}
$$

and

$$
\begin{gathered}
y''=-\frac{(1+y'(t))^2}{2(y(t)-\alpha)},\quad t\in(a,b),\\
y(a)=\alpha,\quad y(b)=\beta
\end{gathered}
$$

which are very well known and were proposed in 1690 and 1696, respectively. In 1964, the BVPs studied by Zhidkov and Shirikov [66] and by Lee [31] are also nonhomogeneous.

There are also several papers concerned with the existence of positive solutions of BVPs for differential equations with non-homogeneous BCs. Ma Ruyun [45] studied existence of positive solutions of the following BVP consisting of second order differential equations and three-point BC

$$
\begin{gathered}
x''(t))+a(t)f(x(t))=0,\quad t\in(0,1),\\
x(0)=0,\quad x(1)-\alpha x(\eta)=b,
\end{gathered}\tag{1.5}
$$

Kwong and Kong [29] studied the BVP

$$\begin{gathered} y''(t) = -f(t, y(t)), \quad 0 < t < 1, \\ \sin\theta y(0) - \cos\theta y'(0) = 0, \\ y(1) - \sum_{i=1}^{m-2} \alpha_i y(\xi_i) = b \geq 0, \end{gathered} \tag{1.6}$$

where $\xi_i \in (0,1), \alpha_i \geq 0, \theta \in [0, 3\pi/4]$, f is a nonnegative and continuous function. Under some assumptions, it was proved that there exists a constant $b^* > 0$ such that: (1.6) has at least two positive solutions if $b \in (0, b^*)$; (1.6) has at least one solution if $b = 0$ or $b = b^*$; (1.6) has no positive solution if $b > b^*$.

Palamides [[51]], under superlinear and/or sublinear growth rate in f, proved the existence of positive solutions (and monotone in some cases) of the boundary-value problem

$$\begin{gathered} y''(t) = -f(t, y(t), y'(t)), \quad 0 \leq t \leq 1, \\ \alpha y(0) - \beta y'(0) = 0, \quad y(1) - \sum_{i=1}^{m-2} \alpha_i y(\xi_i) = b \geq 0, \end{gathered} \tag{1.7}$$

where $\alpha > 0$, $\beta > 0$, the function f is continuous, and $f(t, y, y') \geq 0$, for all $t \in [0,1]$, $y \geq 0$, $y' \in \mathbb{R}$. The approach is based on an analysis of the corresponding vector field on the face-plane and on Kneser's property for the solution's funnel.

Sun, Chen, Zhang and Wang [53] studied the existence of positive solutions for the three-point boundary-value problem

$$\begin{gathered} u''(t) + a(t) f(u(t)) = 0, \quad 0 \leq t \leq 1, \\ u'(0) = 0, \quad u(1) - \sum_{i=1}^{m-2} \alpha_i u(\xi_i) = b \geq 0, \end{gathered} \tag{1.8}$$

where $\xi_i \in (0,1)$, $\alpha_i \geq 0$ are given. It was proved that there exists $b^* > 0$ such that (1.8) has at least one positive solution if $b \in (0, b^*)$ and no positive solution if $b > b^*$. To study the existence of positive solutions of BVPs (1.5), (1.6), (1.7), (1.8), the Green's functions of the corresponding problems are established and play an important role in the proofs of the main results.

In recent papers, using lower and upper solutions methods, Kong and Kong [23, 24, 25] established results for solutions and positive solutions of the following two problems

$$\begin{gathered} x''(t)) + f(t, x(t), x'(t)) = 0, \quad t \in (0,1), \\ x'(0) - \sum_{i=1}^{m} \alpha_i x'(\xi_i) = \lambda_1, \quad x(1) - \sum_{i=1}^{m} \beta_i x(\xi_i) = \lambda_2, \end{gathered} \tag{1.9}$$

and

$$\begin{gathered} x''(t)) + f(t, x(t), x'(t)) = 0, \quad t \in (0,1), \\ x(0) - \sum_{i=1}^{m} \alpha_i x(\xi_i) = \lambda_1, \quad x(1) - \sum_{i=1}^{m} \beta_i x(\xi_i) = \lambda_2, \end{gathered} \tag{1.10}$$

respectively. We note that the boundary conditions in [23, 24] are two-parameter non-homogeneous BCs. There, the existence of lower and upper solutions with certain relations are assumed.

In recent years, there have been many exciting results concerning the existence of positive solutions of BVPs of second order differential equations with p-Laplacian subjected to different multi-point boundary conditions:

$$\begin{gathered}
[\phi(x'(t))]' + f(t, x(t), x'(t)) = 0, \quad t \in (0,1), \\
x(0) = \sum_{i=1}^{m} \alpha_i x(\xi_i), \quad x(1) = \sum_{i=1}^{m} \beta_i x(\xi_i),
\end{gathered} \tag{1.11}$$

$$\begin{gathered}
[\phi(x'(t))]' + f(t, x(t), x'(t)) = 0, \quad t \in (0,1), \\
x'(0) = \sum_{i=1}^{m} \alpha_i x'(\xi_i), \quad x(1) = \sum_{i=1}^{m} \beta_i x(\xi_i),
\end{gathered} \tag{1.12}$$

$$\begin{gathered}
[\phi(x'(t))]' + f(t, x(t), x'(t)) = 0, \quad t \in (0,1), \\
x(0) = \sum_{i=1}^{m} \alpha_i x(\xi_i), \quad x'(1) = \sum_{i=1}^{m} \beta_i x'(\xi_i).
\end{gathered} \tag{1.13}$$

These results, can be found in [6, 8, 9, 10, 11, 13, 20, 21, 22, 37, 54, 55, 56, 58, 59, 60, 61, 62, 65]. In above mentioned papers, to obtain positive solutions, two kinds of assumptions are supposed. The first one is imposed on α_i, β_i, the other one called growth conditions is imposed on the nonlinearity f. To define a cone P in Banach spaces and to define the nonlinear operator $T : P \to P$ are important steps in the the proofs of the results.

It is easy to see that

$$\begin{gathered}
x''(t) = -2, \quad t \in (0,1), \\
x(0) = x(1) = 0
\end{gathered}$$

has unique positive solution $x(t) = -t^2 + t$, but the BVP

$$\begin{gathered}
x''(t) = -2, \quad t \in (0,1), \\
x(0) = A, \quad x(1) = B
\end{gathered}$$

has positive solution $x(t) = -t^2 + (B - A + 1)t + A$ if and only if $A \geq 0$ and $B \geq 0$. It shows us that the presence of nonhomogeneous BCs can induce nonexistence of positive solutions of a BVP.

Motivated by the facts mentioned above, this paper is concerned with the more generalized BVPs for second order differential equation with p-Laplacian coupled with nonhomogeneous multi-point BCs; i.e.,

$$\begin{gathered}
[\phi(x'(t))]' + f(t, x(t), x'(t)) = 0, \quad t \in (0,1), \\
x'(0) = \sum_{i=1}^{m} \alpha_i x'(\xi_i) + A, \quad x(1) = \sum_{i=1}^{m} \beta_i x(\xi_i) + B,
\end{gathered} \tag{1.14}$$

$$\begin{gathered}
[\phi(x'(t))]' + f(t, x(t), x'(t)) = 0, \quad t \in (0,1), \\
x(0) = \sum_{i=1}^{m} \alpha_i x(\xi_i) + A, \quad x'(1) = \sum_{i=1}^{m} \beta_i x'(\xi_i) + B,
\end{gathered} \tag{1.15}$$

$$\begin{gathered}
[\phi(x'(t))]' + f(t, x(t), x'(t)) = 0, \quad t \in (0,1), \\
x(0) = \alpha x'(0) + A, \quad x(1) = \sum_{i=1}^{m} \beta_i x(\xi_i) + B,
\end{gathered} \tag{1.16}$$

$$\begin{gathered}[\phi(x'(t))]' + f(t, x(t), x'(t)) = 0, \quad t \in (0,1), \\ x(0) = \sum_{i=1}^{m} \alpha_i x'(\xi_i) + A, \quad x'(1) = \sum_{i=1}^{m} \beta_i x'(\xi_i) + B,\end{gathered} \tag{1.17}$$

and

$$\begin{gathered}[\phi(x'(t))]' + f(t, x(t), x'(t)) = 0, \quad t \in (0,1), \\ x(0) = \sum_{i=1}^{m} \alpha_i x(\xi_i) + A, \quad x(1) = \sum_{i=1}^{m} \beta_i x(\xi_i) + B,\end{gathered} \tag{1.18}$$

where $0 < \xi_1 < \dots < \xi_m < 1$, $A, B \in \mathbb{R}$, $\alpha_i \geq 0, \alpha \geq 0$, $\beta_i \geq 0$ for all $i = 1, \dots, m$, f is continuous and nonnegative, ϕ is called p-Laplacian, $\phi(x) = |x|^{p-2}x$ for $x \neq 0$ and $\phi(0) = 0$ with $p > 1$, its inverse function is denoted by $\phi^{-1}(x)$ with $\phi^{-1}(x) = |x|^{q-2}x$ for $x \neq 0$ and $\phi^{-1}(0) = 0$ with $1/p + 1/q = 1$.

The purpose is to establish sufficient conditions for the existence of at least three positive solutions for (1.14)–(1.18). The results in this paper are new since there exists no paper concerned with the existence of at least three positive solutions of these nonhomogeneous multi-point BVPs even when $\phi(x) = x$. Maybe it is the first time to use the multi-fixed-point theorem to solve these kinds of BVPs.

The remainder of this paper is organized as follows: The main results are presented in Section 2 (Theorems 2.10, 2.16, 2.21, 2.25, 2.30). Some examples to show the main results are given in Section 3.

2. Main Results

In this section, we first present some background definitions in Banach spaces and state an important three fixed point theorem. Then the main results are given and proved.

Definition 2.1. Let X be a semi-ordered real Banach space. The nonempty convex closed subset P of X is called a cone in X if $ax \in P$ for all $x \in P$ and $a \geq 0$ and $x \in X$ and $-x \in X$ imply $x = 0$.

Definition 2.2. A map $\psi : P \to [0, +\infty)$ is a nonnegative continuous concave or convex functional map provided ψ is nonnegative, continuous and satisfies

$$\psi(tx + (1-t)y) \geq t\psi(x) + (1-t)\psi(y),$$

or

$$\psi(tx + (1-t)y) \leq t\psi(x) + (1-t)\psi(y),$$

for all $x, y \in P$ and $t \in [0,1]$.

Definition 2.3. An operator $T; X \to X$ is completely continuous if it is continuous and maps bounded sets into relative compact sets.

Definition 2.4. Let $a, b, c, d, h > 0$ be positive constants, α, ψ be two nonnegative continuous concave functionals on the cone P, γ, β, θ be three nonnegative continuous convex functionals on the cone P. Define the convex sets:

$$\begin{gathered}P_c = \{x \in P : \|x\| < c\}, \\ P(\gamma, \alpha; a, c) = \{x \in P : \alpha(x) \geq a,\ \gamma(x) \leq c\}, \\ P(\gamma, \theta, \alpha; a, b, c) = \{x \in P : \alpha(x) \geq a,\ \theta(x) \leq b,\ \gamma(x) \leq c\}, \\ Q(\gamma, \beta; , d, c) = \{x \in P : \beta(x) \leq d,\ \gamma(x) \leq c\},\end{gathered}$$

$$Q(\gamma, \beta, \psi; h, d, c) = \{x \in P : \psi(x) \geq h,\ \beta(x) \leq d,\ \gamma(x) \leq c\}.$$

Lemma 2.5 ([2]). *Let X be a real Banach space, P be a cone in X, α, ψ be two nonnegative continuous concave functionals on the cone P, γ, β, θ be three nonnegative continuous convex functionals on the cone P. There exist constant $M > 0$ such that*

$$\alpha(x) \leq \beta(x), \quad \|x\| \leq M\gamma(x) \quad \text{for all } x \in P.$$

Furthermore, Suppose that $h, d, a, b, c > 0$ are constants with $d < a$. Let $T : \overline{P_c} \to \overline{P_c}$ be a completely continuous operator. If

- (C1) *$\{y \in P(\gamma, \theta, \alpha; a, b, c) | \alpha(x) > a\} \neq \emptyset$ and $\alpha(Tx) > a$ for every x in $P(\gamma, \theta, \alpha; a, b, c)$;*
- (C2) *$\{y \in Q(\gamma, \theta, \psi; h, d, c) | \beta(x) < d\} \neq \emptyset$ and $\beta(Tx) < d$ for every x in $Q(\gamma, \theta, \psi; h, d, c)$;*
- (C3) *$\alpha(Ty) > a$ for $y \in P(\gamma, \alpha; a, c)$ with $\theta(Ty) > b$;*
- (C4) *$\beta(Tx) < d$ for each $x \in Q(\gamma, \beta; , d, c)$ with $\psi(Tx) < h$,*

then T has at least three fixed points y_1, y_2 and y_3 such that

$$\beta(y_1) < d, \quad \alpha(y_2) > a, \quad \beta(y_3) > d, \quad \alpha(y_3) < a.$$

Choose $X = C^1[0, 1]$. We call $x \leq y$ for $x, y \in X$ if $x(t) \leq y(t)$ for all $t \in [0, 1]$, define the norm $\|x\| = \max\{\max_{t\in[0,1]} |x(t)|,\ \max_{t\in[0,1]} |x'(t)|\}$. It is easy to see that X is a semi-ordered real Banach space.

Choose $k \in (0, 1/2)$, let $\sigma_0 = \min\{k, 1 - k\} = k$. For a cone $P \subseteq X$ of the Banach space $X = C^1[0, 1]$, define the functionals on $P : P \to \mathbb{R}$ by

$$\gamma(y) = \max_{t\in[0,1]} |y'(t)|,\ y \in P, \quad \beta(y) = \max_{t\in[0,1]} |y(t)|,\ y \in P,$$
$$\theta(y) = \max_{t\in[k,1-k]} |y(t)|,\ y \in P, \quad \alpha(y) = \min_{t\in[k,1-k]} |y(t)|,\ y \in P,$$
$$\psi(y) = \min_{t\in[k,1-k]} |y(t)|,\ y \in P.$$

It is easy to see that α, ψ are two nonnegative continuous concave functionals on the cone P, γ, β, θ are three nonnegative continuous convex functionals on the cone P and $\alpha(y) \leq \beta(y)$ for all $y \in P$.

Lemma 2.6. *Suppose that $x \in X$, $x(t) \geq 0$ for all $t \in [0, 1]$ and $x'(t)$ is decreasing on $[0, 1]$. Then*

$$x(t) \geq \min\{t, 1 - t\} \max_{t\in[0,1]} x(t), \quad t \in [0, 1]. \tag{2.1}$$

Proof. Suppose $x(t_0) = \max_{t\in[0,1]} x(t)$. If $t \in (0, t_0)$, we get that there exist $0 \leq \eta \leq t \leq \xi \leq t_0$ such that

$$\begin{aligned} \frac{x(t) - x(0)}{t - 0} - \frac{x(t_0) - x(0)}{t_0 - 0} &= -\frac{t[x(t_0) - x(t)] - (t_0 - t)[x(t) - x(0)]}{tt_0} \\ &= -\frac{t(t_0 - t)x'(\xi) - (t_0 - t)tx'(\eta)}{tt_0} \\ &\geq -\frac{t(t_0 - t)x'(\eta) - (t_0 - t)tx'(\eta)}{tt_0} = 0. \end{aligned}$$

Then

$$x(t) \geq \frac{t}{t_0} x(t_0) + (1 - \frac{t}{t_0} x(0) \geq \frac{t}{t_0} x(t_0) \geq tx(t_0), \quad t \in (0, t_0).$$

Similarly $x(t) \geq (1-t)x(t_0)$, for $t \in (t_0, 1)$. It follows that $x(t) \geq \min\{t, 1-t\} \max_{t\in[0,1]} x(t)$ for all $t \in [0,1]$. The proof is complete. $\square$

2.1. **Positive Solutions of** (1.14)**.** First, we establish an existence result for three decreasing positive solutions of (1.14). We use the following assumptions:

- (H1) $f : [0,1]\times[h,+\infty)\times(-\infty,0] \to [0,+\infty)$ is continuous with $f(t, c+h, 0) \not\equiv 0$ on each sub-interval of [0,1], where $h = \frac{B}{1-\sum_{i=1}^{m}\beta_i}$;
- (H2) $A \leq 0, B \geq 0$;
- (H3) $\alpha_i \geq 0$, $\beta_i \geq 0$ satisfy $\sum_{i=1}^{m}\alpha_i < 1$, $\sum_{i=1}^{m}\beta_i < 1$;
- (H4) $h : [0,1] \to [0,+\infty)$ is a continuous function and $h(t) \not\equiv 0$ on each subinterval of [0,1].

Consider the boundary-value problem

$$\begin{gathered}
[\phi(y'(t))]' + h(t) = 0, \quad t \in (0,1), \\
y'(0) - \sum_{i=1}^{m}\alpha_i y'(\xi_i) = A, \quad y(1) - \sum_{i=1}^{m}\beta_i y(\xi_i) = 0,
\end{gathered} \tag{2.2}$$

Lemma 2.7. *Suppose that* (H2)–(H4) *hold. If y is a solution of* (2.2)*, then y is decreasing and positive on* $(0,1)$.

Proof. Suppose y satisfies (2.2). It follows from the assumptions that y' is decreasing on $[0,1]$. Then the BCs in (2.2) and (H3) imply

$$y'(0) = \sum_{i=1}^{m}\alpha_i y'(\xi_i) + A \leq \sum_{i=1}^{m}\alpha_i y'(0) + A.$$

It follows that $y'(0) \leq A\big(1 - \sum_{i=1}^{m}\alpha_i\big)^{-1} \leq 0$. We get $y'(t) \leq 0$ for $t \in [0,1]$. Then

$$y(1) = \sum_{i=1}^{m}\beta_i y(\xi_i) \geq \sum_{i=1}^{m}\beta_i y(1).$$

So $y(1) \geq 0$. Then $y(t) > y(1) \geq 0$ for $t \in (0,1)$ since $y'(t) \leq 0$ on $[0,1]$. The proof is complete. $\square$

Lemma 2.8. *Suppose that* (H2)–(H4) *hold. If y is a solution of* (2.2)*, then*

$$y(t) = B_h + \int_0^t \phi^{-1}\Big(A_h - \int_0^s h(u)du\Big)ds,$$

with

$$\phi^{-1}(A_h) = \sum_{i=1}^{m}\alpha_i\phi^{-1}\Big(A_h - \int_0^{\xi_i} h(s)ds\Big) + A, \tag{2.3}$$

and

$$\begin{aligned}
B_h = \frac{1}{1-\sum_{i=1}^{m}\beta_i}\Big[&-\int_0^1 \phi^{-1}\Big(A_h - \int_0^s h(u)du\Big)ds \\
&+ \sum_{i=1}^{m}\beta_i\int_0^{\xi_i}\phi^{-1}\Big(A_h - \int_0^s h(u)du\Big)ds\Big],
\end{aligned}$$

where $a=\phi\left(\frac{A}{1-\sum_{i=1}^m\alpha_i}\right)$ *and*

$$b=\phi\Big(\frac{A\big(1+\sum_{i=1}^m\alpha_i\big)}{1-\sum_{i=1}^m\alpha_i}\Big)-\frac{\phi\big(\frac{1+\sum_{i=1}^m\alpha_i}{2}\big)}{1-\phi\big(\frac{1+\sum_{i=1}^m\alpha_i}{2}\big)}\int_0^1 h(s)ds.$$

Proof. It follows from (2.2) that

$$y(t)=y(0)+\int_0^t\phi^{-1}\Big(\phi(y'(0))-\int_0^s h(u)du\Big)ds,$$

and BCs in (2.2) imply that

$$y'(0)=\sum_{i=1}^m\alpha_i\phi^{-1}\Big(\phi(y'(0))-\int_0^{\xi_i}h(s)ds\Big)+A, \tag{2.4}$$

and

$$\begin{aligned}y(0)=\frac{1}{1-\sum_{i=1}^m\beta_i}\Big[&-\int_0^1\phi^{-1}\Big(\phi(y'(0))-\int_0^s h(u)du\Big)ds\\&+\sum_{i=1}^m\beta_i\int_0^{\xi_i}\phi^{-1}\Big(\phi(y'(0))-\int_0^s h(u)du\Big)ds\Big].\end{aligned}$$

Let

$$G(c)=\phi^{-1}(c)-\sum_{i=1}^m\alpha_i\phi^{-1}\Big(c-\int_0^{\xi_i}h(s)ds\Big)-A.$$

It is easy to see that

$$\begin{aligned}G(a)&=G\Big(\phi\Big(\frac{A}{1-\sum_{i=1}^m\alpha_i}\Big)\Big)\\&\ge\frac{A}{1-\sum_{i=1}^m\alpha_i}-\sum_{i=1}^m\alpha_i\phi^{-1}\Big(\phi\Big(\frac{A}{1-\sum_{i=1}^m\alpha_i}\Big)\Big)-A=0.\end{aligned}$$

On the other hand, one sees that

$$\begin{aligned}\frac{G(b)}{\phi^{-1}(b)}&=1-\sum_{i=1}^m\alpha_i\phi^{-1}\Big(1-\frac{\int_0^{\xi_i}h(s)ds}{b}\Big)-\frac{A}{\phi^{-1}(b)}\\&=1-\sum_{i=1}^m\alpha_i\phi^{-1}\Big(1-\frac{\int_0^{\xi_i}h(s)ds}{\phi\Big(\frac{A\big(1+\sum_{i=1}^m\alpha_i\big)}{1-\sum_{i=1}^m\alpha_i}\Big)-\frac{\phi\big(\frac{1+\sum_{i=1}^m\alpha_i}{2}\big)}{1-\phi\big(\frac{1+\sum_{i=1}^m\alpha_i}{2}\big)}\int_0^1h(s)ds}\Big)\\&\quad-\frac{A}{\phi^{-1}\Big(\phi\Big(\frac{A\big(1+\sum_{i=1}^m\alpha_i\big)}{1-\sum_{i=1}^m\alpha_i}\Big)-\frac{\phi\big(\frac{1+\sum_{i=1}^m\alpha_i}{2}\big)}{1-\phi\big(\frac{1+\sum_{i=1}^m\alpha_i}{2}\big)}\int_0^1h(s)ds\Big)}\\&\ge1-\sum_{i=1}^m\alpha_i\phi^{-1}\Big(1+\frac{1-\phi\big(\frac{1+\sum_{i=1}^m\alpha_i}{2}\big)}{\phi\big(\frac{1+\sum_{i=1}^m\alpha_i}{2}\big)}\Big)-\frac{1-\sum_{i=1}^m\alpha_i}{1+\sum_{i=1}^m\alpha_i}\\&=1-\sum_{i=1}^m\alpha_i\frac{2}{1+\sum_{i=1}^m\alpha_i}-\frac{1-\sum_{i=1}^m\alpha_i}{1+\sum_{i=1}^m\alpha_i}=0.\end{aligned}$$

Hence $G(b) \le 0$. It is easy to know that $\frac{G(c)}{\phi^{-1}(c)}$ is continuous and decreasing on $(-\infty, 0)$ and continuous and increasing on $(0, +\infty)$, Hence $G(a) \ge 0$ and $G(b) \le 0$ and

$$\lim_{c\to 0^+} \frac{G(c)}{\phi^{-1}(c)} = +\infty, \quad \lim_{c\to +\infty} \frac{G(c)}{\phi^{-1}(c)} = 1 - \sum_{i=1}^m \alpha_i > 0,$$

we get that there exists unique constant $A_h = \phi(y'(0)) \in [b, a]$ such that (2.3) holds. The proof is completed. □

Note $h = B/(1 - \sum_{i=1}^m \beta_i)$, and let $x(t) - h = y(t)$. Then (1.14) is transformed into the boundary-value problem

$$\begin{gathered} [\phi(y'(t))]' + f\big(t, y(t) + h, y'(t)\big) = 0, \quad t \in (0,1), \\ y'(0) - \sum_{i=1}^m \alpha_i y'(\xi_i) = A, \quad y(1) - \sum_{i=1}^m \beta_i y(\xi_i) = 0, \end{gathered} \tag{2.5}$$

Let

$$\begin{aligned} P_1 = \Big\{ & y \in X : y(t) \ge 0 \text{ for all } t \in [0,1],\ y'(t) \le 0 \text{ is decreasing on } [0,1], \\ & y(t) \ge \min\{t, (1-t)\} \max_{t\in[0,1]} y(t) \text{ for all } t \in [0,1] \Big\}. \end{aligned}$$

Then P_1 is a cone in X. Since

$$\begin{aligned} |y(t)| &= \Big| \frac{\sum_{i=1}^m \beta_i y(\xi_i) - \sum_{i=1}^m \beta_i y(1)}{1 - \sum_{i=1}^m \beta_i} \Big| \\ &\le \frac{\sum_{i=1}^m \beta_i (1-\xi_i)}{1 - \sum_{i=1}^m \beta_i} \max_{t\in[0,1]} |y'(t)| \\ &= \frac{\sum_{i=1}^m \beta_i (1-\xi_i)}{1 - \sum_{i=1}^m \beta_i} \gamma(y), \end{aligned}$$

we obtain

$$\max_{t\in[0,1]} |y(t)| \le \frac{\sum_{i=1}^m \beta_i (1-\xi_i)}{1 - \sum_{i=1}^m \beta_i} \gamma(y).$$

It is easy to see that there exists a constant $M > 0$ such that $\|y\| \le M\gamma(y)$ for all $y \in P_1$.

Define the nonlinear operator $T_1 : P_1 \to X$ by

$$(T_1 y)(t) = B_y + \int_0^t \phi^{-1}\Big(A_y - \int_0^s f(u, y(u) + h, y'(u))du\Big) ds, \quad y \in P_1,$$

where A_y satisfies

$$\phi^{-1}(A_y) = \sum_{i=1}^m \alpha_i \phi^{-1}\Big(A_y - \int_0^{\xi_i} f(s, y(s) + h, y'(s))ds\Big) + A, \tag{2.6}$$

and B_y satisfies

$$\begin{aligned} B_y = \frac{1}{1 - \sum_{i=1}^m \beta_i} \Big(& -\int_0^1 \phi^{-1}\Big(A_y - \int_0^s f(u, y(u) + h, y'(u))du\Big) ds \\ & + \sum_{i=1}^m \beta_i \int_0^{\xi_i} \phi^{-1}\Big(A_y - \int_0^s f(u, y(u) + h, y'(u))du\Big) ds\Big). \end{aligned}$$

Then for $y \in P_1$,

$$\begin{aligned}(T_1y)(t) = &-\int_t^1 \phi^{-1}\Big(A_y - \int_0^s f(u, y(u)+h, y'(u))du\Big)ds\\ &- \frac{1}{1-\sum_{i=1}^m \beta_i}\sum_{i=1}^m \beta_i \int_{\xi_i}^1 \phi^{-1}\Big(A_y - \int_0^s f(u, y(u)+h, y'(u))du\Big)ds.\end{aligned}$$

Lemma 2.9. *Suppose that* (H1)–(H3) *hold. It is easy to show that*

- (i) *the following equalities hold:*
$$[\phi((T_1y)'(t))]' + f\big(t, y(t)+h, y'(t)\big) = 0, \quad t \in (0,1),$$
$$(T_1y)'(0) - \sum_{i=1}^m \alpha_i (T_1y)'(\xi_i) = A, \quad (T_1y)(1) - \sum_{i=1}^m \beta_i (T_1y)(\xi_i) = 0;$$
- (ii) $T_1y \in P_1$ *for each* $y \in P_1$*;*
- (iii) x *is a solution of* (1.14) *if and only if* $x = y+h$ *and* y *is a solution of the operator equation* $y = T_1y$ *in* P_1*;*
- (iv) $T_1 : P_1 \to P_1$ *is completely continuous.*

Proof. The proofs of (i), (ii) and (iii) are simple. To prove (iv), it suffices to prove that T_1 is continuous and T_1 is compact. We divide the proof into two steps:
Step 1. Prove that T_1 is continuous about y. Suppose $y_n \in X$ and $y_n \to y_0 \in X$. Let A_{y_n} be decided by (2.3) corresponding to y_n for $n = 0, 1, 2, \dots$. We will prove that $A_{y_n} \to A_{y_0}$ as n tends to infinity.

Since $y_n \to y_0$ uniformly on [0,1] and f is continuous, we have that for $\epsilon = 1$, there exists N, when $n > N$, for each $t \in [0,1]$, such that

$$0 \le f(t, y_n(t)+h, y_n'(t)) \le 1 + f(t, y_0(t)+h, y_0'(t)) \le 1 + \max_{t\in[0,1]} f(t, y_0(t)+h, y_0'(t)).$$

Hence Lemma 2.8 implies that A_{y_n} is an element in the interval

$$\begin{aligned}&\Big[\phi\Big(\frac{A(1+\sum_{i=1}^m \alpha_i)}{1-\sum_{i=1}^m \alpha_i}\Big) - \frac{\phi\big(\frac{1+\sum_{i=1}^m \alpha_i}{2}\big)}{1-\phi\big(\frac{1+\sum_{i=1}^m \alpha_i}{2}\big)}\int_0^1 f(s, y_n(s)+h, y_n'(s))ds,\\ &\phi\Big(\frac{A}{1-\sum_{i=1}^m \alpha_i}\Big)\Big]\\ &\subseteq \Big[\phi\Big(\frac{A\big(1+\sum_{i=1}^m \alpha_i\big)}{1-\sum_{i=1}^m \alpha_i}\Big) - \frac{\phi\big(\frac{1+\sum_{i=1}^m \alpha_i}{2}\big)}{1-\phi\big(\frac{1+\sum_{i=1}^m \alpha_i}{2}\big)}\Big(1 + \max_{t\in[0,1]} f(t, y_0(t)+h, y_0'(t))\Big),\\ &\quad \phi\Big(\frac{A}{1-\sum_{i=1}^m \alpha_i}\Big)\Big].\end{aligned}$$

It follows that $\{A_{y_n}\}$ is bounded. If $\{A_{y_n}\}$ does not converge to A_{y_0}, then there exist two subsequences $\{A^1_{y_{n_k}}\}$ and $\{A^2_{y_{n_k}}\}$ of $\{A_{y_n}\}$ with

$$A^1_{y_{n_k}} \to C_1, \ \ A^2_{y_{n_k}} \to C_2, \quad k \to +\infty, \quad C_1 \ne C_2.$$

By the construction of A_{y_n},

$$\phi^{-1}(A^1_{y_{n_k}}) = \sum_{i=1}^m \alpha_i \phi^{-1}\Big(A^1_{y_{n_k}} - \int_0^{\xi_i} f(s, y_{n_k}(s)+h, y'_{n_k}(s))ds\Big) + A.$$

Since $f(t, y_n(t)+h, y_n'(t))$ is uniformly bounded, by Lebesgue's dominated convergence theorem, letting $k \to +\infty$, we get

$$\phi^{-1}(C_1) = \sum_{i=1}^{m} \alpha_i \phi^{-1}\Big(C_1 - \int_0^{\xi_i} f(s, y_0(s)+h, y_0'(s))ds\Big) + A.$$

Since A_{y_0} satisfies

$$\phi^{-1}(A_{y_0}) = \sum_{i=1}^{m} \alpha_i \phi^{-1}\Big(A_{y_0} - \int_0^{\xi_i} f(s, y_0(s)+h, y_0'(s))ds\Big) + A,$$

Lemma 2.8 implies that $A_{y_0} = C_1$. Similarly, we can prove that $A_{y_0} = C_2$. This contradicts to $C_1 \neq C_2$. Therefore for each $y_n \to y_0$, we have $A_{y_n} \to A_{y_0}$. It follows that A_y is continuous about y. So the continuity of T_1 is obvious.

Step 2. Prove that T_1 is compact. Let $\Omega \subseteq P_1$ bet a bounded set. Suppose that $\Omega \subseteq \{y \in P_1 : \|y\| \leq M\}$. For $y \in \Omega$, we have

$$0 \leq \int_0^1 f(s, y(s)+h, y'(s))ds \leq \max_{t\in[0,1], u\in[h,M+h], v\in[-M,M]} f(t,u,v) =: D.$$

It follows from the definition of T_1 and Lemma 2.8 that

$$\begin{aligned}
&|(T_1y)(t)| \\
&= \Big| -\int_t^1 \phi^{-1}\Big(A_y - \int_0^s f(u, y(u)+h, y'(u))du\Big)ds \\
&\quad - \frac{1}{1-\sum_{i=1}^m \beta_i} \sum_{i=1}^{m} \beta_i \int_{\xi_i}^1 \phi^{-1}\Big(A_y - \int_0^s f(u, y(u)+h, y'(u))du\Big)ds\Big| \\
&\leq \int_0^1 \phi^{-1}\Big(\phi\Big(\frac{-A\big(1+\sum_{i=1}^m \alpha_i\big)}{1-\sum_{i=1}^m \alpha_i}\Big) + \frac{\phi\big(\frac{1+\sum_{i=1}^m \alpha_i}{2}\big)}{1-\phi\big(\frac{1+\sum_{i=1}^m \alpha_i}{2}\big)} \int_0^1 f(u, y(u)+h, y'(u))du \\
&\quad + \int_0^s f(u, y(u)+h, y'(u))du\Big)ds \\
&\quad + \frac{1}{1-\sum_{i=1}^m \beta_i} \sum_{i=1}^{m} \beta_i \int_{\xi_i}^1 \phi^{-1}\Big(\phi\Big(\frac{-A\big(1+\sum_{i=1}^m \alpha_i\big)}{1-\sum_{i=1}^m \alpha_i}\Big) \\
&\quad + \frac{\phi\big(\frac{1+\sum_{i=1}^m \alpha_i}{2}\big)}{1-\phi\big(\frac{1+\sum_{i=1}^m \alpha_i}{2}\big)} \int_0^1 f(u, y(u)+h, y'(u))du + \int_0^s f(u, y(u)+h, y'(u))du\Big)ds \\
&\leq \phi^{-1}\Big(\phi\Big(\frac{-A\big(1+\sum_{i=1}^m \alpha_i\big)}{1-\sum_{i=1}^m \alpha_i}\Big) + \frac{D}{1-\phi\big(\frac{1+\sum_{i=1}^m \alpha_i}{2}\big)}\Big) \\
&\quad + \frac{\sum_{i=1}^m \beta_i(1-\xi_i)}{1-\sum_{i=1}^m \beta_i} \phi^{-1}\Big(\phi\Big(\frac{-A\big(1+\sum_{i=1}^m \alpha_i\big)}{1-\sum_{i=1}^m \alpha_i}\Big) + \frac{D}{1-\phi\big(\frac{1+\sum_{i=1}^m \alpha_i}{2}\big)}\Big) \\
&= \phi^{-1}(E) + \frac{\sum_{i=1}^m \beta_i(1-\xi_i)}{1-\sum_{i=1}^m \beta_i} \phi^{-1}(E),
\end{aligned}$$

$$\begin{aligned}
&|(T_1y)'(t)| \\
&= \Big|\phi^{-1}\Big(\phi(A_y) - \int_0^t f(u, y(u)+h, y'(u))du\Big) ds\Big|
\end{aligned}$$

$$
\begin{aligned}
&\le \phi^{-1}\Big(\phi\left(\frac{-A\big(1+\sum_{i=1}^m\alpha_i\big)}{1-\sum_{i=1}^m\alpha_i}\right)\\
&\quad+\frac{\phi\big(\frac{1+\sum_{i=1}^m\alpha_i}{2}\big)}{1-\phi\big(\frac{1+\sum_{i=1}^m\alpha_i}{2}\big)}\int_0^1 f(u,y(u)+h,y'(u))du+\int_0^1 f(u,y(u)+h,y'(u))du\Big)ds\\
&\le \phi^{-1}\Big(\phi\Big(\frac{-A\big(1+\sum_{i=1}^m\alpha_i\big)}{1-\sum_{i=1}^m\alpha_i}\Big)+\frac{D}{1-\phi\big(\frac{1+\sum_{i=1}^m\alpha_i}{2}\big)}\Big)\\
&=\phi^{-1}(E),
\end{aligned}
$$

where

$$
E=\phi\Big(\frac{-A\big(1+\sum_{i=1}^m\alpha_i\big)}{1-\sum_{i=1}^m\alpha_i}\Big)+\frac{D}{1-\phi\big(\frac{1+\sum_{i=1}^m\alpha_i}{2}\big)}.
$$

For the uniform continuity of ϕ on the interval $[-E,E]$, for each $\epsilon>0$, there exists a $\rho>0$ such that

$$
|\phi^{-1}(Y_1)-\phi^{-1}(Y_2)|<\epsilon,\quad Y_1,Y_2\in[-E,E],\quad |Y_1-Y_2|<\rho.
$$

Put

$$
Y_1=\phi(A_y)-\int_0^{t_1} f(u,y(u)+h,y'(u))du,\quad Y_2=\phi(A_y)-\int_0^{t_2} f(u,y(u)+h,y'(u))du.
$$

Since $|Y_1-Y_2|=|\int_{t_1}^{t_2} f(u,y(u)+h,y'(u))du|\le D|t_1-t_2|$, it is easy to see that there exists $\delta>0$ (independent of ϵ) such that $|Y_1-Y_2|<\rho$ for all t_1,t_2 with $|t_1-t_2|<\delta$. Hence there is $\delta>0$ (independent of ϵ) such that

$$
|(Ty)'(t_1)-(Ty)'(t_2)|=\phi^{-1}(Y_1)-\phi^{-1}(Y_2)|<\epsilon,
$$

whenever $t_1,t_2\in[0,1]$ and $|t_1-t_2|<\delta$. This shows that $(Ty)(t)$ is equi-continuous on [0,1]. The Arzela-Askoli theorem guarantees that $T(\Omega)$ is relative compact, which means that T is compact. Hence the continuity and the compactness of T imply that T is completely continuous. □

Theorem 2.10. *Suppose that* (H1)–(H3) *hold and there exist positive constants* e_1,e_2,c*,*

$$
\begin{aligned}
L=&\int_0^1\phi^{-1}\left(1+\frac{\phi\big(1+\sum_{i=1}^m\alpha_i\big)}{\phi(2)-\phi\left(1+\sum_{i=1}^m\alpha_i\right)}+s\right)ds\\
&+\frac{1}{1-\sum_{i=1}^m\beta_i}\sum_{i=1}^m\beta_i\int_{\xi_i}^1\phi^{-1}\left(1+\frac{\phi\big(1+\sum_{i=1}^m\alpha_i\big)}{\phi(2)-\phi\left(1+\sum_{i=1}^m\alpha_i\right)}+s\right)ds,\\
&Q=\min\Big\{\phi(\frac{c}{L}),\ \frac{\phi(c)}{1+\frac{\phi(2)}{\phi(2)-\phi\left(1+\sum_{i=1}^m\alpha_i\right)}}\Big\};\\
&W=\phi\Big(\frac{e_2}{\sigma_0\int_k^{1-k}\phi^{-1}\left(s-k\right)ds}\Big);\quad E=\phi(\frac{e_1}{L}).
\end{aligned}
$$

such that

$$
c\ge\frac{e_2}{\sigma_0^2}>e_2>\frac{e_1}{\sigma_0}>e_1>0,\quad Q\ge\phi\Big(\frac{-A\big(1+\sum_{i=1}^m\alpha_i\big)}{1-\sum_{i=1}^m\alpha_i}\Big),\quad Q>W.
$$

If

(A1) $f(t,u,v) < Q$ *for all* $t \in [0,1], u \in [h, c+h], v \in [-c, c]$;
(A2) $f(t,u,v) > W$ *for all* $t \in [k, 1-k], u \in [e_2 + h, e_2/\sigma_0^2 + h], v \in [-c, c]$;
(A3) $f(t,u,v) < E$ *for all* $t \in [0,1], u \in [h, e_1/\sigma_0 + h], v \in [-c, c]$;

then (1.14) *has at least three decreasing positive solutions* x_1, x_2, x_3 *such that*

$$x_1(0) < e_1 + h, \quad x_2(1-k) > e_2 + h, \quad x_3(0) > e_1 + h, \quad x_3(1-k) < e_2 + h.$$

Remark 2.11. In paper [23], sufficient conditions are found for the existence of solutions of (1.9) based on the existence of lower and upper solutions with certain relations. Using the obtained results, under some other assumptions, the explicit ranges of values of λ_1 and λ_2 are presented with which (1.9) has a solution, has a positive solution, and has no solution, respectively. Furthermore, it is proved that the whole plane for λ_1 and λ_2 can be divided into two disjoint connected regions $\wedge E$ and $\wedge N$ such that (1.9) has a solution for $(\lambda_1, \lambda_2) \in \wedge E$ and has no solution for $(\lambda_1, \lambda_2) \in \wedge N$. When applying Theorem 2.10 to (1.9), it shows us that (1.9) has at least three decreasing positive solutions under the assumptions $\lambda_1 \le 0, \lambda_2 \ge 0$ and some other assumptions.

Remark 2.12. Consider the case $A \le 0$ and $B < 0$, when (H3) and (H4) hold, we can prove similarly that Lemma 2.7 and Lemma 2.8 are valid. Define the same operator T_1 on the cone P_1. Theorem 2.10 shows that (2.5) has at least three decreasing and positive solutions y_1, y_2, y_3. Hence (1.14) has at least three decreasing solutions $x_1 = y_1 + h, x_2 = y_2 + h$ and $x_3 = y_3 + h$, which need not be positive since $h = \frac{B}{1 - \sum_{i=1}^m \beta_i} < 0$. In cases $A > 0, B \le 0$ and $A > 0, B < 0$, the author could not get the sufficient conditions guaranteeing the existence of multiple positive solutions of (1.14).

Proof of Theorem 2.10. To apply Lemma 2.5, we prove that its hypotheses are satisfied. By the definitions, it is easy to see that α, ψ are two nonnegative continuous concave functionals on the cone P_1, γ, β, θ are three nonnegative continuous convex functionals on the cone P_1 and $\alpha(y) \le \beta(y)$ for all $y \in P_1$, there exist constants $M > 0$ such that $\|y\| \le M\gamma(y)$ for all $y \in P_1$. Lemma 2.8 implies that $x = x(t)$ is a positive solution of (1.14) if and only if $x(t) = y(t) + h$ and $y(t)$ is a solution of the operator equation $y = T_1 y$ and $T_1 : P_1 \to P_1$ is completely continuous.

Corresponding to Lemma 2.5,

$$c = c, \quad h = \sigma_0 e_1, \quad d = e_1, \quad a = e_2, \quad b = \frac{e_2}{\sigma_0}.$$

Now, we prove that all conditions of Lemma 2.5 hold. One sees that $0 < d < a$. The remainder is divided into four steps.

Step 1. Prove that $T_1 : \overline{P_{1c}} \to \overline{P_{1c}}$; For $y \in \overline{P_{1c}}$, we have $\|y\| \le c$. Then $0 \le y(t) \le c$ for $t \in [0,1]$ and $-c \le y'(t) \le c$ for all $t \in [0,1]$. So (A_1) implies that

$$f(t, y(t) + h, y'(t)) \le Q, \quad t \in [0,1].$$

It follows from Lemma 2.9 that $T_1 y \in P_1$. Then Lemma 2.8 implies

$$\begin{aligned} 0 &\le (T_1 y)(t) \\ &= -\int_t^1 \phi^{-1}\Big(A_y - \int_0^s f(u, y(u) + h, y'(u))du\Big)ds \\ &\quad - \frac{1}{1 - \sum_{i=1}^m \beta_i} \sum_{i=1}^m \beta_i \int_{\xi_i}^1 \phi^{-1}\Big(A_y - \int_0^s f(u, y(u) + h, y'(u))du\Big)ds \end{aligned}$$

$$\begin{aligned}
&\le \int_0^1 \phi^{-1}\Big(\phi\Big(\frac{-A\big(1+\sum_{i=1}^m \alpha_i\big)}{1-\sum_{i=1}^m \alpha_i}\Big) + \frac{\phi\big(\frac{1+\sum_{i=1}^m \alpha_i}{2}\big)}{1-\phi\big(\frac{1+\sum_{i=1}^m \alpha_i}{2}\big)} \\
&\quad \times \int_0^1 f(u,y(u)+h,y'(u))du + \int_0^s f(u,y(u)+h,y'(u))du\Big)ds \\
&\quad + \frac{1}{1-\sum_{i=1}^m \beta_i}\sum_{i=1}^m \beta_i \int_{\xi_i}^1 \phi^{-1}\Big(\phi\Big(\frac{-A(1+\sum_{i=1}^m \alpha_i)}{1-\sum_{i=1}^m \alpha_i}\Big) \\
&\quad + \frac{\phi\big(\frac{1+\sum_{i=1}^m \alpha_i}{2}\big)}{1-\phi\big(\frac{1+\sum_{i=1}^m \alpha_i}{2}\big)}\int_0^1 f(u,y(u)+h,y'(u))du + \int_0^s f(u,y(u)+h,y'(u))du\Big)ds \\
&\le \int_0^1 \phi^{-1}\Big(\phi\Big(\frac{-A\big(1+\sum_{i=1}^m \alpha_i\big)}{1-\sum_{i=1}^m \alpha_i}\Big) + \frac{\phi\big(\frac{1+\sum_{i=1}^m \alpha_i}{2}\big)}{1-\phi\Big(\frac{1+\sum_{i=1}^m \alpha_i}{2}\Big)}Q + Qs\Big)ds \\
&\quad + \frac{1}{1-\sum_{i=1}^m \beta_i}\sum_{i=1}^m \beta_i \int_{\xi_i}^1 \phi^{-1}\Big(\phi\Big(\frac{-A\big(1+\sum_{i=1}^m \alpha_i\big)}{1-\sum_{i=1}^m \alpha_i}\Big) \\
&\quad + \frac{\phi\big(\frac{1+\sum_{i=1}^m \alpha_i}{2}\big)}{1-\phi\big(\frac{1+\sum_{i=1}^m \alpha_i}{2}\big)}Q + Qs\Big)ds \\
&\le \int_0^1 \phi^{-1}\Big(Q + \frac{\phi\big(\frac{1+\sum_{i=1}^m \alpha_i}{2}\big)}{1-\phi\big(\frac{1+\sum_{i=1}^m \alpha_i}{2}\big)}Q + Qs\Big)ds \\
&\quad + \frac{1}{1-\sum_{i=1}^m \beta_i}\sum_{i=1}^m \beta_i \int_{\xi_i}^1 \phi^{-1}\Big(Q + \frac{\phi\big(\frac{1+\sum_{i=1}^m \alpha_i}{2}\big)}{1-\phi\big(\frac{1+\sum_{i=1}^m \alpha_i}{2}\big)}Q + Qs\Big)ds \\
&= \phi^{-1}(Q)\Big[\int_0^1 \phi^{-1}\Big(1 + \frac{\phi\Big(1+\sum_{i=1}^m \alpha_i\Big)}{\phi(2)-\phi\Big(1+\sum_{i=1}^m \alpha_i\Big)} + s\Big)ds \\
&\quad + \frac{1}{1-\sum_{i=1}^m \beta_i}\sum_{i=1}^m \beta_i \int_{\xi_i}^1 \phi^{-1}\Big(1 + \frac{\phi\big(1+\sum_{i=1}^m \alpha_i\big)}{\phi(2)-\phi\left(1+\sum_{i=1}^m \alpha_i\right)} + s\Big)ds\Big] \\
&\le c.
\end{aligned}$$

Similarly to the above discussion, we have from Lemma 2.7 that

$$|A_y| \le \Big|\phi\Big(\frac{-A\big(1+\sum_{i=1}^m \alpha_i\big)}{1-\sum_{i=1}^m \alpha_i}\Big) - \frac{1}{1-\phi\big(\frac{1+\sum_{i=1}^m \alpha_i}{2}\big)}\int_0^1 f(u,y(u)+h,y'(u))du\Big|.$$

Then

$$\begin{aligned}
&|(T_1y)'(t)| \\
&\le |(T_1y)'(0)| = |\phi^{-1}(A_y)\| \\
&\le \phi^{-1}\Big(\phi\Big(\frac{-A\big(1+\sum_{i=1}^m \alpha_i\big)}{1-\sum_{i=1}^m \alpha_i}\Big) + \frac{1}{1-\phi\big(\frac{1+\sum_{i=1}^m \alpha_i}{2}\big)}\int_0^1 f(u,y(u)+h,y'(u))du\Big) \\
&\le \phi^{-1}\Big(\phi\Big(\frac{-A\big(1+\sum_{i=1}^m \alpha_i\big)}{1-\sum_{i=1}^m \alpha_i}\Big) + \frac{1}{1-\phi\big(\frac{1+\sum_{i=1}^m \alpha_i}{2}\big)}Q\Big)
\end{aligned}$$

$$\le \phi^{-1}\Big(Q + \frac{\phi(2)}{\phi(2) - \phi\big(1 + \sum_{i=1}^m \alpha_i\big)} Q\Big) \le c.$$

It follows that $\|T_1 y\| = \max\{\max_{t\in[0,1]} |(T_1y)(t)|, \max_{t\in[0,1]} |(T_1y)'(t)|\} \le c$. Then $T_1 : \overline{P_{1c}} \to \overline{P_{1c}}$.

Step 2. Prove that

$$\{y \in P_1(\gamma,\theta,\alpha;a,b,c) | \alpha(y) > a\} = \{y \in P_1\big(\gamma,\theta,\alpha;e_2,\frac{e_2}{\sigma_0},c\big) | \alpha(y) > e_2\} \ne \emptyset$$

and $\alpha(T_1y) > e_2$ for every $y \in P_1\big(\gamma,\theta,\alpha;e_2,\frac{e_2}{\sigma_0},c\big)$.

Choose $y(t) = \frac{e_2}{2\sigma_0}$ for all $t \in [0,1]$. Then $y \in P_1$ and

$$\alpha(y) = \frac{e_2}{2\sigma_0} > e_2,\ \theta(y) = \frac{e_2}{2\sigma_0} \le \frac{e_2}{\sigma_0},\ \gamma(y) = 0 < c.$$

It follows that $\{y \in P_1(\gamma,\theta,\alpha;a,b,c) : \alpha(y) > a\} \ne \emptyset$. For $y \in P_1(\gamma,\theta,\alpha;a,b,c)$, one has

$$\alpha(y) = \min_{t\in[k,1-k]} y(t) \ge e_2, \quad \theta(y) = \max_{t\in[k,1-k]} y(t) \le \frac{e_2}{\sigma_0}, \quad \gamma(y) = \max_{t\in[0,1]} |y'(t)| \le c.$$

Then

$$e_2 \le y(t) \le \frac{e_2}{\sigma_0^2}, \quad t \in [k,1-k],\ |y'(t)| \le c.$$

Thus (A2) implies

$$f(t, y(t)+h, |y'(t)|) \ge W, \quad n \in [k,1-k].$$

Since

$$\alpha(T_1y) = \min_{t\in[k,1-k]} (T_1y)(t) \ge \sigma_0 \max_{t\in[0,1]} (T_1y)(t),$$

we get

$$\begin{aligned}
\alpha(T_1y) &\ge \sigma_0\Big[-\int_0^1 \phi^{-1}\Big(A_y - \int_0^s f(u,y(u)+h,y'(u))du\Big)ds \\
&\quad - \frac{1}{1-\sum_{i=1}^m \beta_i}\sum_{i=1}^m \beta_i \int_{\xi_i}^1 \phi^{-1}\Big(A_y - \int_0^s f(u,y(u)+h,y'(u))du\Big)ds\Big] \\
&\ge \sigma_0\Big[-\int_0^1 \phi^{-1}\Big(A_y - \int_0^s f(u,y(u)+h,y'(u))du\Big)ds\Big] \\
&\ge \sigma_0\Big[\int_0^1 \phi^{-1}\left(\phi\left(\frac{-A}{1-\sum_{i=1}^m \alpha_i}\right) + \int_0^s f(u,y(u)+h,y'(u))du\right)ds\Big] \\
&\ge \sigma_0\Big[\int_k^{1-k} \phi^{-1}\left(\phi\left(\frac{-A}{1-\sum_{i=1}^m \alpha_i}\right) + \int_k^s f(u,y(u)+h,y'(u))du\right)ds\Big] \\
&\ge \sigma_0 \int_k^{1-k} \phi^{-1}\big(W(s-k)\big)ds \\
&= e_2.
\end{aligned}$$

This completes Step 2.

Step 3. Prove that $\{y \in Q(\gamma,\theta,\psi;h,d,c) : \beta(y) < d\}$ which is equal to $\{y \in Q(\gamma,\theta,\psi;\sigma_0 e_1,e_1,c) : \beta(y) < e_1\}$ is not empty, and

$$\beta(T_1y) < e_1 \quad \text{for every } y \in Q(\gamma,\theta,\psi;h,d,c) = Q\left(\gamma,\theta,\psi;\sigma_0 e_1,e_1,c\right);$$

Choose $y(t)=\sigma_0 e_1$. Then $y\in P_1$, and

$$\psi(y)=\sigma_0 e_1\ge h,\quad \beta(y)=\theta(y)=\sigma_0 e_1<e_1=d,\quad \gamma(y)=0\le c.$$

It follows that $\{y\in Q(\gamma,\theta,\psi;h,d,c)|\beta(y)<d\}\ne\emptyset$.

For $y\in Q(\gamma,\theta,\psi;h,d,c)$, one has

$$\psi(y)=\min_{t\in[k,1-k]}y(t)\ge h=e_1\sigma_0,\quad \theta(y)=\max_{t\in[k,1-k]}y(t)\le d=e_1,$$
$$\gamma(y)=\max_{t\in[0,1]}|y'(t)|\le c.$$

Hence $0\le y(t)\le\frac{e_1}{\sigma_0}$ and $-c\le y'(t)\le c$, for $t\in[0,1]$. Then (A3) implies

$$f(t,y(t)+h,|y'(t)|)\le E,\quad t\in[0,1].$$

So

$$\begin{aligned}
&\beta(T_1y)\\
&=\max_{t\in[0,1]}(T_1y)(t)\\
&=-\int_0^1\phi^{-1}\Big(A_y-\int_0^s f(u,y(u)+h,y'(u))du\Big)ds\\
&\quad-\frac{1}{1-\sum_{i=1}^m\beta_i}\sum_{i=1}^m\beta_i\int_{\xi_i}^1\phi^{-1}\Big(A_y-\int_0^s f(u,y(u)+h,y'(u))du\Big)ds\\
&\le\int_0^1\phi^{-1}\Big(\phi\Big(\frac{-A\big(1+\sum_{i=1}^m\alpha_i\big)}{1-\sum_{i=1}^m\alpha_i}\Big)+\frac{\phi\big(\frac{1+\sum_{i=1}^m\alpha_i}{2}\big)}{1-\phi\big(\frac{1+\sum_{i=1}^m\alpha_i}{2}\big)}\int_0^1 f(u,y(u)+h,y'(u))du\\
&\quad+\int_0^s f(u,y(u)+h,y'(u))du\Big)ds\\
&\quad+\frac{1}{1-\sum_{i=1}^m\beta_i}\sum_{i=1}^m\beta_i\int_{\xi_i}^1\phi^{-1}\Big(\phi\Big(\frac{-A\big(1+\sum_{i=1}^m\alpha_i\big)}{1-\sum_{i=1}^m\alpha_i}\Big)\\
&\quad+\frac{\phi\big(\frac{1+\sum_{i=1}^m\alpha_i}{2}\big)}{1-\phi\big(\frac{1+\sum_{i=1}^m\alpha_i}{2}\big)}\int_0^1 f(u,y(u)+h,y'(u))du+\int_0^s f(u,y(u)+h,y'(u))du\Big)ds\\
&\le\int_0^1\phi^{-1}\Big(\phi\Big(\frac{-A\big(1+\sum_{i=1}^m\alpha_i\big)}{1-\sum_{i=1}^m\alpha_i}\Big)+\frac{\phi\left(\frac{1+\sum_{i=1}^m\alpha_i}{2}\right)}{1-\phi\left(\frac{1+\sum_{i=1}^m\alpha_i}{2}\right)}E+Es\Big)ds\\
&\quad+\frac{1}{1-\sum_{i=1}^m\beta_i}\sum_{i=1}^m\beta_i\int_{\xi_i}^1\phi^{-1}\Big(\phi\Big(\frac{-A\big(1+\sum_{i=1}^m\alpha_i\big)}{1-\sum_{i=1}^m\alpha_i}\Big)\\
&\quad+\frac{\phi\big(\frac{1+\sum_{i=1}^m\alpha_i}{2}\big)}{1-\phi\big(\frac{1+\sum_{i=1}^m\alpha_i}{2}\big)}E+Es\Big)ds\\
&\le\int_0^1\phi^{-1}\Big(E+\frac{\phi\big(\frac{1+\sum_{i=1}^m\alpha_i}{2}\big)}{1-\phi\big(\frac{1+\sum_{i=1}^m\alpha_i}{2}\big)}E+Es\Big)ds\\
&\quad+\frac{1}{1-\sum_{i=1}^m\beta_i}\sum_{i=1}^m\beta_i\int_{\xi_i}^1\phi^{-1}\Big(E+\frac{\phi\big(\frac{1+\sum_{i=1}^m\alpha_i}{2}\big)}{1-\phi\big(\frac{1+\sum_{i=1}^m\alpha_i}{2}\big)}E+Es\Big)ds
\end{aligned}$$

$$
\begin{aligned}
&= \phi^{-1}(E)\Big[\int_0^1 \phi^{-1}\Big(1 + \frac{\phi\big(1 + \sum_{i=1}^m \alpha_i\big)}{\phi(2) - \phi\left(1 + \sum_{i=1}^m \alpha_i\right)} + s\Big)ds \\
&\quad + \frac{1}{1 - \sum_{i=1}^m \beta_i} \sum_{i=1}^m \beta_i \int_{\xi_i}^1 \phi^{-1}\Big(1 + \frac{\phi\big(1 + \sum_{i=1}^m \alpha_i\big)}{\phi(2) - \phi\left(1 + \sum_{i=1}^m \alpha_i\right)} + s\Big)ds\Big] \\
&= e_1 = d.
\end{aligned}
$$

This completes Step 3.

Step 4. Prove that $\alpha(T_1y) > a$ for $y \in P_1(\gamma, \alpha; a, c)$ with $\theta(T_1y) > b$; For $y \in P_1(\gamma, \alpha; a, c) = P_1(\gamma, \alpha; e_2, c)$ with $\theta(T_1y) > b = \frac{e_2}{\sigma_0}$, we have that $\alpha(y) = \min_{t\in[k,1-k]} y(t) \geq e_2$ and $\gamma(y) = \max_{t\in[0,1]} |y(t)| \leq c$ and $\max_{t\in[k,1-k]}(T_1y)(t) > \frac{e_2}{\sigma_0}$. Then

$$
\alpha(T_1y) = \min_{t\in[k,1-k]} (T_1y)(t) \geq \sigma_0\beta(T_1y) > \sigma_0\frac{e_2}{\sigma_0} = e_2 = a.
$$

This completes Step 4.

Step 5. Prove that $\beta(T_1y) < d$ for each $y \in Q(\gamma, \beta; d, c)$ with $\psi(T_1y) < h$. For $y \in Q(\gamma, \beta; d, c)$ with $\psi(T_1y) < d$, we have $\gamma(y) = \max_{t\in[0,1]} |y(t)| \leq c$ and $\beta(y) = \max_{t\in[0,1]} y(t) \leq d = e_1$ and $\psi(T_1y) = \min_{t\in[k,1-k]}(T_1y)(t) < h = e_1\sigma_0$. Then

$$
\beta(T_1y) = \max_{t\in[0,1]} (T_1y)(t) \leq \frac{1}{\sigma_0} \min_{t\in[k,1-k]} (T_1y)(t) < \frac{1}{\sigma_0} e_1\sigma_0 = e_1 = d.
$$

This completes the Step 5.

Then Lemma 2.5 implies that T_1 has at least three fixed points y_1, y_2 and y_3 such that

$$
\beta(y_1) < e_1, \quad \alpha(y_2) > e_2, \quad \beta(y_3) > e_1, \quad \alpha(y_3) < e_2.
$$

Hence (1.14) has three decreasing positive solutions x_1, x_2 and x_3 such that

$$
\begin{aligned}
&\max_{t\in[0,1]} x_1(t) < e_1 + h, \qquad \min_{t\in[k,1-k]} x_2(t) > e_2 + h, \\
&\max_{t\in[0,1]} x_3(t) > e_1 + h, \qquad \min_{t\in[k,1-k]} x_3(t) < e_2 + h.
\end{aligned}
$$

Hence $x_1(0) < e_1 + h$, $x_2(1-k) > e_2 + h$, $x_3(0) > e_1 + h$, $x_3(1-k) < e_2 + h$. □

2.2. **Positive Solutions of** (1.15)**.** Now we prove the existence of three positive increasing solutions of (1.15). We use the assumptions:

- (H5) $f : [0, 1] \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is continuous with $f(t, c+h, 0) \not\equiv 0$ on each sub-interval of [0,1] for all $c \geq 0$, where $h = \frac{A}{1-\sum_{i=1}^m \alpha_i}$;
- (H6) $A \geq 0, B \geq 0$;
- (H7) $\alpha_i \geq 0$, $\beta_i \geq 0$ satisfy $\sum_{i=1}^m \alpha_i \leq 1$, $\sum_{i=1}^m \beta_i < 1$;

Consider the boundary-value problem

$$
\begin{gathered}
[\phi(y'(t))]' + h(t) = 0, \quad t \in (0, 1), \\
y(0) - \sum_{i=1}^m \alpha_i y(\xi_i) = 0, \quad y'(1) - \sum_{i=1}^m \beta_i y'(\xi_i) = B.
\end{gathered} \tag{2.7}
$$

Lemma 2.13. *Suppose that* (H4), (H6), (H7) *hold. If* y *is a solution of* (2.7)*, then* y *is increasing and positive on* $(0, 1)$.

Proof. Suppose y satisfies (2.7). It follows from the assumptions that y' is decreasing on $[0,1]$. Then the BCs in (2.7) and (H4) imply

$$y'(1)=\sum_{i=1}^m \beta_i y'(\xi_i)+B \geq \sum_{i=1}^m \beta_i y'(1).$$

It follows that $y'(1)\geq 0$. We get that $y'(t)\geq 0$ for $t\in[0,1]$. Then

$$y(0)=\sum_{i=1}^m \alpha_i y(\xi_i)\geq \sum_{i=1}^m \alpha_i y(0).$$

It follows that $y(0)\geq 0$. Then $y(t)>y(0)\geq 0$ for $t\in(0,1)$ since $y'(t)\geq 0$ for all $t\in[0,1]$. The proof is complete. □

Lemma 2.14. *Suppose that* (H4), (H6), (H7) *hold. If y is a solution of* (2.7)*, then*

$$y(t)=B_h+\int_0^t \phi^{-1}\Big(A_h+\int_s^1 h(u)du\Big)ds,$$

with

$$\phi^{-1}(A_h)=\sum_{i=1}^m \beta_i\phi^{-1}\Big(A_h+\int_{\xi_i}^1 h(s)ds\Big)+B, \tag{2.8}$$

and

$$B_h=\frac{1}{1-\sum_{i=1}^m\alpha_i}\sum_{i=1}^m \alpha_i\int_0^{\xi_i}\phi^{-1}\Big(A_h+\int_s^1 h(u)du\Big)ds,$$

where

$$a=\phi\Big(\frac{B}{1-\sum_{i=1}^m\beta_i}\Big),$$

$$b=\phi\Big(\frac{B\big(1+\sum_{i=1}^m\beta_i\big)}{1-\sum_{i=1}^m\beta_i}\Big)+\frac{\phi\big(\frac{1+\sum_{i=1}^m\beta_i}{2}\big)}{1-\phi\big(\frac{1+\sum_{i=1}^m\beta_i}{2}\big)}\int_0^1 h(s)ds.$$

Proof. It follows from (2.7) that

$$y(t)=y(0)+\int_0^t\phi^{-1}\Big(\phi(y'(1))+\int_s^1 h(u)du\Big)ds,$$

and the BCs in (2.7) imply

$$y'(1)=\sum_{i=1}^m \beta_i\phi^{-1}\left(\phi(y'(1))+\int_{\xi_i}^1 h(s)ds\right)+B,$$

$$y(0)=\frac{1}{1-\sum_{i=1}^m\alpha_i}\sum_{i=1}^m \alpha_i\int_0^{\xi_i}\phi^{-1}\Big(\phi(y'(1))+\int_s^1 h(u)du\Big)ds.$$

Let

$$G(c)=\phi^{-1}(c)-\sum_{i=1}^m\beta_i\phi^{-1}\Big(c+\int_{\xi_i}^1 h(s)ds\Big)-B.$$

It is easy to see that

$$\begin{aligned}G(a)&=G\Big(\phi\Big(\frac{B}{1-\sum_{i=1}^m\beta_i}\Big)\Big)\\&\leq\frac{B}{1-\sum_{i=1}^m\beta_i}-\sum_{i=1}^m\beta_i\phi^{-1}\Big(\phi\Big(\frac{B}{1-\sum_{i=1}^m\beta_i}\Big)\Big)-B\end{aligned}$$

$$= 0.$$

On the other hand, one sees that

$$\begin{aligned}
\frac{G(b)}{\phi^{-1}(b)} &= 1-\sum_{i=1}^m \beta_i\phi^{-1}\Big(1+\frac{\int_{\xi_i}^1 h(s)ds}{b}\Big)-\frac{B}{\phi^{-1}(b)}\\
&= 1-\sum_{i=1}^m \beta_i\phi^{-1}\Big(1+\frac{\int_{\xi_i}^1 h(s)ds}{\phi\Big(\frac{B\left(1+\sum_{i=1}^m\beta_i\right)}{1-\sum_{i=1}^m\beta_i}\Big)+\frac{\phi\left(\frac{1+\sum_{i=1}^m\beta_i}{2}\right)}{1-\phi\left(\frac{1+\sum_{i=1}^m\beta_i}{2}\right)}\int_0^1 h(s)ds}\Big)\\
&\quad -\frac{B}{\phi^{-1}\Big(\phi\Big(\frac{B\left(1+\sum_{i=1}^m\beta_i\right)}{1-\sum_{i=1}^m\beta_i}\Big)+\frac{\phi\left(\frac{1+\sum_{i=1}^m\beta_i}{2}\right)}{1-\phi\left(\frac{1+\sum_{i=1}^m\beta_i}{2}\right)}\int_0^1 h(s)ds\Big)}\\
&\ge 1-\sum_{i=1}^m \beta_i\phi^{-1}\Big(1+\frac{1-\phi\Big(\frac{1+\sum_{i=1}^m\beta_i}{2}\Big)}{\phi\big(\frac{1+\sum_{i=1}^m\beta_i}{2}\big)}\Big)-\frac{1-\sum_{i=1}^m\beta_i}{1+\sum_{i=1}^m\beta_i}=0.
\end{aligned}$$

Hence $G(b)\ge 0$. It is easy to know that $\frac{G(c)}{\phi^{-1}(c)}$ is continuous and decreasing on $(-\infty,0)$ and continuous and increasing on $(0,+\infty)$, Hence $G(a)\le 0$ and $G(b)\ge 0$ and

$$\lim_{c\to 0^-}\frac{G(c)}{\phi^{-1}(c)}=+\infty,\quad \lim_{c\to-\infty}\frac{G(c)}{\phi^{-1}(c)}=1-\sum_{i=1}^m\beta_i>0.$$

Then there exists unique constant $A_h=\phi(y'(1))\in[a,b]$ such that (2.8) holds. The proof is complete. □

Note $h=\frac{A}{1-\sum_{i=1}^m\alpha_i}$, and $x(t)-h=y(t)$. Then (1.15) is transformed into the boundary-value problem

$$\begin{gathered}
[\phi(y'(t))]'+f\big(t,y(t)+h,y'(t)\big)=0,\quad t\in(0,1),\\
y(0)-\sum_{i=1}^m\alpha_i y(\xi_i)=0,\quad y'(1)-\sum_{i=1}^m\beta_i y'(\xi_i)=B,
\end{gathered}\tag{2.9}$$

Let

$$\begin{aligned}
P_2=\big\{y\in X: y(t)\ge 0 \text{ for all } t\in[0,1], y'(t) \text{ is decreasing on } [0,1],\\
y(t)\ge\min\{t,(1-t)\}\max_{t\in[0,1]}y(t) \text{ for all } t\in[0,1]\big\}.
\end{aligned}$$

Then P_2 is a cone in X. Since

$$\begin{aligned}
|y(t)| &= \Big|\frac{\sum_{i=1}^m\alpha_i y(\xi_i)-\sum_{i=1}^m\alpha_i y(0)}{1-\sum_{i=1}^m\alpha_i}\Big|\\
&\le \frac{\sum_{i=1}^m\alpha_i\xi_i}{1-\sum_{i=1}^m\alpha_i}\max_{t\in[0,1]}|y'(t)|=\frac{\sum_{i=1}^m\alpha_i\xi_i}{1-\sum_{i=1}^m\alpha_i}\gamma(y),
\end{aligned}$$

we have

$$\max_{t\in[0,1]}|y(t)|\le\frac{\sum_{i=1}^m\alpha_i\xi_i}{1-\sum_{i=1}^m\alpha_i}\gamma(y).$$

It is easy to see that there exists a constant $M>0$ such that $\|y\|\le M\gamma(y)$ for all $y\in P_1$.

Define the nonlinear operator $T_2 : P_2 \to X$ by

$$(T_2y)(t) = B_y + \int_0^t \phi^{-1}\Big(A_y + \int_s^1 f(u, y(u) + h, y'(u))du\Big)ds, \quad y \in P_2,$$

where A_y satisfies

$$\phi^{-1}(A_y) = \sum_{i=1}^m \beta_i\phi^{-1}\left(A_y + \int_{\xi_i}^1 f(s, y(s) + h, y'(s))ds\right) + B, \tag{2.10}$$

and B_y satisfies

$$B_y = \frac{1}{1 - \sum_{i=1}^m \alpha_i}\sum_{i=1}^m \alpha_i \int_0^{\xi_i} \phi^{-1}\left(A_y + \int_s^1 f(u, y(u) + h, y'(u))ds\right)ds.$$

Then for $y \in P_2$,

$$\begin{aligned}(T_2y)(t) &= \int_0^t \phi^{-1}\Big(A_y + \int_s^1 f(u, y(u) + h, y'(u))du\Big)ds \\ &\quad + \frac{1}{1 - \sum_{i=1}^m \alpha_i}\sum_{i=1}^m \alpha_i \int_0^{\xi_i} \phi^{-1}\left(A_y + \int_s^1 f(u, y(u) + h, y'(u))ds\right)ds\,.\end{aligned}$$

Lemma 2.15. *Suppose that* (H5), (H6), (H7) *hold. Then*

(i) *the following equalities hold:*

$$[\phi((T_2y)'(t))]' + f\big(t, y(t) + h, y'(t)\big) = 0, \quad t \in (0, 1),$$

$$(T_2y)(0) - \sum_{i=1}^m \alpha_i(T_2y)(\xi_i) = 0, \quad (T_2y)'(1) - \sum_{i=1}^m \beta_i(T_2y)'(\xi_i) = B;$$

(ii) $T_2y \in P_2$ *for each* $y \in P_2$*;*

(iii) x *is a solution of* (1.15) *if and only if* $x = y + h$ *and* y *is a solution of the operator equation* $y = T_2y$ *in cone* P_2*;*

(iv) $T_2 : P_2 \to P_2$ *is completely continuous.*

The proofs of the above lemma is similar to that of Lemma 2.9 and it is omitted.

Theorem 2.16. *Suppose that* (H5)–(H7) *hold and there exist positive constants* e_1, e_2, c*,*

$$\begin{aligned}L &= \int_0^1 \phi^{-1}\Big(1 + \frac{\phi\big(\frac{1+\sum_{i=1}^m \beta_i}{2}\big)}{1 - \phi\big(\frac{1+\sum_{i=1}^m \beta_i}{2}\big)} + (1 - s)\Big)ds \\ &\quad + \frac{1}{1 - \sum_{i=1}^m \alpha_i}\sum_{i=1}^m \alpha_i \int_0^{\xi_i} \phi^{-1}\Big(1 + \frac{\phi\big(\frac{1+\sum_{i=1}^m \beta_i}{2}\big)}{1 - \phi\big(\frac{1+\sum_{i=1}^m \beta_i}{2}\big)} + (1 - s)\Big)ds,\end{aligned}$$

$$Q = \min\Big\{\phi\Big(\frac{c}{L}\Big),\ \frac{\phi(c)}{1 + \frac{\phi(2)}{\phi(2) - \phi\big(1+\sum_{i=1}^m \beta_i\big)}}\Big\}; \quad W = \phi\Big(\frac{e_2}{\sigma_0 \int_k^{1-k} \phi^{-1}(1 - k - s)ds}\Big);$$

$$E = \phi(\frac{e_1}{L}).$$

such that

$$c \geq \frac{e_2}{\sigma_0^2} > e_2 > \frac{e_1}{\sigma_0} > e_1 > 0, \quad Q \geq \phi\Big(\frac{B\big(1 + \sum_{i=1}^m \beta_i\big)}{1 - \sum_{i=1}^m \beta_i}\Big), \quad Q > W.$$

If (A1), (A2) and (A3) from in Theorem 2.10 hold, then (1.15) *has at least three increasing positive solutions* x_1, x_2, x_3 *such that*

$$x_1(1) < e_1 + h, \quad x_2(k) > e_2 + h, \quad x_3(1) > e_1 + h, \quad x_3(k) < e_2 + h.$$

Proof. To apply Lemma 2.5, we prove that all its conditions are satisfied. By the definitions, it is easy to see that α, ψ are two nonnegative continuous concave functionals on the cone P_2, γ, β, θ are three nonnegative continuous convex functionals on the cone P_2 and $\alpha(y) \le \beta(y)$ for all $y \in P_2$, there exist constants $M > 0$ such that $\|y\| \le M\gamma(y)$ for all $y \in P_2$. Lemma 2.15 implies that $x = x(t)$ is a positive solution of (1.15) if and only if $x(t) = y(t) + h$ and $y(t)$ is a solution of the operator equation $y = T_2 y$ and $T_2 : P_2 \to P_2$ is completely continuous.

Corresponding to Lemma 2.5,

$$c = c, \quad h = \sigma_0 e_1, \quad d = e_1, \quad a = e_2, \quad b = \frac{e_2}{\sigma_0}.$$

Now, we prove that all conditions of Lemma 2.5 hold. One sees that $0 < d < a$. The remainder is divided in four steps.

Step 1. Prove that $T_2 : \overline{P_{2c}} \to \overline{P_{2c}}$;

For $y \in \overline{P_{2c}}$, we have $\|y\| \le c$. Then $0 \le y(t) \le c$ for $t \in [0, 1]$ and $-c \le y'(t) \le c$ for all $t \in [0, 1]$. So (A_1) implies that

$$f(t, y(t) + h, y'(t)) \le Q, \quad t \in [0, 1].$$

It follows from Lemma 2.15 that $T_2 y \in P_2$. Then Lemma 2.14 implies

$$\begin{aligned}
0 &\le (T_2y)(t) \\
&\le \int_0^1 \phi^{-1}\Big(A_y + \int_s^1 f(u, y(u) + h, y'(u))du\Big)ds \\
&\quad + \frac{1}{1 - \sum_{i=1}^m \alpha_i} \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \phi^{-1}\Big(A_y + \int_s^1 f(u, y(u) + h, y'(u))du\Big)ds \\
&\le \int_0^1 \phi^{-1}\Big(\phi\Big(\frac{B\big(1 + \sum_{i=1}^m \beta_i\big)}{1 - \sum_{i=1}^m \beta_i}\Big) + \frac{\phi\big(\frac{1+\sum_{i=1}^m \beta_i}{2}\big)}{1 - \phi\Big(\frac{1+\sum_{i=1}^m \beta_i}{2}\Big)} \int_0^1 f(u, y(u) + h, y'(u))du \\
&\quad + \int_s^1 f(u, y(u) + h, y'(u))du\Big)ds \\
&\quad + \frac{1}{1 - \sum_{i=1}^m \alpha_i} \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \phi^{-1}\Big(\phi\Big(\frac{B\big(1 + \sum_{i=1}^m \beta_i\big)}{1 - \sum_{i=1}^m \beta_i}\Big) \\
&\quad + \frac{\phi\big(\frac{1+\sum_{i=1}^m \beta_i}{2}\big)}{1 - \phi\big(\frac{1+\sum_{i=1}^m \beta_i}{2}\big)} \int_0^1 f(u, y(u) + h, y'(u))du + \int_s^1 f(u, y(u) + h, y'(u))du\Big)ds \\
&\le \int_0^1 \phi^{-1}\Big(\phi\left(\frac{B\big(1 + \sum_{i=1}^m \beta_i\big)}{1 - \sum_{i=1}^m \beta_i}\right) + \frac{\phi\big(\frac{1+\sum_{i=1}^m \beta_i}{2}\big)}{1 - \phi\big(\frac{1+\sum_{i=1}^m \beta_i}{2}\big)} Q + Q(1 - s)\Big)ds \\
&\quad + \frac{1}{1 - \sum_{i=1}^m \alpha_i} \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \phi^{-1}\Big(\phi\Big(\frac{B\big(1 + \sum_{i=1}^m \beta_i\big)}{1 - \sum_{i=1}^m \beta_i}\Big) \\
&\quad + \frac{\phi\big(\frac{1+\sum_{i=1}^m \beta_i}{2}\big)}{1 - \phi\big(\frac{1+\sum_{i=1}^m \beta_i}{2}\big)} Q + Q(1 - s)\Big)ds
\end{aligned}$$

$$
\begin{aligned}
&\le \int_0^1 \phi^{-1}\Big(Q+\frac{\phi\big(\frac{1+\sum_{i=1}^m\beta_i}{2}\big)}{1-\phi\big(\frac{1+\sum_{i=1}^m\beta_i}{2}\big)}Q+Q(1-s)\Big)ds\\
&\quad+\frac{1}{1-\sum_{i=1}^m\alpha_i}\sum_{i=1}^m\alpha_i\int_0^{\xi_i}\phi^{-1}\Big(Q+\frac{\phi\big(\frac{1+\sum_{i=1}^m\beta_i}{2}\big)}{1-\phi\big(\frac{1+\sum_{i=1}^m\beta_i}{2}\big)}Q+Q(1-s)\Big)ds\\
&=\phi^{-1}(Q)\Big[\int_0^1 \phi^{-1}\Big(1+\frac{\phi\big(\frac{1+\sum_{i=1}^m\beta_i}{2}\big)}{1-\phi\big(\frac{1+\sum_{i=1}^m\beta_i}{2}\big)}+(1-s)\Big)ds\\
&\quad+\frac{1}{1-\sum_{i=1}^m\alpha_i}\sum_{i=1}^m\alpha_i\int_0^{\xi_i}\phi^{-1}\Big(1+\frac{\phi\big(\frac{1+\sum_{i=1}^m\beta_i}{2}\big)}{1-\phi\big(\frac{1+\sum_{i=1}^m\beta_i}{2}\big)}+(1-s)\Big)ds\Big]\\
&\le c.
\end{aligned}
$$

Similarly to above discussion, from Lemma 2.14, we have

$$
\begin{aligned}
&|(T_2y)'(t)|\\
&\le (T_2y)'(0)=\phi^{-1}\Big(A_y+\int_0^1 f(u,y(u)+h,y'(u))du\Big)\\
&\le \phi^{-1}\Big(\phi\Big(\frac{B\big(1+\sum_{i=1}^m\beta_i\big)}{1-\sum_{i=1}^m\beta_i}\Big)+\frac{\phi\big(\frac{1+\sum_{i=1}^m\beta_i}{2}\big)}{1-\phi\big(\frac{1+\sum_{i=1}^m\beta_i}{2}\big)}\int_0^1 f(u,y(u)+h,y'(u))du\\
&\quad+\int_0^1 f(u,y(u)+h,y'(u))du\Big)\\
&\le \phi^{-1}\Big(\phi\Big(\frac{B\big(1+\sum_{i=1}^m\beta_i\big)}{1-\sum_{i=1}^m\beta_i}\Big)+\frac{1}{1-\phi\big(\frac{1+\sum_{i=1}^m\beta_i}{2}\big)}Q\Big)\\
&\le \phi^{-1}\Big(Q+\frac{\phi(2)}{\phi(2)-\phi\left(1+\sum_{i=1}^m\beta_i\right)}Q\Big)\le c.
\end{aligned}
$$

It follows that $\|T_2y\|=\max\{\max_{t\in[0,1]}|(T_2y)(t)|,\max_{t\in[0,1]}|(T_2y)'(t)|\}\le c$. Then $T_2:\overline{P_{2c}}\to\overline{P_{2c}}$.

Step 2. Prove that

$$
\{y\in P_2(\gamma,\theta,\alpha;a,b,c)|\alpha(y)>a\}=\{y\in P_2\big(\gamma,\theta,\alpha;e_2,\frac{e_2}{\sigma_0},c\big):\alpha(y)>e_2\}\ne\emptyset
$$

and $\alpha(T_2y)>e_2$ for every $y\in P_2\big(\gamma,\theta,\alpha;e_2,\frac{e_2}{\sigma_0},c\big)$; Choose $y(t)=\frac{e_2}{2\sigma_0}$ for all $t\in[0,1]$. Then $y\in P_2$ and

$$
\alpha(y)=\frac{e_2}{2\sigma_0}>e_2,\quad \theta(y)=\frac{e_2}{2\sigma_0}\le\frac{e_2}{\sigma_0},\quad \gamma(y)=0<c.
$$

It follows that $\{y\in P_2(\gamma,\theta,\alpha;a,b,c)|\alpha(y)>a\}\ne\emptyset$.

For $y\in P_2(\gamma,\theta,\alpha;a,b,c)$, one has

$$
\alpha(y)=\min_{t\in[k,1-k]}y(t)\ge e_2,\quad \theta(y)=\max_{t\in[k,1-k]}y(t)\le\frac{e_2}{\sigma_0},\quad \gamma(y)=\max_{t\in[0,1]}|y'(t)|\le c.
$$

Then

$$
e_2\le y(t)\le\frac{e_2}{\sigma_0^2},\quad t\in[k,1-k],\quad |y'(t)|\le c.
$$

Thus (A2) implies $f(t,y(t)+h,|y'(t)|)\ge W$, $t\in[k,1-k]$. Since

$$
\alpha(T_2y)=\min_{t\in[k,1-k]}(T_2y)(t)\ge\sigma_0\max_{t\in[0,1]}(T_2y)(t),
$$

from Lemma 2.14, we have

$$
\begin{aligned}
\alpha(T_2y) &\geq \sigma_0 \max_{t\in[0,1]}(T_2y)(t)\\
&= \sigma_0\Big[\int_0^1 \phi^{-1}\Big(A_y + \int_s^1 f(u,y(u)+h,y'(u))du\Big)ds\\
&\quad + \frac{1}{1-\sum_{i=1}^m \alpha_i}\sum_{i=1}^m \alpha_i \int_0^{\xi_i} \phi^{-1}\Big(A_h + \int_s^1 f(u,y(u)+h,y'(u))ds\Big)ds\Big]\\
&\geq \sigma_0\Big[\int_0^1 \phi^{-1}\Big(\phi\Big(\frac{B}{1-\sum_{i=1}^m \beta_i}\Big) + \int_s^1 f(u,y(u)+h,y'(u))du\Big)ds\\
&\quad + \frac{1}{1-\sum_{i=1}^m \alpha_i}\sum_{i=1}^m \alpha_i \int_0^{\xi_i} \phi^{-1}\Big(\phi\Big(\frac{B}{1-\sum_{i=1}^m \beta_i}\Big)\\
&\quad + \int_s^1 f(u,y(u)+h,y'(u))ds\Big)ds\Big]\\
&\geq \sigma_0 \int_0^1 \phi^{-1}\left(\phi\Big(\frac{B}{1-\sum_{i=1}^m \beta_i}\Big) + \int_s^1 f(u,y(u)+h,y'(u))du\right)ds\\
&\geq \sigma_0 \int_0^1 \phi^{-1}\Big(\int_s^1 f(u,y(u)+h,y'(u))du\Big)ds\\
&\geq \sigma_0 \int_k^{1-k} \phi^{-1}\Big(\int_s^{1-k} f(u,y(u)+h,y'(u))du\Big)ds\\
&\geq \sigma_0 \int_k^{1-k} \phi^{-1}\left(W(1-k-s)\right)ds = e_2.
\end{aligned}
$$

This completes Step 2.

Step 3. Prove that the set $\{y \in Q(\gamma,\theta,\psi;h,d,c)|\beta(y) < d\}$ which is equal to $\{y \in Q\big(\gamma,\theta,\psi;\sigma_0e_1,e_1,c\big) : \beta(y) < e_1\}$ is not empty, and $\beta(T_2y) < e_1$ for every $y \in Q(\gamma,\theta,\psi;h,d,c) = Q\big(\gamma,\theta,\psi;\sigma_0e_1,e_1,c\big)$. Choose $y(t) = \sigma_0e_1$. Then $y \in P_2$, and

$$
\psi(y) = \sigma_0e_1 \geq h, \quad \beta(y) = \theta(y) = \sigma_0e_1 < e_1 = d, \quad \gamma(y) = 0 \leq c.
$$

It follows that $\{y \in Q(\gamma,\theta,\psi;h,d,c) : \beta(y) < d\} \neq \emptyset$.

For $y \in Q(\gamma,\theta,\psi;h,d,c)$, one has

$$
\psi(y) = \min_{t\in[k,1-k]} y(t) \geq h = e_1\sigma_0, \quad \theta(y) = \max_{t\in[k,1-k]} y(t) \leq d = e_1,
$$
$$
\gamma(y) = \max_{t\in[0,1]} |y'(t)| \leq c.
$$

Hence we have $0 \leq y(t) \leq \frac{e_1}{\sigma_0}$ and $-c \leq y'(t) \leq c$ for $t \in [0,1]$. Then (A3) implies $f(t,y(t)+h,|y'(t)|) \leq E, \quad t \in [0,1]$. So that

$$
\begin{aligned}
\beta(T_2y) &= \max_{t\in[0,1]}(T_2y)(t)\\
&= \int_0^1 \phi^{-1}\Big(A_y + \int_s^1 f(u,y(u)+h,y'(u))du\Big)ds\\
&\quad + \frac{1}{1-\sum_{i=1}^m \alpha_i}\sum_{i=1}^m \alpha_i \int_0^{\xi_i} \phi^{-1}\Big(A_y + \int_s^1 f(u,y(u)+h,y'(u))du\Big)ds
\end{aligned}
$$

$$
\begin{aligned}
&\le \int_0^1 \phi^{-1}\Big(\phi\Big(\frac{B\big(1+\sum_{i=1}^m \beta_i\big)}{1-\sum_{i=1}^m \beta_i}\Big)\\
&\quad + \frac{\phi\big(\frac{1+\sum_{i=1}^m \beta_i}{2}\big)}{1-\phi\Big(\frac{1+\sum_{i=1}^m \beta_i}{2}\Big)} \int_0^1 f(u,y(u)+h,y'(u))du\\
&\quad + \int_s^1 f(u,y(u)+h,y'(u))du\Big)ds\\
&\quad + \frac{1}{1-\sum_{i=1}^m \alpha_i}\sum_{i=1}^m \alpha_i \int_0^{\xi_i} \phi^{-1}\Big(\phi\Big(\frac{B\big(1+\sum_{i=1}^m \beta_i\big)}{1-\sum_{i=1}^m \beta_i}\Big)\\
&\quad + \frac{\phi\big(\frac{1+\sum_{i=1}^m \beta_i}{2}\big)}{1-\phi\Big(\frac{1+\sum_{i=1}^m \beta_i}{2}\Big)} \int_0^1 f(u,y(u)+h,y'(u))du\\
&\quad + \int_s^1 f(u,y(u)+h,y'(u))du\Big)ds\\
&\le \int_0^1 \phi^{-1}\Big(\phi\Big(\frac{B\big(1+\sum_{i=1}^m \beta_i\big)}{1-\sum_{i=1}^m \beta_i}\Big) + \frac{\phi\big(\frac{1+\sum_{i=1}^m \beta_i}{2}\big)}{1-\phi\big(\frac{1+\sum_{i=1}^m \beta_i}{2}\big)}E + E(1-s)\Big)ds\\
&\quad + \frac{1}{1-\sum_{i=1}^m \alpha_i}\sum_{i=1}^m \alpha_i \int_0^{\xi_i} \phi^{-1}\Big(\phi\Big(\frac{B\big(1+\sum_{i=1}^m \beta_i\big)}{1-\sum_{i=1}^m \beta_i}\Big)\\
&\quad + \frac{\phi\big(\frac{1+\sum_{i=1}^m \beta_i}{2}\big)}{1-\phi\big(\frac{1+\sum_{i=1}^m \beta_i}{2}\big)}E + E(1-s)\Big)ds\\
&\le \int_0^1 \phi^{-1}\Big(E + \frac{\phi\big(\frac{1+\sum_{i=1}^m \beta_i}{2}\big)}{1-\phi\big(\frac{1+\sum_{i=1}^m \beta_i}{2}\big)}E + E(1-s)\Big)ds\\
&\quad + \frac{1}{1-\sum_{i=1}^m \alpha_i}\sum_{i=1}^m \alpha_i \int_0^{\xi_i} \phi^{-1}\Big(E + \frac{\phi\big(\frac{1+\sum_{i=1}^m \beta_i}{2}\big)}{1-\phi\big(\frac{1+\sum_{i=1}^m \beta_i}{2}\big)}E + E(1-s)\Big)ds\\
&= \phi^{-1}(E)\Big[\int_0^1 \phi^{-1}\Big(1 + \frac{\phi\big(\frac{1+\sum_{i=1}^m \beta_i}{2}\big)}{1-\phi\big(\frac{1+\sum_{i=1}^m \beta_i}{2}\big)} + (1-s)\Big)ds\\
&\quad + \frac{1}{1-\sum_{i=1}^m \alpha_i}\sum_{i=1}^m \alpha_i \int_0^{\xi_i} \phi^{-1}\Big(1 + \frac{\phi\big(\frac{1+\sum_{i=1}^m \beta_i}{2}\big)}{1-\phi\big(\frac{1+\sum_{i=1}^m \beta_i}{2}\big)} + (1-s)\Big)ds\Big]\\
&= e_1 = d.
\end{aligned}
$$

This completes Step 3.

Step 4. Prove that $\alpha(T_2y) > a$ for $y \in P_2(\gamma,\alpha;a,c)$ with $\theta(T_2y) > b$; For $y \in P_2(\gamma,\alpha;a,c) = P_2(\gamma,\alpha;e_2,c)$ with $\theta(T_2y) > b = \frac{e_2}{\sigma_0}$, we have that $\alpha(y) = \min_{t\in[k,1-k]} y(t) \ge e_2$ and $\gamma(y) = \max_{t\in[0,1]} |y(t)| \le c$ and $\max_{t\in[k,1-k]}(T_2y)(t) > \frac{e_2}{\sigma_0}$. Then

$$
\alpha(T_2y) = \min_{t\in[k,1-k]}(T_2y)(t) \ge \sigma_0\beta(T_2y) > \sigma_0\frac{e_2}{\sigma_0} = e_2 = a.
$$

This completes Step 4.

Step 5. Prove that $\beta(T_2y) < d$ for each $y \in Q(\gamma,\beta;d,c)$ with $\psi(T_2y) < h$. For $y \in Q(\gamma,\beta;d,c)$ with $\psi(T_2y) < d$, we have $\gamma(y) = \max_{t\in[0,1]} |y(t)| \le c$ and

$\beta(y) = \max_{t\in[0,1]} y(t) \le d = e_1$ and $\psi(T_2y) = \min_{t\in[k,1-k]}(T_1y)(t) < h = e_1\sigma_0$. Then

$$\beta(T_2y) = \max_{t\in[0,1]}(T_2y)(t) \le \frac{1}{\sigma_0}\min_{t\in[k,1-k]}(T_2y)(t) < \frac{1}{\sigma_0}e_1\sigma_0 = e_1 = d.$$

This completes the Step 5.

Then Lemma 2.5 implies that T_2 has at least three fixed points y_1, y_2 and y_3 in P_2 such that

$$\beta(y_1) < e_1, \quad \alpha(y_2) > e_2, \quad \beta(y_3) > e_1, \quad \alpha(y_3) < e_2.$$

Hence (1.15) has three increasing positive solutions x_1, x_2 and x_3 such that

$$\begin{gathered}\max_{t\in[0,1]} x_1(t) < e_1 + h, \qquad \min_{t\in[k,1-k]} x_2(t) > e_2 + h,\\ \max_{t\in[0,1]} x_3(t) > e_1 + h, \qquad \min_{t\in[k,1-k]} x_3(t) < e_2 + h.\end{gathered}$$

Hence

$$x_1(1) < e_1 + h, \quad x_2(k) > e_2 + h, \quad x_3(1) > e_1 + h, \quad x_3(k) < e_2 + h.$$

The proof is complete. □

Remark 2.17. For (1.15), when $A < 0, B \ge 0$, we can also get the existence results for three increasing solutions of (1.15) similarly, but the solutions need not be positive. By the way, it is interesting to establish sufficient conditions guarantee the existence of positive solutions of (1.15) when $B < 0$.

2.3. **Positive Solutions of** (1.16)**.** Now we prove an existence result for three positive solutions of (1.16). We use the following there conditions:

- (H8) $f : [0,1] \times [h, +\infty) \times \mathbb{R} \to [0, +\infty)$ is continuous with $f(t, c + h, 0) \not\equiv 0$ on each sub-interval of [0,1] for all $c \ge 0$, where $h = \frac{B}{1-\sum_{i=1}^m \beta_i}$;
- (H9) $A \ge 0, B \ge 0$;
- (H10) $\alpha \ge 0$, $\beta_i \ge 0$ satisfy $\sum_{i=1}^m \beta_i < 1$ and $A \ge \frac{B}{1-\sum_{i=1}^m \beta_i}$;

Consider the following boundary-value problem

$$\begin{gathered}[\phi(y'(t))]' + h(t) = 0, \quad t \in (0,1),\\ y(0) - \alpha y'(0) = D, \quad y(1) - \sum_{i=1}^m \beta_i y(\xi_i) = 0,\end{gathered} \tag{2.11}$$

Lemma 2.18. *Suppose that* (H4), (H10) *hold and* $D \ge 0$. *If* y *is a solution of* (2.11)*, then* y *is positive on* $(0,1)$.

Proof. Suppose y satisfies (2.11). It follows from assumption (H4) that y' is decreasing on $[0,1]$.

If $y'(1) > 0$, the BCs in (2.11) and (H4) imply $y'(t) > 0$ for all $t \in [0,1]$. Then

$$y(1) = \sum_{i=1}^m \beta_i y(\xi_i) \le \sum_{i=1}^m \beta_i y(1).$$

Then $y(1) \le 0$. On the other hand, $y(0) = \alpha y'(0) + D \ge 0$. This is a contradiction since $y'(t) > 0$.

If $y'(1) \le 0$ and $y'(0) \le 0$, then we get that $y'(t) \le 0$ for all $t \in [0,1]$. The BCs in (2.11) and (H4) imply

$$y(1) = \sum_{i=1}^{m} \beta_i y(\xi_i) \ge \sum_{i=1}^{m} \beta_i y(1).$$

It follows that $y(1) \ge 0$. Then $y(t) \ge y(1) \ge 0$ for all $t \in [0,1]$. (H4) implies $y(t) > 0$ for all $t \in (0,1)$.

If $y'(1) \le 0$ and $y'(0) > 0$, then $y(0) = \alpha y'(0) + D \ge 0$. It follows from (2.11) and (H4) that

$$y(1) = \sum_{i=1}^{m} \beta_i y(\xi_i) \ge \sum_{i=1}^{m} \beta_i \min\{y(0), y(1)\}.$$

If $y(1) \le y(0)$, then $y(1) \ge \sum_{i=1}^{m} \beta_i y(1)$ implies that $y(1) \ge 0$. If $y(1) > y(0)$, then $y(1) > 0$ since $y(0) \ge 0$. It follows from (H4) that $y(t) \ge \min\{y(0), y(1)\} \ge 0$. Then y is positive on $(0,1)$. The proof is complete. □

Lemma 2.19. *Assume* (H4), (H10) *hold and* $D \ge 0$. *If* y *is a solution of* (2.11), *then*

$$y(t) = B_h + \int_0^t \phi^{-1}\Big(A_h + \int_s^1 h(u)du\Big), \quad t \in [0,1],$$

where $A_h \in [b, 0]$,

$$\begin{aligned}
&\Big[\alpha\phi^{-1}\Big(A_h + \int_\xi^1 h(u)du\Big) + D\Big]\Big(1 - \sum_{i=1}^{m} \beta_i\Big) \\
&= -\Big(1 - \sum_{i=1}^{m} \beta_i\Big) \int_0^1 \phi^{-1}\Big(A_h + \int_s^1 h(u)du\Big)ds \\
&\quad - \sum_{i=1}^{m} \beta_i \int_{\xi_i}^1 \phi^{-1}\Big(A_h + \int_s^1 h(u)du\Big)ds,
\end{aligned}$$

$$\begin{gathered}
B_h = \alpha\phi^{-1}\Big(A_h + \int_\xi^1 h(u)du\Big) + D, \\
b = -\int_0^1 h(u)du - \phi\Big(\frac{D\Big(1 - \sum_{i=1}^{m} \beta_i\Big)}{a}\Big), \\
a = 1 + \alpha\Big(1 - \sum_{i=1}^{m} \beta_i\Big) - \sum_{i=1}^{m} \beta_i \xi_i.
\end{gathered}$$

Proof. From (2.11) it follows that

$$\begin{aligned}
y(t) &= y(0) + \int_0^t \phi^{-1}\Big(\phi(y'(1)) + \int_s^1 (u)du\Big)ds, \\
y(0)\Big(1 - \sum_{i=1}^{m} \beta_i\Big) &= -\int_0^1 \phi^{-1}\Big(\phi(y'(1)) + \int_s^1 h(u)du\Big)ds \\
&\quad + \sum_{i=1}^{m} \beta_i \int_0^{\xi_i} \phi^{-1}\Big(\phi(y'(1)) + \int_s^1 h(u)du\Big)ds,
\end{aligned}$$

$$y(0) = \alpha\phi^{-1}\Big(\phi(y'(1)) + \int_0^1 h(u)du\Big) + D.$$

Thus

$$\begin{aligned}
&\Big[\alpha\phi^{-1}\Big(\phi(y'(1)) + \int_0^1 h(u)du\Big) + D\Big]\Big(1 - \sum_{i=1}^m \beta_i\Big) \\
&= -\int_0^1 \phi^{-1}\Big(\phi(y'(1)) + \int_s^1 h(u)du\Big)ds \\
&\quad + \sum_{i=1}^m \beta_i \int_0^{\xi_i} \phi^{-1}\Big(\phi(y'(1)) + \int_s^1 h(u)du\Big)ds.
\end{aligned}$$

Let

$$\begin{aligned}
G(c) &= \Big[\alpha\phi^{-1}\Big(c + \int_0^1 h(u)du\Big) + D\Big]\Big(1 - \sum_{i=1}^m \beta_i\Big) \\
&\quad + \int_0^1 \phi^{-1}\Big(c + \int_s^1 h(u)du\Big)ds \\
&\quad - \sum_{i=1}^m \beta_i \int_{\xi_i}^1 \phi^{-1}\Big(c + \int_s^1 h(u)du\Big)ds \\
&= \Big[\alpha\phi^{-1}\Big(c + \int_0^1 h(u)du\Big) + D\Big]\Big(1 - \sum_{i=1}^m \beta_i\Big) \\
&\quad + \Big(1 - \sum_{i=1}^m \beta_i\Big) \int_0^1 \phi^{-1}\Big(c + \int_s^1 h(u)du\Big)ds \\
&\quad + \sum_{i=1}^m \beta_i \int_{\xi_i}^1 \phi^{-1}\Big(c + \int_s^1 h(u)du\Big)ds
\end{aligned}$$

.

It is easy to see that $G(c)$ is increasing on $(-\infty, +\infty)$, $G(0) \geq 0$. Since

$$a = 1 + \alpha\Big(1 - \sum_{i=1}^m \beta_i\Big) - \sum_{i=1}^m \beta_i\xi_i,$$

we get

$$\begin{aligned}
G(b) &= G\Big(-\phi\Big(\frac{D\Big(1 - \sum_{i=1}^m \beta_i\Big)}{a}\Big) - \int_0^1 h(u)du\Big) \\
&= \Big[\alpha\phi^{-1}\Big(-\phi\Big(\frac{D\Big(1 - \sum_{i=1}^m \beta_i\Big)}{a}\Big)\Big) + D\Big]\Big(1 - \sum_{i=1}^m \beta_i\Big) \\
&\quad + \Big(1 - \sum_{i=1}^m \beta_i\Big) \int_0^1 \phi^{-1}\Big(-\phi\Big(\frac{D\Big(1 - \sum_{i=1}^m \beta_i\Big)}{a}\Big) - \int_0^s h(u)du\Big)ds \\
&\quad + \sum_{i=1}^m \beta_i \int_{\xi_i}^1 \phi^{-1}\Big(-\phi\Big(\frac{D\Big(1 - \sum_{i=1}^m \beta_i\Big)}{a}\Big) - \int_0^s h(u)du\Big)ds
\end{aligned}$$

$$
\begin{aligned}
&\le \Big[\alpha\phi^{-1}\Big(-\phi\Big(\frac{D\Big(1-\sum_{i=1}^m\beta_i\Big)}{a}\Big)\Big)+D\Big]\Big(1-\sum_{i=1}^m\beta_i\Big)\\
&\quad+\Big(1-\sum_{i=1}^m\beta_i\Big)\int_0^1\phi^{-1}\Big(-\phi\Big(\frac{D\Big(1-\sum_{i=1}^m\beta_i\Big)}{a}\Big)\Big)ds\\
&\quad+\sum_{i=1}^m\beta_i\int_{\xi_i}^1\phi^{-1}\Big(-\phi\Big(\frac{D\Big(1-\sum_{i=1}^m\beta_i\Big)}{a}\Big)\Big)ds=0,
\end{aligned}
$$

we get $\phi(y'(1))\ge -\int_0^1 h(u)du-\phi\Big(\frac{D\big(1-\sum_{i=1}^m\beta_i\big)}{a}\Big)=b$. The proof is complete. □

Let

$$
\begin{aligned}
P_3=\big\{&y\in X: y(t)\ge 0 \text{ for all } t\in[0,1], y'(t) \text{ is decreasing on } [0,1],\\
&y(t)\ge \min\{t,(1-t)\}\max_{t\in[0,1]} y(t) \text{ for all } t\in[0,1]\big\}.
\end{aligned}
$$

Then P_3 is a cone in X. Since, for $y\in P_3$, we have

$$
\begin{aligned}
|y(t)|&=|y(t)-y(1)+y(1)|\\
&\le |y'(\theta)|(1-t)+|y(1)|\\
&\le \max_{t\in[0,1]}|y'(t)|+\Big|\frac{\sum_{i=1}^m\beta_i y(\xi_i)-\sum_{i=1}^m\beta_i y(1)}{1-\sum_{i=1}^m\beta_i}\Big|\\
&\le \Big(1+\frac{\sum_{i=1}^m\beta_i(1-\xi_i)}{1-\sum_{i=1}^m\beta_i}\Big)\max_{t\in[0,1]}|y'(t)|\\
&=\Big(1+\frac{\sum_{i=1}^m\beta_i(1-\xi_i)}{1-\sum_{i=1}^m\beta_i}\Big)\gamma(y),
\end{aligned}
$$

we get

$$
\max_{t\in[0,1]}|y(t)|\le\Big(1+\frac{\sum_{i=1}^m\beta_i(1-\xi_i)}{1-\sum_{i=1}^m\beta_i}\Big)\gamma(y).
$$

It is easy to see that there exists a constant $M>0$ such that $\|y\|\le M\gamma(y)$ for all $y\in P_3$.

Suppose that (H10) holds. Let $x(t)-h=y(t)$. Then (1.16) is transformed into

$$
\begin{gathered}
[\phi(y'(t))]'+f(t,y(t)+h,y'(t))=0,\quad t\in(0,1),\\
y(0)-\alpha y'(0)=A-\frac{B}{1-\sum_{i=1}^m\beta_i}\ge 0,\\
y(1)-\sum_{i=1}^m\beta_i y(\xi_i)=0,
\end{gathered}
$$

Define the nonlinear operator $T_3:P_3\to X$ by

$$
(T_3y)(t)=B_y+\int_0^t\phi^{-1}\Big(A_y+\int_s^1 f(u,y(u)+h,y'(u))du\Big)ds,\quad y\in P_3,
$$

where

$$
\Big[\alpha\phi^{-1}\Big(A_y+\int_0^1 f(u,y(u)+h,y'(u))du\Big)+A-\frac{B}{1-\sum_{i=1}^m\beta_i}\Big]\Big(1-\sum_{i=1}^m\beta_i\Big)
$$

$$
\begin{aligned}
&= -\Big(1-\sum_{i=1}^m \beta_i\Big)\int_0^1 \phi^{-1}\Big(A_y+\int_s^1 f(u,y(u)+h,y'(u))du\Big)ds\\
&\quad -\sum_{i=1}^m \beta_i\int_{\xi_i}^1 \phi^{-1}\Big(A_y+\int_s^1 f(u,y(u)+h,y'(u))du\Big)ds,
\end{aligned}
$$

and

$$
B_h=\alpha\phi^{-1}\Big(A_y+\int_0^1 f(u,y(u)+h,y'(u))duu\Big)+A-\frac{B}{1-\sum_{i=1}^m\beta_i}.
$$

For $y\in P_3$, the definition of A_y implies

$$
\begin{aligned}
(T_3y)(t)&=\alpha\phi^{-1}\Big(A_y+\int_0^1 f(u,y(u)+h,y'(u))du\Big)+A-\frac{B}{1-\sum_{i=1}^m\beta_i}\\
&\quad+\int_0^t \phi^{-1}\Big(A_y+\int_s^1 f(u,y(u)+h,y'(u))du\Big)ds\\
&=-\int_t^1 \phi^{-1}\Big(A_y+\int_s^1 f(u,y(u)+h,y'(u))du\Big)ds\\
&\quad-\frac{1}{1-\sum_{i=1}^m\beta_i}\sum_{i=1}^m\beta_i\int_{\xi_i}^1\phi^{-1}\Big(A_y+\int_s^1 f(u,y(u)+h,y'(u))du\Big)ds.
\end{aligned}
$$

Lemma 2.20. *Suppose that* (H8)–(H10) *hold. Then*

(i) *$T_3y\in P_3$ for each $y\in P_3$;*
(ii) *x is a solution of* (1.17) *if and only if $x=y+h$ and y is a solution of the operator equation $T_3y=y$ in P_3;*
(iii) *$T_3:P_3\to P_3$ is completely continuous;*
(iv) *the following equalities hold:*

$$
\begin{gathered}
[\phi((T_3y)'(t))]'+f\big(t,y(t)+h,y'(t)\big)=0,\quad t\in(0,1),\\
(T_3y)(0)-\alpha(T_3y)'(\xi)=A-\frac{B}{1-\sum_{i=1}^m\beta_i},\\
(T_3y)(1)-\sum_{i=1}^m\beta_i(T_3y)(\xi_i)=0;
\end{gathered}
$$

The proof of the above lemma is similar to that of Lemma 2.9, so it is omitted.

Theorem 2.21. *Suppose that* (H8)–(H10) *hold and that there exist positive constants e_1,e_2,c,*

$$
\begin{gathered}
L=\phi^{-1}(2)+\frac{\phi^{-1}(2)}{1-\sum_{i=1}^m\beta_i}\sum_{i=1}^m\beta_i(1-\xi_i),\\
Q=\min\big\{\phi\big(\frac{c}{L}\big),\frac{\phi(c)}{2}\big\};\\
W=\phi\Big(\frac{e_2}{\sigma_0\int_k^{1-k}\phi^{-1}(s-k)ds}\Big);\quad E=\phi\big(\frac{e_1}{L}\big).
\end{gathered}
$$

such that $c\ge\frac{e_2}{\sigma_0^2}>e_2>\frac{e_1}{\sigma_0}>e_1>0$,

$$
Q\ge\phi\Big(\frac{\big(A-\frac{B}{1-\sum_{i=1}^m\beta_i}\big)\big(1-\sum_{i=1}^m\beta_i\big)}{1+\alpha-\sum_{i=1}^m\beta_i\xi_i}\Big),\quad Q>W.
$$

If (A1)–(A3) *in Theorem 2.10 hold, then* (1.16) *has at least three positive solutions* x_1, x_2, x_3 *such that*

$$\max_{t\in[0,1]} x_1(t) < e_1 + h, \quad \min_{t\in[k,1-k]} x_2(t) > e_2 + h,$$
$$\min_{t\in[k,1-k]} x_3(t) > e_1 + h, \quad \min_{t\in[k,1-k]} x_3(t) < e_2 + h.$$

The proof of the above theorem is similar to that of Theorem 2.10, so it is omitted.

Assume the following three conditions:

(H11) $f : [0,1] \times [h, +\infty) \times \mathbb{R} \to [0, +\infty)$ is continuous with $f(t, c+h, 0) \not\equiv 0$ on each sub-interval of [0,1] for all $c \ge 0$, where $h = A$;

H12) $A \ge 0, B \ge 0$;

(H13) $\alpha \ge 0$, $\beta_i \ge 0$ satisfy $\sum_{i=1}^m \beta_i < 1$ and $B \ge A\big(1 - \sum_{i=1}^m \beta_i\big)$;

Consider the boundary-value problem

$$\begin{gathered} [\phi(y'(t))]' + h(t) = 0, \quad t \in (0,1), \\ y(0) - \alpha y'(0) = 0, \quad y(1) - \sum_{i=1}^m \beta_i y(\xi_i) = D, \end{gathered} \tag{2.12}$$

Lemma 2.22. *Assume* (H4), (H13) *and* $D \ge 0$. *If* y *is a solution of* (2.12)*, then* y *is positive on* $(0,1)$.

Proof. Suppose y satisfies (2.12). It follows from the assumptions that y' is decreasing on $[0,1]$.

If $y'(0) < 0$, the BCs in (2.12) and (H4) imply that $y'(t) < 0$ for all $t \in [0,1]$ and $y(0) \le 0$. Then

$$y(1) = \sum_{i=1}^m \beta_i y(\xi_i) + D \ge \sum_{i=1}^m \beta_i y(1) + D.$$

Then $y(1) \ge \frac{D}{1-\sum_{i=1}^m \beta_i}$. It follows from $D \ge 0$ that $y(1) \ge y(0)$, a contradiction to $y'(t) < 0$ for all $t \in [0,1]$.

If $y'(0) \ge 0$ and $y'(1) > 0$, we get that $y'(t) > 0$ for all $t \in [0,1]$. The BCs in (2.12) and (H4) imply that $y(0) = \alpha y'(0) \ge 0$. Then $y(t) > y(0) \ge 0$ for all $t \in [0,1]$.

If $y'(0) \ge 0$ and $y'(1) \le 0$, then $y(0) = \alpha y'(0) \ge 0$. It follows from (2.12) that

$$y(1) = \sum_{i=1}^m \beta_i y(\xi_i) + D \ge \sum_{i=1}^m \beta_i \min\{y(0), y(1)\}.$$

If $y(1) \le y(0)$, then $y(1) \ge \sum_{i=1}^m \beta_i y(1)$ implies that $y(1) \ge 0$. If $y(1) > y(0)$, then $y(1) > 0$ since $y(0) \ge 0$. It follows that $y(t) \ge \min\{y(0), y(1)\} \ge 0$. Then y is positive on $(0,1)$. The proof is complete. □

Lemma 2.23. *Assume* (H4), (H13)*,* $D \ge 0$. *If* y *is a solution of* (2.12)*, then*

$$y(t) = B_h - \int_t^1 \phi^{-1}\Big(A_h - \int_0^s h(u)du\Big), \quad t \in [0,1],$$

where $A_h \in [a,b]$,

$$\alpha\phi^{-1}\Big(A_h - \int_0^\xi h(u)du\Big)\Big(1 - \sum_{i=1}^m \beta_i\Big)$$

$$
\begin{aligned}
&+\Big(1-\sum_{i=1}^{m}\beta_i\Big)\int_0^1\phi^{-1}\Big(A_h-\int_0^s h(u)du\Big)ds\\
&+\sum_{i=1}^{m}\beta_i\int_{\xi_i}^1\phi^{-1}\Big(A_h-\int_0^s h(u)du\Big)ds-D=0,
\end{aligned}
$$

and

$$
B_h=\alpha\phi^{-1}\Big(\phi(y'(0))-\int_0^{\xi} h(u)du\Big),
$$

$$
a=\phi(\frac{D}{c}),\quad b=\phi(\frac{D}{c})+\int_0^1 h(u)du,\quad c=\alpha\big(1-\sum_{i=1}^{m}\beta_i\big)+1-\sum_{i=1}^{m}\beta_i\xi_i.
$$

Proof. It follows from (2.12) that

$$
\begin{gathered}
y(t)=y(1)-\int_t^1\phi^{-1}\Big(\phi(y'(0))-\int_0^s h(u)du\Big)ds,\\
y(1)\Big(1-\sum_{i=1}^{m}\beta_i\Big)=-\sum_{i=1}^{m}\beta_i\int_{\xi_i}^1\phi^{-1}\Big(\phi(y'(0))-\int_0^s h(u)du\Big)ds+D,\\
y(1)-\int_0^1\phi^{-1}\Big(\phi(y'(0))-\int_0^s h(u)du\Big)ds=\alpha y'(0).
\end{gathered}
$$

Thus

$$
\begin{aligned}
&\Big(1-\sum_{i=1}^{m}\beta_i\Big)\Big(\int_0^1\phi^{-1}\Big(\phi(y'(0))-\int_0^s h(u)du\Big)ds+\alpha y'(0)\Big)\\
&=-\sum_{i=1}^{m}\beta_i\int_{\xi_i}^1\phi^{-1}\Big(\phi(y'(0))-\int_0^s h(u)du\Big)ds+D.
\end{aligned}
$$

It follows from the proof of Lemma 2.22 that $y'(0)\ge 0$. Let

$$
\begin{aligned}
G(c)=&\Big(1-\sum_{i=1}^{m}\beta_i\Big)\int_0^1\phi^{-1}\Big(c-\int_0^s h(u)du\Big)ds+\alpha\Big(1-\sum_{i=1}^{m}\beta_i\Big)\phi^{-1}(c)\\
&+\sum_{i=1}^{m}\beta_i\int_{\xi_i}^1\phi^{-1}\Big(c-\int_0^s h(u)du\Big)ds-D.
\end{aligned}
$$

It is easy to see that $G(c)$ is increasing on $(-\infty,+\infty)$ and $G(a)=G\left(\phi(\frac{D}{a})\right)\le 0$. Since

$$
\begin{aligned}
G(b)&=G\Big(\phi(\frac{D}{c})+\int_0^1 h(u)du\Big)\\
&\ge\Big(1-\sum_{i=1}^{m}\beta_i\Big)\int_0^1\phi^{-1}\Big(\phi(\frac{D}{c})+\int_s^1 h(u)du\Big)ds\\
&\quad+\alpha\Big(1-\sum_{i=1}^{m}\beta_i\Big)\phi^{-1}\Big(\phi(\frac{D}{c})+\int_0^1 h(u)du\Big)\\
&\quad+\sum_{i=1}^{m}\beta_i\int_{\xi_i}^1\phi^{-1}\Big(\phi(\frac{D}{c})+\int_s^1 h(u)du\Big)ds-D
\end{aligned}
$$

$$\geq \Big(1-\sum_{i=1}^m \beta_i\Big)\frac{D}{c} + \alpha\Big(1-\sum_{i=1}^m \beta_i\Big)\frac{D}{c} + \sum_{i=1}^m \beta_i(1-\xi_i)\frac{D}{c} - D$$
$$= 0,$$

we get $a \leq \phi(y'(0)) \leq \phi\big(\frac{D}{c}\big) + \int_0^1 h(u)du$. The proof is complete. □

Suppose that (H13) holds. Let $x(t) - h = y(t)$. Then (1.16) is transformed into

$$\begin{gathered}
[\phi(y'(t))]' + f(t, y(t)+h, y'(t)) = 0, \quad t \in (0,1), \\
y(0) - \alpha y'(0) = 0, \\
y(1) - \sum_{i=1}^m \beta_i y(\xi_i) = B - \frac{A}{1-\sum_{i=1}^m \beta_i} \geq 0.
\end{gathered}$$

Let

$$\begin{aligned}
P_3' = \{y \in X : {}& y(t) \geq 0 \text{ for all } t \in [0,1], y'(t) \text{ is decreasing on } [0,1], \\
& y(t) \geq \min\{t, (1-t)\} \max_{t\in[0,1]} y(t) \text{ for all } t \in [0,1]\}.
\end{aligned}$$

Then P_3' is a cone in X. Since, for $y \in P_3'$, we have

$$|y(t)| \leq |y(t) - y(0)| + |y(0)| \leq \max_{t\in[0,1]} |y'(t)| + \alpha \max_{t\in[0,1]} |y'(t)| = (1+\alpha)\gamma(y),$$

we get $\max_{t\in[0,1]} |y(t)| \leq (1+\alpha)\gamma(y)$. It is easy to see that there exists a constant $M > 0$ such that $\|y\| \leq M\gamma(y)$ for all $y \in P_3'$.

Define the nonlinear operator $T_3' : P_3' \to X$ by

$$(T_3'y)(t) = B_y - \int_t^1 \phi^{-1}\Big(A_y - \int_0^s f(u, y(u)+h, y'(u))du\Big)ds, \quad y \in P_3',$$

where

$$\begin{aligned}
&\Big(1-\sum_{i=1}^m \beta_i\Big)\Big(\int_0^1 \phi^{-1}\Big(A_y - \int_0^s f(u, y(u)+h, y'(u))du\Big)ds + \alpha\phi^{-1}(A_y)\Big) \\
&= -\sum_{i=1}^m \beta_i \int_{\xi_i}^1 \phi^{-1}\Big(A_y - \int_0^s f(u, y(u)+h, y'(u))du\Big)ds + B - \frac{A}{1-\sum_{i=1}^m \beta_i}.
\end{aligned}$$

and

$$B_y = \int_0^1 \phi^{-1}\Big(A_y - \int_0^s f(u, y(u)+h, y'(u))du\Big)ds + \alpha\phi^{-1}(A_y).$$

Then

$$\begin{aligned}
(T_3'y)(t) &= \int_0^1 \phi^{-1}\Big(A_y - \int_0^s f(u, y(u)+h, y'(u))du\Big)ds + \alpha\phi^{-1}(A_y) \\
&\quad - \int_t^1 \phi^{-1}\Big(A_y - \int_0^s f(u, y(u)+h, y'(u))du\Big)ds \\
&= \int_0^t \phi^{-1}\Big(A_y - \int_0^s f(u, y(u)+h, y'(u))du\Big)ds + \alpha\phi^{-1}(A_y), \quad y \in P_3'.
\end{aligned}$$

Lemma 2.24. *Assume* (H11)–(H13)*. Then the following holds:*

(i) $T_3'y \in P_3'$ *for each* $y \in P_3'$*;*
(ii) x *is a solution of* (1.16)*' if and only if* $x = y + h$ *and* y *is a solution of the operator equation* $T_3'y = y$ *in* P_3'*;*

(iii) $T_3' : P_3' \to P_3'$ *is completely continuous;*
(iv) *the following equalities hold:*

$$\begin{gathered}[\phi((T_3'y)'(t))]' + f\big(t, y(t)+h, y'(t)\big) = 0, \quad t \in (0,1), \\ (T_3'y)(0) - \alpha(T_3'y)(\xi) = 0, \\ (T_3'y)(1) - \sum_{i=1}^m \beta_i (T_3'y)(\xi_i) = B - \Big(1 - \sum_{i=1}^m \beta_i\Big)A;\end{gathered}$$

The proof of the above lemma is similar to that of Lemma 2.9, so it is omitted.

Theorem 2.25. *Suppose that* H11)–(H13) *hold and there exist positive constants* e_1, e_2, c,

$$\begin{gathered}Q = \min\Big\{\phi\Big(\frac{c}{\int_0^1 \phi^{-1}(2-s)ds + \alpha\phi^{-1}(2)}\Big),\ \frac{\phi(c)}{2}\Big\}; \\ W = \phi\Big(\frac{e_2}{\sigma_0 \min\big\{\int_k^{1-k} \phi^{-1}\,(1-k-s)\,ds,\ \int_k^{1-k}\phi^{-1}(s-k)ds\big\}}\Big); \\ E = \phi\Big(\frac{e_1}{\int_0^1 \phi^{-1}(2-s)ds + \alpha\phi^{-1}(2)}\Big).\end{gathered}$$

such that

$$c \ge \frac{e_2}{\sigma_0^2} > e_2 > \frac{e_1}{\sigma_0} > e_1 > \frac{\alpha}{\alpha\Big(1 - \sum_{i=1}^m \beta_i\Big) + 1 - \sum_{i=1}^m \beta_i\xi_i}\Big(B - \frac{A}{1-\sum_{i=1}^m \beta_i}\Big),$$

and

$$Q \ge \phi\Big(\frac{B\big(1+\sum_{i=1}^m \beta_i\big)}{1-\sum_{i=1}^m \beta_i}\Big), \quad Q > W.$$

If (A1)–(A3) *defined in Theorem 2.10 hold, then* (1.16) *has at least three positive solutions* x_1, x_2, x_3 *such that*

$$\begin{gathered}\max_{t\in[0,1]} x_1(t) < e_1 + h, \quad \min_{t\in[k,1-k]} x_2(t) > e_2 + h, \\ \min_{t\in[k,1-k]} x_3(t) > e_1 + h, \quad \min_{t\in[k,1-k]} x_3(t) < e_2 + h.\end{gathered}$$

The proof of the above theroem is similar to that Theorem 2.16 and it is omitted.

Remark 2.26. Kwong and Wong [29], Palamides [51] studied the existence of positive solutions of (1.6) and (1.7) (the main results may be seen in Section 1). When applying Theorem 2.21 to (1.6), we get three positive solutions if $\theta \in (0, \pi/2]$ and the other assumptions in Theorem 2.21 hold.

2.4. **Positive Solutions of** (1.17)**.** We prove existence results for three positive solutions of (1.17). The following assumptions are used in this sub-section.

(H14) $f : [0,1]\times[0,+\infty)\times[0,+\infty) \to [0,+\infty)$ is continuous with $f(t, c+h, 0) \not\equiv 0$ on each sub-interval of [0,1] for all $c \ge 0$, where $h = A$;
(H15) $\alpha_i \ge 0$, $\beta_i \ge 0$ satisfy $\sum_{i=1}^m \alpha_i < 1$, $\sum_{i=1}^m \beta_i < 1$;
(H16) $A \ge 0, B \ge 0$ with $B \ge \frac{A}{1-\sum_{i=1}^m \beta_i}$.

Suppose that (H4)–(H16) hold and consider the boundary-value problem

$$
\begin{gathered}
[\phi(y'(t))]'+h(t)=0, \quad t\in(0,1),\\
y(0)-\sum_{i=1}^m \alpha_i y'(\xi_i)=0, \quad y'(1)-\sum_{i=1}^m \beta_i y'(\xi_i)=D,
\end{gathered}
\tag{2.13}
$$

Lemma 2.27. *Assume* (H4), (H15), (H16) *and* $D\ge 0$. *If* y *is a solution of* (2.13), *then* y *is positive on* $(0,1)$.

Proof. Suppose y satisfies (2.13). It follows from the assumptions that y' is decreasing on $[0,1]$. Then the BCs in (2.13) and (H4) imply

$$
y'(1)=\sum_{i=1}^m \beta_i y'(\xi_i)+D\ge \sum_{i=1}^m \beta_i y'(1).
$$

It follows that $y'(1)\ge 0$. We get $y'(t)\ge 0$ for $t\in[0,1]$. Then

$$
y(0)=\sum_{i=1}^m \alpha_i y(\xi_i)\ge \sum_{i=1}^m \alpha_i y(0).
$$

It follows that $y(0)\ge 0$. Then $y(t)>y(0)\ge 0$ for $t\in(0,1)$ since $y'(t)\ge 0$ for all $t\in[0,1]$. The proof is complete. □

Lemma 2.28. *Assume* (H4)–(H16). *If* y *is a solution of* (2.13), *then*

$$
y(t)=B_h+\int_0^t \phi^{-1}\Big(A_h+\int_s^1 h(u)du\Big)ds
$$

and there exists unique $A_h\in[a,b]$ *such that*

$$
\phi^{-1}(A_h)=\sum_{i=1}^m \beta_i\phi^{-1}\left(A_h+\int_{\xi_i}^1 h(s)ds\right)+D,
\tag{2.14}
$$

and

$$
\begin{gathered}
B_h=\sum_{i=1}^m \alpha_i\phi^{-1}\Big(A_h+\int_{\xi_i}^1 h(u)du\Big),\\
a=\phi\Big(\frac{D}{1-\sum_{i=1}^m \beta_i}\Big),\\
b=\phi\Big(\frac{D\big(1+\sum_{i=1}^m \beta_i\big)}{1-\sum_{i=1}^m \beta_i}\Big)+\frac{\phi\big(\frac{1+\sum_{i=1}^m \beta_i}{2}\big)}{1-\phi\big(\frac{1+\sum_{i=1}^m \beta_i}{2}\big)}\int_0^1 h(s)ds.
\end{gathered}
$$

Proof. It follows from (2.13) that

$$
y(t)=y(0)+\int_0^t \phi^{-1}\Big(\phi(y'(1))+\int_s^1 h(u)du\Big)ds
$$

The BCs in (2.13) imply

$$
\begin{gathered}
y'(1)=\sum_{i=1}^m \beta_i\phi^{-1}\Big(\phi(y'(1))+\int_{\xi_i}^1 h(s)ds\Big)+D,\\
y(0)=\sum_{i=1}^m \alpha_i\phi^{-1}\Big(\phi(y'(1))+\int_{\xi_i}^1 h(u)du\Big).
\end{gathered}
$$

Let

$$G(c)=\phi^{-1}(c)-\sum_{i=1}^{m}\beta_i\phi^{-1}\Big(c+\int_{\xi_i}^{1}h(s)ds\Big)-D.$$

It is easy to see that

$$\begin{aligned}G(a)&=\frac{D}{1-\sum_{i=1}^m\beta_i}-\sum_{i=1}^{m}\beta_i\phi^{-1}\Big(\phi\Big(\frac{D}{1-\sum_{i=1}^m\beta_i}\Big)+\int_{\xi_i}^{1}h(s)ds\Big)-D\\&\le\frac{D}{1-\sum_{i=1}^m\beta_i}-\sum_{i=1}^{m}\beta_i\frac{D}{1-\sum_{i=1}^m\beta_i}-D=0.\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{G(b)}{\phi^{-1}(b)}&=1-\sum_{i=1}^{m}\beta_i\phi^{-1}\Big(1+\frac{\int_{\xi_i}^1h(s)ds}{b}\Big)-\frac{D}{\phi^{-1}(b)}\\&=1-\sum_{i=1}^{m}\beta_i\phi^{-1}\Bigg(1+\frac{\int_{\xi_i}^1h(s)ds}{\phi\Big(\frac{D\left(1+\sum_{i=1}^m\beta_i\right)}{1-\sum_{i=1}^m\beta_i}\Big)+\frac{\phi\left(\frac{1+\sum_{i=1}^m\beta_i}{2}\right)}{1-\phi\left(\frac{1+\sum_{i=1}^m\beta_i}{2}\right)}\int_0^1h(s)ds}\Bigg)\\&\quad-\frac{D}{\phi^{-1}\Big(\phi\Big(\frac{D\left(1+\sum_{i=1}^m\beta_i\right)}{1-\sum_{i=1}^m\beta_i}\Big)+\frac{\phi\left(\frac{1+\sum_{i=1}^m\beta_i}{2}\right)}{1-\phi\left(\frac{1+\sum_{i=1}^m\beta_i}{2}\right)}\int_0^1h(s)ds\Big)}\\&\le1-\sum_{i=1}^{m}\beta_i\phi^{-1}\Big(1+\frac{1-\phi\left(\frac{1+\sum_{i=1}^m\beta_i}{2}\right)}{\phi\left(\frac{1+\sum_{i=1}^m\beta_i}{2}\right)}\Big)-\frac{1-\sum_{i=1}^m\beta_i}{1+\sum_{i=1}^m\beta_i}=0.\end{aligned}$$

Hence $G(b)\ge0$. It is easy to know that $\frac{G(c)}{\phi^{-1}(c)}$ is continuous and decreasing on $(-\infty,0)$ and continuous and increasing on $(0,+\infty)$, Hence $G(a)\le0$ and $G(b)\ge0$ and

$$\lim_{c\to0^-}\frac{G(c)}{\phi^{-1}(c)}=+\infty,\quad\lim_{c\to-\infty}\frac{G(c)}{\phi^{-1}(c)}=1-\sum_{i=1}^m\beta_i>0.$$

Then there exists unique constant $A_h=\phi(y'(1))\in[a,b]$ such that (2.14) holds. The proof is complete. □

Suppose (H16) holds. Let $x(t)-A=y(t)$. Then (1.17) is transformed into

$$\begin{gathered}[\phi(y'(t))]'+f\big(t,y(t)+h,y'(t)\big)=0,\quad t\in(0,1),\\y(0)-\sum_{i=1}^m\alpha_iy'(\xi_i)=0,\\y'(1)-\sum_{i=1}^m\beta_iy'(\xi_i)=B-\frac{A}{1-\sum_{i=1}^m\beta_i}\ge0,\end{gathered}\tag{2.15}$$

Let

$$\begin{aligned}P_4=\{y\in X:y(t)\ge0\text{ for all }t\in[0,1],y'(t)\text{ is decreasing on }[0,1],\\y(t)\ge ty(1)\text{ for all }t\in[0,1]\}.\end{aligned}$$

Then P_4 is a cone in X. For $y \in P_4$, since

$$|y(t)| = |y(t) - y(0)| + |y(0)| \le \Big(1 + \sum_{i=1}^m \alpha_i\Big) \max_{t\in[0,1]} |y'(t)| = \Big(1 + \sum_{i=1}^m \alpha_i\Big)\gamma(y),$$

we get

$$\max_{t\in[0,1]} |y(t)| \le \Big(1 + \sum_{i=1}^m \alpha_i\Big)\gamma(y).$$

It is easy to see that there exists a constant $M > 0$ such that $\|y\| \le M\gamma(y)$ for all $y \in P_1$.

Define the operator $T_4 : P_4 \to X$ by

$$(T_4y)(t) = B_y + \int_0^t \phi^{-1}\Big(A_y + \int_s^1 f(u, y(u) + h, y'(u))du\Big)ds, \quad y \in P_4,$$

where

$$\phi^{-1}(A_h) = \sum_{i=1}^m \beta_i\phi^{-1}\Big(A_h + \int_{\xi_i}^1 f(u, y(u) + h, y'(u))du\Big) + B - \frac{A}{1 - \sum_{i=1}^m \beta_i}, \tag{2.16}$$

and

$$B_y = \sum_{i=1}^m \alpha_i\phi^{-1}\left(A_y + \int_{\xi_i}^1 f(u, y(u) + h, y'(u))du\right).$$

Then

$$\begin{aligned}(T_4y)(t) = &\sum_{i=1}^m \alpha_i\phi^{-1}\Big(A_y + \int_{\xi_i}^1 f(u, y(u) + h, y'(u))du\Big)\\ &+ \int_0^t \phi^{-1}\Big(A_y + \int_s^1 f(u, y(u) + h, y'(u))du\Big)ds.\end{aligned}$$

It follows from Lemma 2.28 that

$$\begin{aligned}\phi\Big(\frac{\Big(B - \frac{A}{1-\sum_{i=1}^m \beta_i}\Big)}{1 - \sum_{i=1}^m \beta_i}\Big) &\le A_y\\ &\le \phi\Big(\frac{\Big(B - \frac{A}{1-\sum_{i=1}^m \beta_i}\Big)\Big(1 + \sum_{i=1}^m \beta_i\Big)}{1 - \sum_{i=1}^m \beta_i}\Big)\\ &\quad + \frac{\phi\big(\frac{1+\sum_{i=1}^m \beta_i}{2}\big)}{1 - \phi\big(\frac{1+\sum_{i=1}^m \beta_i}{2}\big)} \int_0^1 f(u, y(u) + h, y'(u))du.\end{aligned}$$

Lemma 2.29. *Suppose that* (H14)–(H16). *Then*

(i) *the following equalities hold:*

$$\begin{gathered}[\phi((T_4y)'(t))]' + f\big(t, y(t) + h, y'(t)\big) = 0, \quad t \in (0, 1),\\ (T_4y)(0) - \sum_{i=1}^m \alpha_i(T_4y)'(\xi_i) = 0,\\ (T_4y)'(1) - \sum_{i=1}^m \beta_i(T_4y)'(\xi_i) = B - \frac{A}{1 - \sum_{i=1}^m \beta_i};\end{gathered}$$

(ii) $T_4y \in P_4$ *for each* $y \in P_4$;

(iii) *x is a solution of* (1.17) *if and only if $x = y + h$ and y is a solution of the operator equation $y = T_4 y$ in P_4;*
(iv) *$T_4 : P_4 \to P_4$ is completely continuous.*

The proof of the above lemma is similar to that of Lemma 2.9; so we omit it.

Theorem 2.30. *Suppose that* (H14)–(H16) *hold, and there exist positive constants e_1, e_2, c, and*

$$
\begin{aligned}
L = &\int_0^1 \phi^{-1}\Big(1 + \frac{\phi\big(\frac{1+\sum_{i=1}^m \beta_i}{2}\big)}{1-\phi\big(\frac{1+\sum_{i=1}^m \beta_i}{2}\big)} + 1 - s\Big) ds \\
&+ \sum_{i=1}^m \alpha_i \phi^{-1}\Big(1 + \frac{\phi\big(\frac{1+\sum_{i=1}^m \beta_i}{2}\big)}{1-\phi\big(\frac{1+\sum_{i=1}^m \beta_i}{2}\big)} + 1 - \xi_i\Big), \\
Q = &\min\big\{\phi(\frac{c}{L}),\ \frac{\phi(c)}{1+\frac{\phi(2)}{\phi(2)-\phi\big(1+\sum_{i=1}^m \beta_i\big)}}\big\}; \\
W = &\phi\Big(\frac{e_2}{\sigma_0 \int_k^{1-k} \phi^{-1}(1-k-s)\, ds}\Big); \quad E = \phi(\frac{e_1}{L}).
\end{aligned}
$$

such that $c \ge \frac{e_2}{\sigma_0^2} > e_2 > \frac{e_1}{\sigma_0} > e_1 > 0$,

$$
Q \ge \phi\Big(\frac{\big(B - \frac{A}{1-\sum_{i=1}^m \beta_i}\big)\,(1+\sum_{i=1}^m \beta_i)}{1-\sum_{i=1}^m \beta_i}\Big), \quad Q > W.
$$

If (A1)–(A3) *in Theorem 2.10 hold, then* (1.17) *has at least three increasing positive solutions x_1, x_2, x_3 such that*

$$
\begin{gathered}
\max_{t\in[0,1]} x_1(t) < e_1 + h, \qquad \min_{t\in[k,1-k]} x_2(t) > e_2 + h, \\
\min_{t\in[k,1-k]} x_3(t) > e_1 + h, \qquad \min_{t\in[k,1-k]} x_3(t) < e_2 + h.
\end{gathered}
$$

The proof of the above theorem is similar to that of Theorem 2.16; Therefore, it is omitted.

2.5. **Positive Solutions of** (1.18)**.** Finally, we prove an existence result for three positive solutions of (1.18). The following assumptions will be used in the proofs of all results in this subsection.

(H17) $f : [0,1] \times [h, +\infty) \times \mathbb{R} \to [0, +\infty)$ is continuous with $f(t, c+h, 0) \not\equiv 0$ on each sub-interval of [0,1] for all $c \ge 0$, where $h = \frac{A}{1-\sum_{i=1}^m \alpha_i}$;
(H18) $\alpha_i \ge 0$, $\beta_i \ge 0$ satisfy $\sum_{i=1}^m \alpha_i < 1$, $\sum_{i=1}^m \beta_i < 1$;
(H19) $A \ge 0, B \ge 0$ with $B \ge \frac{1-\sum_{i=1}^m \beta_i}{1-\sum_{i=1}^m \alpha_i} A$.

Consider the boundary-value problem

$$
\begin{gathered}
[\phi(y'(t))]' + h(t) = 0, \quad t \in (0,1), \\
y(0) - \sum_{i=1}^m \alpha_i y(\xi_i) = 0, \quad y(1) - \sum_{i=1}^m \beta_i y(\xi_i) = B,
\end{gathered} \tag{2.17}
$$

Lemma 2.31. *Assume* (H4), (H18), (H19)*. If y is a solution of* (2.17)*, then y is positive on* $(0,1)$.

Proof. Suppose y satisfies (2.17). It follows from the assumptions that y' is decreasing on $[0,1]$. Then the BCs in (2.17) and (H4) imply that $y(t) \geq \min\{y(0), y(1)\}$ for all $t \in [0,1]$. Then

$$y(0) \geq \sum_{i=1}^{m} \alpha_i \min\{y(0), y(1)\}. \tag{2.18}$$

Similarly, we get

$$y(1) \geq \sum_{i=1}^{m} \beta_i \min\{y(0), y(1)\}. \tag{2.19}$$

It follows from (2.18) and (2.19) that

$$\min\{y(0), y(1)\} \geq \min\Big\{\sum_{i=1}^{m} \alpha_i,\ \sum_{i=1}^{m} \beta_i\Big\} \min\{y(0), y(1)\}.$$

Then (H18) implies that $\min\{y(0), y(1)\} \geq 0$. So $y(t) \geq \min\{y(0), y(1)\} \geq 0$ for all $t \in [0,1]$. The proof is complete. □

Lemma 2.32. *Assume* (H4), (H18), (H19). *If y is a solution of* (2.17)*, then there exists unique $A_h \in [0,b]$ such that*

$$\begin{aligned}
&\frac{1-\sum_{i=1}^m \beta_i}{1-\sum_{i=1}^m \alpha_i} \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} \phi^{-1}\Big(A_h - \int_0^s h(u)du\Big)ds \\
&+ \int_0^1 \phi^{-1}\Big(A_h - \int_0^s h(u)du\Big)ds \\
&- \sum_{i=1}^{m} \beta_i \int_0^{\xi_i} \phi^{-1}\Big(A_h - \int_0^s h(u)du\Big)ds - B = 0.
\end{aligned}$$

where

$$b = \int_0^1 h(u)du + \phi\big(\frac{B}{a}\big), \quad a = \sum_{i=1}^{m} \alpha_i \xi_i + \Big(1 - \sum_{i=1}^{m} \beta_i\Big) + \sum_{i=1}^{m} \beta_i (1-\xi_i).$$

Proof. From (2.17) it follows that

$$y(t) = y(0) + \int_0^t \phi^{-1}\Big(\phi(y'(0)) - \int_0^s h(u)du\Big)ds.$$

The BCs in (2.17) implies

$$y(0) = y(0)\sum_{i=1}^{m} \alpha_i + \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} \phi^{-1}\Big(\phi(y'(0)) - \int_0^s h(u)du\Big)ds$$

and

$$\begin{aligned}
&y(0) + \int_0^1 \phi^{-1}\Big(\phi(y'(0)) - \int_0^s h(u)du\Big)ds \\
&= y(0)\sum_{i=1}^{m} \beta_i + \sum_{i=1}^{m} \beta_i \int_0^{\xi_i} \phi^{-1}\Big(\phi(y'(0)) - \int_0^s h(u)du\Big)ds + B.
\end{aligned}$$

Then

$$\frac{1-\sum_{i=1}^m \beta_i}{1-\sum_{i=1}^m \alpha_i} \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} \phi^{-1}\Big(\phi(y'(0)) - \int_0^s h(u)du\Big)ds$$

$$
\begin{aligned}
&+\int_0^1 \phi^{-1}\Big(\phi(y'(0)) - \int_0^s h(u)du\Big)ds \\
&-\sum_{i=1}^m \beta_i \int_0^{\xi_i} \phi^{-1}\Big(\phi(y'(0)) - \int_0^s h(u)du\Big)ds - B = 0.
\end{aligned}
$$

Let

$$
\begin{aligned}
G(c) = &\frac{1-\sum_{i=1}^m \beta_i}{1-\sum_{i=1}^m \alpha_i} \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \phi^{-1}\Big(c - \int_0^s h(u)du\Big)ds \\
&+\int_0^1 \phi^{-1}\Big(c - \int_0^s h(u)du\Big)ds \\
&-\sum_{i=1}^m \beta_i \int_0^{\xi_i} \phi^{-1}\Big(c - \int_0^s h(u)du\Big)ds - B \\
=&\frac{1-\sum_{i=1}^m \beta_i}{1-\sum_{i=1}^m \alpha_i} \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \phi^{-1}\Big(c - \int_0^s h(u)du\Big)ds \\
&+\Big(1-\sum_{i=1}^m \beta_i\Big) \int_0^1 \phi^{-1}\Big(c - \int_0^s h(u)du\Big)ds \\
&+\sum_{i=1}^m \beta_i \int_{\xi_i}^1 \phi^{-1}\Big(c - \int_0^s h(u)du\Big)ds - B.
\end{aligned}
$$

It is easy to see that $G(c)$ is increasing on $(-\infty, +\infty)$ and

$$
\begin{aligned}
G(0) = &-\frac{1-\sum_{i=1}^m \beta_i}{1-\sum_{i=1}^m \alpha_i} \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \phi^{-1}\Big(\int_0^s h(u)du\Big)ds \\
&-\Big(1-\sum_{i=1}^m \beta_i\Big) \int_0^1 \phi^{-1}\Big(\int_0^s h(u)du\Big)ds \\
&-\sum_{i=1}^m \beta_i \int_{\xi_i}^1 \phi^{-1}\Big(\int_0^s h(u)du\Big)ds - B < 0,
\end{aligned}
$$

and

$$
\begin{aligned}
G(b) &= G\Big(\int_0^1 h(u)du + \phi(\frac{B}{a})\Big) \\
&= \frac{1-\sum_{i=1}^m \beta_i}{1-\sum_{i=1}^m \alpha_i} \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \phi^{-1}\Big(\phi(\frac{B}{a}) + \int_s^1 h(u)du\Big)ds \\
&\quad+\Big(1-\sum_{i=1}^m \beta_i\Big) \int_0^1 \phi^{-1}\Big(\phi(\frac{B}{a}) + \int_s^1 h(u)du\Big)ds \\
&\quad+\sum_{i=1}^m \beta_i \int_{\xi_i}^1 \phi^{-1}\Big(\phi(\frac{B}{a}) + \int_s^1 h(u)du\Big)ds - B \\
&\ge \frac{1-\sum_{i=1}^m \beta_i}{1-\sum_{i=1}^m \alpha_i} \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \phi^{-1}\Big(\phi(\frac{B}{a})\Big)ds
\end{aligned}
$$

$$
\begin{aligned}
&+\Big(1-\sum_{i=1}^m \beta_i\Big)\int_0^1 \phi^{-1}\left(\phi(\frac{B}{a})\right)ds\\
&+\sum_{i=1}^m \beta_i \int_{\xi_i}^1 \phi^{-1}\left(\phi(\frac{B}{a})\right)ds - B = 0.
\end{aligned}
$$

Then $G(0) < 0$, $G(b) \geq 0$ and that $G(c)$ is increasing on $(-\infty, +\infty)$ imply that $A_h = \phi(y'(0)) \in [0, b]$ and A_h satisfies

$$
\begin{aligned}
&\frac{1-\sum_{i=1}^m \beta_i}{1-\sum_{i=1}^m \alpha_i}\sum_{i=1}^m \alpha_i \int_0^{\xi_i} \phi^{-1}\Big(A_h - \int_0^s h(u)du\Big)ds\\
&+\int_0^1 \phi^{-1}\Big(A_h - \int_0^s h(u)du\Big)ds\\
&-\sum_{i=1}^m \beta_i \int_0^{\xi_i} \phi^{-1}\Big(A_h - \int_0^s h(u)du\Big)ds - B = 0.
\end{aligned}
$$

□

Let $x(t) - h = y(t)$. Then (1.18) is transformed into

$$
\begin{gathered}
[\phi(y'(t))]' + f\big(t, y(t)+h, y'(t)\big) = 0, \quad t \in (0,1),\\
y(0) - \sum_{i=1}^m \alpha_i y(\xi_i) = 0,\\
y(1) - \sum_{i=1}^m \beta_i y(\xi_i) = B - \frac{1-\sum_{i=1}^m \beta_i}{1-\sum_{i=1}^m \alpha_i}A,
\end{gathered}
\tag{2.20}
$$

Let

$$
\begin{aligned}
P_5 = \big\{&y \in X : y(t) \geq 0 \text{ for all } t \in [0,1], y'(t) \text{ is decreasing on } [0,1],\\
&y(t) \geq \min\{t, 1-t\} \max_{t\in[0,1]} y(t) \text{ for all } t \in [0,1]\big\}
\end{aligned}
$$

Then P_5 is a cone in X. For $y \in P_5$, since

$$
\begin{aligned}
|y(t)| &= |y(t) - y(0) + y(0)|\\
&\leq |y'(\theta)|t + |y(0)|\\
&\leq \max_{t\in[0,1]} |y'(t)| + \Big|\frac{\sum_{i=1}^m \alpha_i y(\xi_i) - \sum_{i=1}^m \alpha_i y(0)}{1-\sum_{i=1}^m \alpha_i}\Big|\\
&\leq \Big(1 + \frac{\sum_{i=1}^m \alpha_i \xi_i}{1-\sum_{i=1}^m \alpha_i}\Big)\max_{t\in[0,1]} |y'(t)|\\
&= \Big(1 + \frac{\sum_{i=1}^m \alpha_i \xi_i}{1-\sum_{i=1}^m \alpha_i}\Big)\gamma(y),
\end{aligned}
$$

we get

$$
\max_{t\in[0,1]} |y(t)| \leq \Big(1 + \frac{\sum_{i=1}^m \alpha_i \xi_i}{1-\sum_{i=1}^m \alpha_i}\Big)\gamma(y).
$$

It is easy to see that there exists a constant $M > 0$ such that $\|y\| \leq M\gamma(y)$ for all $y \in P_5$.

Define the operator $T_5 : P_5 \to X$ by

$$(T_5 y)(t) = B_y + \int_0^t \phi^{-1}\Big(A_y - \int_0^s f(u, y(u)+h, y'(u))du\Big)ds, \quad y \in P_5,$$

where

$$\begin{aligned}
&\frac{1-\sum_{i=1}^m \beta_i}{1-\sum_{i=1}^m \alpha_i}\sum_{i=1}^m \alpha_i \int_0^{\xi_i} \phi^{-1}\Big(A_y - \int_0^s f(u, y(u)+h, y'(u))du\Big)ds \\
&+ \int_0^1 \phi^{-1}\Big(A_y - \int_0^s f(u, y(u)+h, y'(u))du\Big)ds \\
&- \sum_{i=1}^m \beta_i \int_0^{\xi_i} \phi^{-1}\left(A_y - \int_0^s f(u, y(u)+h, y'(u))du\right)ds - B = 0.
\end{aligned}$$

and

$$B_y = \frac{1}{1-\sum_{i=1}^m \alpha_i}\sum_{i=1}^m \alpha_i \int_0^{\xi_i} \phi^{-1}\Big(A_y - \int_0^s f(u, y(u)+h, y'(u))du\Big)ds.$$

Lemma 2.33. *Assume* (H17), (H18), (H19). *Then*

(i) *the following equalities hold:*

$$\begin{gathered}
[\phi((T_5 y)'(t))]' + f\big(t, y(t)+h, y'(t)\big) = 0, \quad t \in (0,1), \\
(T_5 y)(0) - \sum_{i=1}^m \alpha_i (T_5 y)(\xi_i) = 0, \\
(T_5 y)(1) - \sum_{i=1}^m \beta_i (T_5 y)(\xi_i) = B - \frac{1-\sum_{i=1}^m \beta_i}{1-\sum_{i=1}^m \alpha_i} A \geq 0;
\end{gathered}$$

(ii) $T_5 y \in P_5$ *for each* $y \in P_5$;

(iii) x *is a solution of* (1.18) *if and only if* $x = y + h$ *and* y *is a solution of the operator equation* $y = T_5 y$;

(iv) $T_5 : P_5 \to P_5$ *is completely continuous.*

The proof is similar to that of Lemma 2.9; therefore, it is omitted.

Theorem 2.34. *Suppose that* (H17), (H18), (H19) *hold, and that there exist positive constants* e_1, e_2, c,

$$\begin{aligned}
L = &\int_0^1 \phi^{-1}\Big(1 + \frac{\phi\big(1+\sum_{i=1}^m \beta_i\big)}{\phi(2) - \phi\big(1+\sum_{i=1}^m \beta_i\big)} + 1 - s\Big)ds \\
&+ \frac{1}{1-\sum_{i=1}^m \alpha_i}\sum_{i=1}^m \alpha_i \int_{\xi_i}^1 \phi^{-1}\Big(1 + \frac{\phi\big(1+\sum_{i=1}^m \beta_i\big)}{\phi(2) - \phi\big(1+\sum_{i=1}^m \beta_i\big)} + 1 - s\Big)ds,
\end{aligned}$$

$$Q = \min\Big\{\phi\Big(\frac{c}{L}\Big),\ \frac{\phi(c)}{1 + \frac{\phi\left(1+\sum_{i=1}^m \beta_i\right)}{\phi(2)-\phi\left(1+\sum_{i=1}^m \beta_i\right)}}\Big\};$$

$$W = \phi\Big(\frac{e_2}{\sigma_0 \int_k^{1-k} \phi^{-1}(1-k-s)\,ds}\Big); \quad E = \phi\Big(\frac{e_1}{L}\Big).$$

such that $c \geq \frac{e_2}{\sigma_0^2} > e_2 > \frac{e_1}{\sigma_0} > e_1 > 0$,

$$Q \geq \phi\Big(\frac{B\big(1+\sum_{i=1}^m \beta_i\big)}{1-\sum_{i=1}^m \beta_i}\Big), \quad Q > W.$$

If (A1)–(A3) *in Theorem 2.10 hold, then* (1.18) *has at least three positive solutions* x_1, x_2, x_3 *such that*

$$\max_{t\in[0,1]} x_1(t) < e_1 + h, \quad \min_{t\in[k,1-k]} x_2(t) > e_2 + h,$$
$$\min_{t\in[k,1-k]} x_3(t) > e_1 + h, \quad \min_{t\in[k,1-k]} x_3(t) < e_2 + h.$$

The proof of the above theorem is similar to that of Theorem 2.10; it is omitted.

For (1.18), we have the following assumptions:

- (H19) $f : [0,1]\times[h,+\infty)\times\mathbb{R} \to [0,+\infty)$ is continuous with $f(t, c+h, 0) \not\equiv 0$ on each sub-interval of [0,1] for all $c \geq 0$, where $h = \frac{B}{1-\sum_{i=1}^m \beta_i}$;
- (H20) $\alpha_i \geq 0$, $\beta_i \geq 0$ satisfy $\sum_{i=1}^m \alpha_i < 1$, $\sum_{i=1}^m \beta_i < 1$;
- (H21) $A \geq 0, B \geq 0$ with $B \leq \frac{1-\sum_{i=1}^m \beta_i}{1-\sum_{i=1}^m \alpha_i} A$.

Consider the boundary-value problem

$$\begin{gathered} [\phi(y'(t))]' + h(t) = 0, \quad t \in (0,1), \\ y(0) - \sum_{i=1}^m \alpha_i y(\xi_i) = A, \quad y(1) - \sum_{i=1}^m \beta_i y(\xi_i) = 0, \end{gathered} \tag{2.21}$$

Lemma 2.35. *Assume* (H4), (H20), (H21). *If* y *is a solution of* (2.21), *then* y *is positive on* $(0,1)$.

The proof of the above lemma is similar to that of Lemma 2.24; it is omitted.

Lemma 2.36. *Assume* (H4), (H20), (H21). *If* y *is a solution of* (2.21), *then*

$$y(t) = B_h - \int_t^1 \phi^{-1}\Big(A_h + \int_s^1 h(u)du\Big)ds.$$

where

$$\begin{aligned} &-\frac{1-\sum_{i=1}^m \alpha_i}{1-\sum_{i=1}^m \beta_i}\sum_{i=1}^m \beta_i \int_{\xi_i}^1 \phi^{-1}\Big(A_h + \int_s^1 h(u)du\Big)ds \\ &-\int_0^1 \phi^{-1}\Big(A_h + \int_s^1 h(u)du\Big)ds \\ &+\sum_{i=1}^m \alpha_i \int_{\xi_i}^1 \phi^{-1}\Big(A_h + \int_s^1 h(u)du\Big)ds - A = 0, \end{aligned}$$

$$b = -\int_0^1 h(u)du - \phi\big(\frac{A}{a}\big),$$

$$a = \frac{1-\sum_{i=1}^m \alpha_i}{1-\sum_{i=1}^m \beta_i}\sum_{i=1}^m \beta_i(1-\xi_i) + \Big(1-\sum_{i=1}^m \alpha_i\Big) + \sum_{i=1}^m \alpha_i \xi_i,$$

$$B_h = -\frac{1}{1-\sum_{i=1}^m \beta_i}\sum_{i=1}^m \beta_i \int_{\xi_i}^1 \phi^{-1}\Big(\phi(y'(1)) + \int_s^1 h(u)du\Big)ds.$$

Proof. It follows from (2.21) that

$$y(t)=y(1)-\int_t^1\phi^{-1}\Big(\phi(y'(1))+\int_s^1 h(u)du\Big)ds.$$

The BCs in (2.21) implies

$$y(1)=y(1)\sum_{i=1}^m\beta_i-\sum_{i=1}^m\beta_i\int_{\xi_i}^1\phi^{-1}\Big(\phi(y'(1))+\int_s^1 h(u)du\Big)ds$$

and

$$\begin{aligned}
&y(1)-\int_0^1\phi^{-1}\Big(\phi(y'(1))+\int_s^1 h(u)du\Big)ds\\
&=y(1)\sum_{i=1}^m\alpha_i-\sum_{i=1}^m\alpha_i\int_{\xi_i}^1\phi^{-1}\Big(\phi(y'(1))+\int_s^1 h(u)du\Big)ds+A.
\end{aligned}$$

Then

$$\begin{aligned}
&-\frac{1-\sum_{i=1}^m\alpha_i}{1-\sum_{i=1}^m\beta_i}\sum_{i=1}^m\beta_i\int_{\xi_i}^1\phi^{-1}\Big(\phi(y'(1))+\int_s^1 h(u)du\Big)ds\\
&-\int_0^1\phi^{-1}\Big(\phi(y'(1))+\int_s^1 h(u)du\Big)ds\\
&+\sum_{i=1}^m\alpha_i\int_{\xi_i}^1\phi^{-1}\Big(\phi(y'(1))+\int_s^1 h(u)du\Big)ds-A=0.
\end{aligned}$$

Let

$$\begin{aligned}
G(c)&=-\frac{1-\sum_{i=1}^m\alpha_i}{1-\sum_{i=1}^m\beta_i}\sum_{i=1}^m\beta_i\int_{\xi_i}^1\phi^{-1}\Big(c+\int_s^1 h(u)du\Big)ds\\
&\quad-\int_0^1\phi^{-1}\Big(c+\int_s^1 h(u)du\Big)ds\\
&\quad+\sum_{i=1}^m\alpha_i\int_{\xi_i}^1\phi^{-1}\Big(c+\int_s^1 h(u)du\Big)ds-A\\
&=-\frac{1-\sum_{i=1}^m\alpha_i}{1-\sum_{i=1}^m\beta_i}\sum_{i=1}^m\beta_i\int_{\xi_i}^1\phi^{-1}\Big(c+\int_s^1 h(u)du\Big)ds\\
&\quad-\Big(1-\sum_{i=1}^m\alpha_i\Big)\int_0^1\phi^{-1}\Big(c+\int_s^1 h(u)du\Big)ds\\
&\quad-\sum_{i=1}^m\alpha_i\int_0^{\xi_i}\phi^{-1}\Big(c+\int_s^1 h(u)du\Big)ds-A.
\end{aligned}$$

It is easy to see that $G(c)$ is decreasing on $(-\infty,+\infty)$ and

$$\begin{aligned}
G(0)=&-\frac{1-\sum_{i=1}^m\alpha_i}{1-\sum_{i=1}^m\beta_i}\sum_{i=1}^m\beta_i\int_{\xi_i}^1\phi^{-1}\Big(\int_s^1 h(u)du\Big)ds\\
&-\Big(1-\sum_{i=1}^m\alpha_i\Big)\int_0^1\phi^{-1}\Big(\int_s^1 h(u)du\Big)ds
\end{aligned}$$

$$-\sum_{i=1}^m\alpha_i\int_0^{\xi_i}\phi^{-1}\Big(\int_s^1 h(u)du\Big)ds - A < 0,$$

and

$$\begin{aligned}
G(b) &= G\Big(-\int_0^1 h(u)du - \phi(\frac{A}{a})\Big)\\
&= -\frac{1-\sum_{i=1}^m\alpha_i}{1-\sum_{i=1}^m\beta_i}\sum_{i=1}^m\beta_i\int_{\xi_i}^1\phi^{-1}\Big(-\int_0^s h(u)du - \phi(\frac{A}{a})\Big)ds\\
&\quad -\Big(1-\sum_{i=1}^m\alpha_i\Big)\int_0^1\phi^{-1}\Big(-\int_0^s h(u)du-\phi(\frac{A}{a})\Big)ds\\
&\quad -\sum_{i=1}^m\alpha_i\int_0^{\xi_i}\phi^{-1}\Big(-\int_0^s h(u)du-\phi(\frac{A}{a})\Big)ds - A\\
&\geq \frac{1-\sum_{i=1}^m\alpha_i}{1-\sum_{i=1}^m\beta_i}\sum_{i=1}^m\beta_i\int_{\xi_i}^1\phi^{-1}\Big(\phi(\frac{A}{a})\Big)ds\\
&\quad +\Big(1-\sum_{i=1}^m\alpha_i\Big)\int_0^1\phi^{-1}\Big(\phi(\frac{A}{a})\Big)ds\\
&\quad +\sum_{i=1}^m\alpha_i\int_0^{\xi_i}\phi^{-1}\Big(\phi(\frac{A}{a})\Big)ds - A = 0.
\end{aligned}$$

Then $G(0) < 0$, $G(b) \geq 0$ and that $G(c)$ is decreasing on $(-\infty,+\infty)$ imply that $A_h = \phi(y'(1)) \in [b,0]$ and A_h satisfies

$$\begin{aligned}
&-\frac{1-\sum_{i=1}^m\alpha_i}{1-\sum_{i=1}^m\beta_i}\sum_{i=1}^m\beta_i\int_{\xi_i}^1\phi^{-1}\Big(A_h+\int_s^1 h(u)du\Big)ds\\
&-\int_0^1\phi^{-1}\Big(A_h+\int_s^1 h(u)du\Big)ds\\
&+\sum_{i=1}^m\alpha_i\int_{\xi_i}^1\phi^{-1}\Big(A_h+\int_s^1 h(u)du\Big)ds - A = 0.
\end{aligned}$$

Let $x(t) - h = y(t)$. Then (1.18) is transformed into

$$\begin{gathered}
[\phi(y'(t))]' + f\big(t, y(t)+h, y'(t)\big) = 0, \quad t\in(0,1),\\
y(0) - \sum_{i=1}^m\alpha_i y(\xi_i) = A - \frac{1-\sum_{i=1}^m\alpha_i}{1-\sum_{i=1}^m\beta_i}B,\\
y(1) - \sum_{i=1}^m\beta_i y(\xi_i) = 0,
\end{gathered} \tag{2.22}$$

Let

$$\begin{aligned}
P_6 = \big\{&y\in X : y(t)\geq 0 \text{ for all } t\in[0,1], y'(t) \text{ is decreasing on } [0,1],\\
&y(t)\geq \min\{t,1-t\}\max_{t\in[0,1]} y(t)\text{for all } t\in[0,1]\big\}
\end{aligned}$$

Then P_6 is a cone in X. For $y \in P_6$, since

$$\begin{aligned}|y(t)| &= \Big|\frac{\sum_{i=1}^m \beta_i y(\xi_i) - \sum_{i=1}^m \beta_i y(1)}{1-\sum_{i=1}^m \beta_i}\Big| \\ &\le \frac{\sum_{i=1}^m \beta_i(1-\xi_i)}{1-\sum_{i=1}^m \beta_i} \max_{t\in[0,1]} |y'(t)| \\ &= \frac{\sum_{i=1}^m \beta_i(1-\xi_i)}{1-\sum_{i=1}^m \beta_i} \beta(y),\end{aligned}$$

we get

$$\max_{t\in[0,1]} |y(t)| \le \frac{\sum_{i=1}^m \beta_i(1-\xi_i)}{1-\sum_{i=1}^m \beta_i} \gamma(y).$$

It is easy to see that there exists a constant $M > 0$ such that $\|y\| \le M\gamma(y)$ for all $y \in P_6$.

Define the operator $T_6 : P_6 \to X$ by

$$(T_6 y)(t) = B_y - \int_t^1 \phi^{-1}\Big(A_y + \int_s^1 f(u, y(u)+h', y'(u))du\Big)ds, \quad y \in P_6,$$

where $A_h \in [b, 0]$ satisfies

$$\begin{aligned}&-\frac{1-\sum_{i=1}^m \alpha_i}{1-\sum_{i=1}^m \beta_i} \sum_{i=1}^m \beta_i \int_{\xi_i}^1 \phi^{-1}\Big(A_h + \int_s^1 h(u)du\Big)ds \\ &- \int_0^1 \phi^{-1}\Big(A_h + \int_s^1 h(u)du\Big)ds \\ &+ \sum_{i=1}^m \alpha_i \int_{\xi_i}^1 \phi^{-1}\Big(A_h + \int_s^1 h(u)du\Big)ds - A = 0,\end{aligned}$$

and B_h satisfies

$$B_h = -\frac{1}{1-\sum_{i=1}^m \beta_i} \sum_{i=1}^m \beta_i \int_{\xi_i}^1 \phi^{-1}\Big(\phi(y'(1)) + \int_s^1 h(u)du\Big)ds.$$

□

Lemma 2.37. *Suppose that* (H19)–(H21) *hold. Then*

(i) *the following equalities hold:*

$$\begin{gathered}[\phi((T_6 y)'(t))]' + f(t, y(t)+h', y'(t)) = 0, \quad t \in (0,1), \\ (T_6 y)(0) - \sum_{i=1}^m \alpha_i (T_6 y)(\xi_i) = A - \frac{1-\sum_{i=1}^m \alpha_i}{1-\sum_{i=1}^m \beta_i} B \ge 0, \\ (T_6 y)(1) - \sum_{i=1}^m \beta_i (T_6 y)(\xi_i) = 0;\end{gathered}$$

(ii) $T_6 y \in P_6$ *for each* $y \in P_6$*;*

(iii) x *is a solution of* (1.18) *if and only if* $x = y + h$ *and* y *is a solution of the operator equation* $y = T_6 y$*;*

(iv) $T_6 : P_6 \to P_6$ *is completely continuous.*

The proof of the above lemma is similar to that of Lemma 2.9; we omit it.

Theorem 2.38. *Suppose that* (H19)–(H21) *hold and that there exist positive constants* e_1, e_2, c,

$$L = \int_0^1 \phi^{-1}\Big(1 + \frac{\phi\big(1+\sum_{i=1}^m \beta_i\big)}{\phi(2) - \phi\big(1+\sum_{i=1}^m \beta_i\big)} + 1 - s\Big) ds$$

$$+ \frac{1}{1-\sum_{i=1}^m \alpha_i} \sum_{i=1}^m \alpha_i \int_{\xi_i}^1 \phi^{-1}\Big(1 + \frac{\phi\big(1+\sum_{i=1}^m \beta_i\big)}{\phi(2) - \phi\big(1+\sum_{i=1}^m \beta_i\big)} + 1 - s\Big) ds,$$

$$Q = \min\Big\{\phi\big(\frac{c}{L}\big),\ \frac{\phi(c)}{1 + \frac{\phi\left(1+\sum_{i=1}^m \beta_i\right)}{\phi(2)-\phi\left(1+\sum_{i=1}^m \beta_i\right)}}\Big\};$$

$$W = \phi\Big(\frac{e_2}{\sigma_0 \int_k^{1-k} \phi^{-1}(1-k-s)\, ds}\Big); \quad E = \phi\big(\frac{e_1}{L}\big).$$

such that $c \ge \frac{e_2}{\sigma_0^2} > e_2 > \frac{e_1}{\sigma_0} > e_1 > 0$,

$$Q \ge \phi\Big(\frac{B\big(1+\sum_{i=1}^m \beta_i\big)}{1-\sum_{i=1}^m \beta_i}\Big), \quad Q > W.$$

If (A1)–(A3) *in Theorem 2.10 hold, then* (1.18) *has at least three positive solutions* x_1, x_2, x_3 *such that*

$$\max_{t\in[0,1]} x_1(t) < e_1 + h, \quad \min_{t\in[k,1-k]} x_2(t) > e_2 + h,$$
$$\min_{t\in[k,1-k]} x_3(t) > e_1 + h, \quad \min_{t\in[k,1-k]} x_3(t) < e_2 + h.$$

The proof of the above theorem is similar to that of Theorem 2.10; it is omitted.

Remark 2.39. In papers [23, 25], sufficient conditions are found for the existence of solutions of (1.10). It was proved that the whole plane is divided by a "continuous decreasing curve" Γ into two disjoint connected regions $\wedge E$ and $\wedge N$ such that (1.10) has at least one solution for $(\lambda_1, \lambda_2) \in \Gamma$, has at least two solutions for $(\lambda_1, \lambda_2) \in \wedge E \setminus \Gamma$, and has no solution for $(\lambda_1, \lambda_2) \in \wedge N$. The explicit subregions of $\wedge E$ where (1.10) has at least two solutions and two positive solutions, respectively. When applying Theorem 2.30 to (1.10), it shows us that (1.10) has at least three positive solutions under the assumptions $\lambda_2 \ge \frac{1-\sum_{i=1}^m \beta_i}{1-\sum_{i=1}^m \alpha_i}\lambda_1$ and some other assumptions. When applying Theorem 2.34 to (1.10), it shows us that (1.10) has at least three positive solutions under the assumptions $\lambda_1 \ge \frac{1-\sum_{i=1}^m \alpha_i}{1-\sum_{i=1}^m \beta_i}\lambda_2$ and some other assumptions.

Remark 2.40. In paper [58], the authors studied the existence of multiple positive solutions of (1.11) under the assumption $\alpha_i \le \beta_i$ for all $i = 1, \dots, m$ and other assumptions, when we apply Theorem 2.34 and Theorem 2.38 to (1.11), the assumptions $\alpha_i \le \beta_i$ for all $i = 1, \dots, m$ are deleted. So Theorem 2.34 and Theorem 2.38 generalize and improve the theorems in [58].

Remark 2.41. The existence problem on multiple positive solutions of (1.18) is solved in the case $A \ge 0, B \ge 0$, but such problems remains unsolved in the cases $A \ge 0, B < 0$, $A < 0, B \ge 0$ and $A < 0, B < 0$.

3. Examples

Now, we present three examples, whose three positive solutions can not be obtained by theorems in known papers, to illustrate the main results.

Example 3.1. Consider the boundary-value problem

$$\begin{gathered} x''(t) + f(t, x(t), x'(t)) = 0, \quad t \in (0,1), \\ x'(0) = \frac{1}{4}x'(1/4) - 2, \quad x(1) = \frac{1}{4}x(1/2) + 1. \end{gathered} \tag{3.1}$$

Corresponding to (1.14), one sees that $\phi(x) = x = \phi^{-1}(x)$, $\alpha = 2$, $\xi_1 = 1/4, \xi_2 = 1/2$, $\alpha_1 = 1/4, \alpha_0 = 0, \beta_1 = 0, \beta_2 = 1/4$, $A = -2$, $B = 1$.

Choose $k = 1/4$, then $\sigma_0 = 4$, choose $e_1 = 10$, $e_2 = 50$, $c = 20000$. Then

$$\begin{aligned} L &= \int_0^1 \phi^{-1}\Big(1 + \frac{\phi\big(1+\sum_{i=1}^m \alpha_i\big)}{\phi(2) - \phi\left(1+\sum_{i=1}^m \alpha_i\right)} + s\Big)ds \\ &\quad + \frac{1}{1-\sum_{i=1}^m \beta_i}\sum_{i=1}^m \beta_i \int_{\xi_i}^1 \phi^{-1}\Big(1 + \frac{\phi\big(1+\sum_{i=1}^m \alpha_i\big)}{\phi(2) - \phi\left(1+\sum_{i=1}^m \alpha_i\right)} + s\Big)ds = \frac{299}{120}, \end{aligned}$$

$$Q = \min\Big\{\phi\big(\frac{c}{L}\big),\ \frac{\phi(c)}{1 + \frac{\phi(2)}{\phi(2)-\phi\left(1+\sum_{i=1}^m \alpha_i\right)}}\Big\} = \frac{120 \times 20000}{299};$$

$$W = \phi\Big(\frac{e_2}{\sigma_0 \int_k^{1-k} \phi^{-1}(s-k)\,ds}\Big) = 1600; \quad E = \phi\big(\frac{e_1}{L}\big) = \frac{1200}{299}.$$

such that $c \geq \frac{e_2}{\sigma_0^2} > e_2 > \frac{e_1}{\sigma_0} > e_1 > 0$,

$$Q \geq \phi\Big(\frac{-A\big(1+\sum_{i=1}^m \alpha_i\big)}{1-\sum_{i=1}^m \alpha_i}\Big), \quad Q > W.$$

If

$$f_0(u) = \begin{cases} \frac{150}{299}x, & x \in [0,4], \\ \frac{600}{299}, & x \in [4,44], \\ (x-44)\frac{4000-\frac{600}{299}}{54-44} + \frac{600}{299}, & x \in [44,54], \\ 4000, & x \in [54, 20004], \\ x - 16004, & x \geq 20004, \end{cases}$$

and

$$f(t,u,v) = f_0(u) + \frac{1+\sin t}{10000} + \frac{u^2+v^2}{2 \times 10^12},$$

it is easy to see that $c \geq \frac{e_2}{\sigma_0^2} > e_2 > \frac{e_1}{\sigma_0} > e_1 > 0$,

$$Q \geq \phi\Big(\frac{-A\big(1+\sum_{i=1}^m \alpha_i\big)}{1-\sum_{i=1}^m \alpha_i}\Big), \quad Q > W$$

and

- (A1) $f(t,u,v) < \frac{120 \times 20000}{299}$ for all $t \in [0,1], u \in [4, 20004], v \in [-20000, 20000]$;
- (A2) $f(t,u,v) > 1600$ for all $t \in [1/4, 3/4], u \in [54, 804], v \in [-20000, 20000]$;
- (A3) $f(t,u,v) \leq \frac{1200}{299}$ for all $t \in [0,1], u \in [4, 44], v \in [-20000, 20000]$;

then Theorem 2.10 implies that (3.1) has at least three decreasing and positive solutions x_1, x_2, x_3 such that $x_1(0) < 14$, $x_2(3/4) > 54$, $x_3(0) > 14$, $x_3(3/4) < 54$.

Example 3.2. Consider the boundary-value problem

$$\begin{gathered} x''(t)+f(t,x(t),x'(t))=0, \quad t\in(0,1),\\ x(0)=\frac{1}{2}x'(1/4)+2, \quad x'(1)=\frac{1}{4}x'(1/4)+\frac{1}{4}x'(1/2)+5. \end{gathered} \tag{3.2}$$

Corresponding to (1.17), one sees that $\phi(x)=x=\phi^{-1}(x)$, $\alpha=2$, $\xi_1=1/4$, $\xi_2=1/2$, $\alpha_1=1/2$, $\alpha_2=0$, $\beta_1=1/4=\beta_2$, $A=2$, $B=5$, $h=2$.

Choose $k=1/3$, then $\sigma_0=1/3$, $e_1=20$, $e_2=80, c=30000$ and

$$\begin{aligned} L=&\int_0^1\phi^{-1}\Big(1+\frac{\phi\big(\frac{1+\sum_{i=1}^m\beta_i}{2}\big)}{1-\phi\Big(\frac{1+\sum_{i=1}^m\beta_i}{2}\Big)}+1-s\Big)ds\\ &+\sum_{i=1}^m\alpha_i\phi^{-1}\Big(1+\frac{\phi\big(\frac{1+\sum_{i=1}^m\beta_i}{2}\big)}{1-\phi\Big(\frac{1+\sum_{i=1}^m\beta_i}{2}\Big)}+1-\xi_i\Big)=\frac{55}{8},\\ Q=&\min\Big\{\phi\left(\frac{c}{L}\right),\ \frac{\phi(c)}{1+\frac{\phi(2)}{\phi(2)-\phi\big(1+\sum_{i=1}^m\beta_i\big)}}\Big\}=\frac{48000}{11};\\ W=&\phi\Big(\frac{e_2}{\sigma_0\int_k^{1-k}\phi^{-1}(1-k-s)\,ds}\Big)=4320; \quad E=\phi\big(\frac{e_1}{L}\big)=\frac{32}{11}. \end{aligned}$$

such that $c\geq\frac{e_2}{\sigma_0^2}>e_2>\frac{e_1}{\sigma_0}>e_1>0$,

$$Q\geq\phi\Big(\frac{\big(B-\frac{A}{1-\sum_{i=1}^m\beta_i}\big)\,(1+\sum_{i=1}^m\beta_i)}{1-\sum_{i=1}^m\beta_i}\Big), \quad Q>W.$$

If

- (A1) $f(t,u,v)<\frac{48000}{11}$ for all $t\in[0,1], u\in[2,30002], v\in[-30000,30000]$;
- (A2) $f(t,u,v)>4320$ for all $t\in[1/3,2/3], u\in[82,722], v\in[-30000,30000]$;
- (A3) $f(t,u,v)\leq\frac{32}{11}$ for all $t\in[0,1], u\in[2,62], v\in[-30000,30000]$;

then Theorem 2.30 implies that (3.2) has at least three positive solutions x_1,x_2,x_3 such that

$$\begin{gathered} \max_{t\in[0,1]}x_1(t)<22, \quad \min_{t\in[1/3,2/3]}x_2(t)>82,\\ \max_{t\in[0,1]}x_3(t)>22, \quad \min_{t\in[k,1-k]}x_3(t)<82. \end{gathered}$$

Example 3.3. Consider the boundary-value problem

$$\begin{gathered} x''(t)+f(t,x(t),x'(t))=0, \quad t\in(0,1),\\ x(0)=\frac{1}{2}x(1/4)+\frac{1}{3}x(1/2)+2,\\ x(1)=\frac{1}{4}x(1/4)+\frac{1}{4}x(1/2)+8. \end{gathered} \tag{3.3}$$

Corresponding to (1.18), one sees that $\phi(x)=x=\phi^{-1}(x)$, $\xi_1=1/4$, $\xi_2=1/2$, $\alpha_1=1/2$, $\alpha_2=1/3$, $\beta_1=1/4=\beta_2$, $A=2, B=8$, $h\frac{2}{1-\sum_{I=1}^m\beta_i}=16$.

Choose $k=1/4$, then $\sigma_0=1/4$. Choose $e_1=50$, $e_2=250$, $c=400000$ and

$$\begin{aligned} L=&\int_0^1 \phi^{-1}\Big(1+\frac{\phi\Big(1+\sum_{i=1}^m\beta_i\Big)}{\phi(2)-\phi\big(1+\sum_{i=1}^m\beta_i\big)}+1-s\Big)ds\\ &+\frac{1}{1-\sum_{i=1}^m\alpha_i}\sum_{i=1}^m\alpha_i\int_{\xi_i}^1\phi^{-1}\Big(1+\frac{\phi\big(1+\sum_{i=1}^m\beta_i\big)}{\phi(2)-\phi\big(1+\sum_{i=1}^m\beta_i\big)}+1-s\Big)ds=\frac{481}{64},\\ Q&=\min\Big\{\phi\Big(\frac{c}{L}\Big),\quad \frac{\phi(c)}{1+\frac{\phi\big(1+\sum_{i=1}^m\beta_i\big)}{\phi(2)-\phi\big(1+\sum_{i=1}^m\beta_i\big)}}\Big\}=\frac{25600000}{481};\\ W&=\phi\Big(\frac{e_2}{\sigma_0\int_k^{1-k}\phi^{-1}\left(1-k-s\right)ds}\Big)=8000;\quad E=\phi\big(\frac{e_1}{L}\big)=\frac{3200}{481}. \end{aligned}$$

such that $c\geq\frac{e_2}{\sigma_0^2}>e_2>\frac{e_1}{\sigma_0}>e_1>0$,

$$Q\geq\phi\Big(\frac{B\left(1+\sum_{i=1}^m\beta_i\right)}{1-\sum_{i=1}^m\beta_i}\Big),\quad Q>W.$$

If

(A1) $f(t,u,v)<\frac{25600000}{481}$ for all $t\in[0,1],u\in[4,400004],v\in[-400000,400000]$;
(A2) $f(t,u,v)>8000$ for all $t\in[1/4,3/4],u\in[254,4004],v\in[-400000,400000]$;
(A3) $f(t,u,v)\leq\frac{3200}{481}$ for all $t\in[0,1],u\in[4,204],v\in[-400000,400000]$;

then Theorem 2.34 implies that (3.3) has at least three positive solutions x_1,x_2,x_3 such that

$$\begin{gathered} \max_{t\in[0,1]}x_1(t)<54,\quad \min_{t\in[k,1-k]}x_2(t)>254,\\ \max_{t\in[0,1]}x_3(t)>54,\quad \min_{t\in[k,1-k]}x_3(t)<254. \end{gathered}$$

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