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POSITIVITY OF THE GREEN FUNCTIONS FOR HIGHER ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. We consider the equation

$$\sum_{k=0}^{n} a_k(t)x^{(n-k)}(t) = 0, \quad t \ge 0,$$

where $a_0(t) \equiv 1$, $a_k(t)$ $(k = 1, \dots, n)$ are real bounded functions. Assuming that all the roots of the polynomial $z^n + a_1(t)z^{n-1} + \dots + a_n(t)$ $(t \ge 0)$ are real, we derive positivity conditions for the Green function for the Cauchy problem. We also establish a lower estimate for the Green function and a comparison theorem for solutions.

1. Introduction and statement of the main result

In this paper we establish positivity conditions of the Green function for the Cauchy problem (the fundamental solution) for the scalar equation

$$\sum_{k=0}^{n} a_k(t)x^{(n-k)}(t) = 0, \quad t > 0, \tag{1.1}$$

where $a_0(t) \equiv 1$; $a_k(t)$ $(k = 1, \dots, n)$ are real continuous functions bounded on $[0, \infty)$.

The literature on the positive and nonoscillating solutions of ordinary differential equations is very rich, cf. [1, 6, 13, 14, 15] and references therein. In particular, Yu and Levin [11, Section 5] among other remarkable results, proved the following result: Suppose that, the roots $r_1(t), \dots, r_n(t)$ of the polynomial

$$P(z, t) := \sum_{k=0}^{n} a_k(t)z^{n-k}, \quad z \in \mathbb{C}.$$

for each $t \ge 0$ are real and satisfy the inequalities

$$\nu_0 \le r_1(t) < \nu_1 \le r_2(t) < \nu_2 \le \dots < \nu_{n-1} \le r_n(t) \le \nu_n, \quad t \ge 0,$$

where ν_j are constants. Then equation (1.1) has non-oscillating solutions. That result is very useful, see for instance [7, 8] and references therein. It should be

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noted that the existence of non-oscillating solutions does not guarantee the positivity of the Green function. Obtaining the positivity conditions for the Green function requires additional restrictions. On the other hand such conditions are very important for various applications, cf. [9, 10]. To the best of our knowledge, the positivity conditions for the Green function were established only in the cases of the second order equations, cf. [10], and equations with constant coefficients [8]; the nonautonomous higher order differential equations were not found in the available literature.

A solution of (1.1) is a function $x(t)$ having continuous derivatives up to n-order satisfying (1.1) for all $t > 0$ and given initial conditions. Recall that the Green function $G(t,\tau)$ for (1.1) is a function defined for $t \geq \tau \geq 0$, satisfying (1.1) for $t > \tau \geq 0$, and the initial conditions

$$\lim_{t\downarrow\tau} \frac{\partial^k G(t,\tau)}{\partial t^k} = 0 \quad (k = 0, \dots, n-2); \quad \lim_{t\downarrow\tau} \frac{\partial^{n-1} G(t,\tau)}{\partial t^{n-1}} = 1. \tag{1.2}$$

Assume that

$$a_k(t) \leq b_k, \quad t \geq 0;\ k = 1, \dots, n, \tag{1.3}$$

where b_k are constant, and introduce the polynomial

$$Q(\lambda) = \lambda^n + b_1\lambda^{n-1} + b_2\lambda^{n-2} + \dots + b_n.$$

The aim of this paper is to prove the following theorem.

Theorem 1.1. *Assume* (1.3), *and let all the roots of polynomial $Q(z)$ be real and non-negative. Then the Green function for* (1.1) *is positive. Moreover,*

$$\frac{\partial^k G(t,s)}{\partial t^k} \geq 0, \quad t > s \geq 0,\ k = 1, \dots, n-1 \tag{1.4}$$

This theorem is proved in the next section. Below we also consider the case when Q has negative roots. Theorem 1.1 supplements the very interesting recent investigations of higher order differential equations, cf. [2, 4, 16].

2. Proof of Theorem 1.1

Denote by $C(\mathbb{R}_+)$ the Banach space of functions continuous and bounded on $\mathbb{R}_+ := [0,\infty)$ and consider the nonhomogeneous equation

$$\sum_{k=0}^{n} a_k(t) D^{n-k} v(t) = f(t) \tag{2.1}$$

with a positive $f \in C(\mathbb{R}_+)$, $D^k x(t) := \frac{d^k v}{dt^k}, t > 0$, and the zero initial conditions

$$v^{(k)}(0) = 0, \quad k = 0, 1, \dots, n-1. \tag{2.2}$$

Since the coefficients of (2.1) are bounded on $\mathbb{R}_+$, a solution $v(t)$ of problem (2.1)–(2.2) satisfies the conditions

$$|v^{(k)}(t)| \leq M e^{\nu t}, \quad t \geq 0,\ k = 0, 1, \dots, n$$

with constants $M \geq 1$ and ν. So $v(t)$ admits the Laplace transform. Let $\tilde{v}(\lambda)$ be the Laplace transform to $v(t)$, λ the dual variable. Put $\tilde{y}(\lambda) = Q(\lambda)\tilde{v}(\lambda)$. Then

$$v(t) = \frac{1}{2\pi i} \int_{c_0 - i\infty}^{c_0 + i\infty} \frac{e^{\lambda t}\tilde{y}(\lambda)}{Q(\lambda)}\, d\lambda \quad (c_0 = \text{const}). \tag{2.3}$$

We can write as

$$f(t) = P(D,t)v(t) = \frac{1}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty} \frac{e^{\lambda t}P(\lambda,t)\tilde{y}(\lambda)d\lambda}{Q(\lambda)}. \tag{2.4}$$

Hence,

$$f(t) = \frac{1}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty} e^{\lambda t}[1 - \frac{Q(\lambda)-P(\lambda,t)}{Q(\lambda)}]\tilde{y}(\lambda)d\lambda = y(t) - Z(t) \tag{2.5}$$

where $y(t)$ is the Laplace original to $\tilde{y}(\lambda)$ and

$$Z(t) = \frac{1}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty} e^{\lambda t}\frac{Q(\lambda)-P(\lambda,t)}{Q(\lambda)}\tilde{y}(\lambda)d\lambda.$$

By the convolution property,

$$Z(t) = \int_0^t K(t,t-s)y(s)ds,$$

where

$$K(\nu,t) = \frac{1}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty} e^{\lambda t}\frac{Q(\lambda)-P(\lambda,\nu)}{Q(\lambda)}d\lambda, \quad \nu \geq 0.$$

So

$$y(t) - \int_0^t K(t,t-s)y(s)ds = f(t).$$

Take into account that

$$K(\nu,t) = \sum_{k=1}^{n}(b_k - a_k(\nu))\mu_k(t),$$

where

$$\mu_k(t) = \frac{1}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty} e^{\lambda t}\frac{\lambda^{n-k}}{Q(\lambda)}d\lambda, \quad k = 1,\dots,n. \tag{2.6}$$

By [8, Lemma 1.11.2, p. 23]

$$\mu_k(t) = \frac{1}{(n-1)!}\frac{d^{n-1}e^{st}s^{n-k}}{ds^{n-1}}\Big|_{s\in[z_1,z_n]} \geq 0 \tag{2.7}$$

where z_1 is the smallest (nonnegative) root of Q and z_n is the largest root of Q.

Since $b_k - a_k(t) \geq 0, t \geq 0$, we can assert the $K(\nu,t) \geq 0$ for all $\nu, t \geq 0$.

Furthermore, denote by C_τ the Banach space of functions continuous on $[0,\tau]$ with a positive $\tau < \infty$. In addition C_τ^+ denotes the cone of positive functions from C_τ. Introduce on C_τ the Volterra operator V by

$$(Vw)(t) = \int_0^t K(t,t-s)w(s)ds.$$

Then $y - Vy = f$. By the Neumann series,

$$(I-V)^{-1}f = \sum_{k=0}^{\infty} V^k f \geq f \geq 0.$$

Here I is the unit operator. Note that the Neumann series of any Volterra operator with a continuous kernel converges in the sup-norm on each finite segment, since the spectral radius of that operator in a space of continuous functions defined on a finite segment is equal to zero, cf. [3]. So $y(t) \geq f(t), t \in [0,\tau]$. But τ is an arbitrary positive number. So we obtain $y(t) \geq f(t), t \in \mathbb{R}_+$. Recall that $y(t)$ is the

Laplace original to $\tilde{y}(\lambda)$; so according to (2.3) and the convolution property, we get

$$v(t) = \int_0^t \mu_n(t-s)y(s)ds \tag{2.8}$$

where μ_n is defined by (2.6). According to (2.7),

$$\mu_n(t) = \frac{1}{(n-1)!}\frac{d^{n-1}e^{st}}{ds^{n-1}}\big|_{s\in[z_1,z_n]} \geq e^{z_1 t}\frac{t^{n-1}}{(n-1)!}. \tag{2.9}$$

Now the inequality $y(t) \geq f(t), t \geq 0$, yields

$$v(t) \geq \int_0^t \mu_n(t-s)f(s)ds \geq 0, \quad t \geq 0.$$

Thus the solution of problem (2.1)–(2.2) is positive, provided f is positive. But

$$v(t) = \int_0^t G(t,s)f(s)ds. \tag{2.10}$$

Hence it follows that $G(t,s) \geq 0$. Furthermore, by (1.2), (2.3) and the convolution property

$$v^{(k)}(t) = \frac{1}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty} \frac{e^{\lambda t}\lambda^k\tilde{y}(\lambda)}{Q(\lambda)}\,d\lambda = \int_0^t \mu_{n-k}(t-s)y(s)ds.$$

But as it was above shown, $\mu_k(t) \geq 0$, $k = 1,\dots,n$. Thus $v^{(k)}(t) \geq 0$. So by (2.10),

$$v^{(k)}(t) = \int_0^t \frac{\partial^k G(t,s)}{\partial t^k} f(s)ds \geq 0, \quad k = 1,\dots,n-1.$$

Hence (1.4) follows. As claimed.

3. Lower solution estimates and comparison of Green's functions

Lemma 3.1. *Under the hypothesis of Theorem 1.1, for any nonnegative f a solution of problem* (2.1)–(2.2) *satisfies the inequality*

$$v(t) \geq \frac{1}{(n-1)!}\int_0^t e^{z_1(t-s)}(t-s)^{n-1}f(s)ds$$

where $z_1 \geq 0$ is the smallest root of $Q(\lambda)$.

Indeed, this result immediately follows from (2.8) and (2.9). Recall also that $G(t,\tau)$ is a solution of the equation

$$P(D,t)y = \delta(t-\tau), \quad t > 0$$

where $\delta(t)$ is the Dirac Delta function. Hence thanks to the previous lemma we easily get the inequality

$$G(t,\tau) \geq \frac{1}{(n-1)!}(t-\tau)^{n-1}e^{z_1(t-\tau)}, \quad t > \tau.$$

Furthermore, together with (1.1), let us consider the equation

$$\sum_{k=0}^{n-1} c_k(t)x^{(n-k)}(t) = 0, \quad t > 0, \tag{3.1}$$

where $c_k(t)$ are bounded real functions satisfying the conditions

$$c_k(t) \leq a_k(t), \quad t \geq 0;\ k = 1,\dots,n. \tag{3.2}$$

Lemma 3.2. *Let the Green function $G(t,s)$ for (1.1) be positive and the inequalities (1.4) and (3.2) hold. Then the Green function $W(t,s)$ for (3.1) satisfies the inequalities*

$$W(t,s) \geq G(t,s) \geq 0; \quad \frac{\partial^k W(t,s)}{\partial t^k} \geq \frac{\partial^k G(t,s)}{\partial t^k} \geq 0 \tag{3.3}$$

for all $t > s \geq 0$ and $k = 1, \dots, n-1$.

Proof. Rewrite (3.1) as

$$\sum_{k=0}^{n} a_k(t) x^{(n-k)}(t) = \sum_{k=1}^{n} (a_n(t) - c_n(t)) x^{(n-k)}(t), \quad t \geq 0.$$

Then with the notation $w(t) = W(t,0)$, we have

$$w(t) = G(t,0) + \int_0^t G(t,s) \sum_{k=1}^{n} (a_{n-k}(s) - c_{n-k}(s)) w^{(k)}(s) ds. \tag{3.4}$$

Hence, according to (1.2),

$$w^{(k)}(t) = \frac{\partial^k G(t,0)}{\partial t^k} + \int_0^t \frac{\partial^k G(t,s)}{\partial t^k} \sum_{k=0}^{n} (a_{n-k}(s) - c_{n-k}(s)) w^{(k)}(s) ds. \tag{3.5}$$

Rewrite (3.4) and (3.5) as the n-vector equation

$$\widehat{w} = \widehat{G} + \tilde{V} \widehat{w}$$

where $\tilde{V}$ is a Volterra equation with a positive continuous matrix kernel

$$\widehat{G}(t) = \text{column}\left[G(t,0), \frac{\partial G(t,0)}{\partial t}, \dots, \frac{\partial^{n-1} G(t,0)}{\partial t^{n-1}}\right],$$
$$\widehat{w}(t) = \text{column}\left[w(t), w'(t), \dots, w^{(n-1)}(t)\right]$$

. Hence by the Neumann series

$$\widehat{w} = \sum_{k=0}^{\infty} \tilde{V}^k \widehat{G} \geq \widehat{G}.$$

So for $s = 0$ the inequalities (3.3) are proved. But the case $s > 0$ can be similarly proved. As it was above mentioned, the Neumann series of any Volterra operator with a continuous kernel converges in the sup-norm on each finite segment, since the spectral radius of that operator in a space of continuous functions defined on a finite segment is equal to zero. This proves the lemma. □

Note that a relatively special but related comparison result is due to MacKenna and Reichel [12].

4. The case of negative roots

Let $Q(\lambda)$ have at least one negative root. With a positive number r, substitute in (1.1) $x(t) = e^{-rt} w(t)$. After simple calculations we get we equation

$$P(D - r, t) w(t) = 0.$$

Take into account that

$$\begin{aligned} P(z-r,t) &= \sum_{k=0}^{n} a_{n-k}(t)(z-r)^k \\ &= \sum_{k=0}^{n} a_{n-k}(t) \sum_{j=0}^{k} C_k^j (-r)^{k-j} z^j \\ &= \sum_{j=0}^{n} z^j \sum_{k=j}^{n} a_{n-k}(t) C_k^j (-r)^{k-j} \\ &= \sum_{j=0}^{n} z^j \tilde{a}_{n-j}(t,r) \end{aligned}$$

where $C_k^j = \frac{k!}{j!(k-j)!}$ and

$$\tilde{a}_{n-j}(t,r) = \sum_{k=j}^{n} a_{n-k}(t) C_k^j (-r)^{k-j}.$$

Thus we have

$$\sum_{k=0}^{n} \tilde{a}_{n-k}(t,r) w^{(k)}(t) = 0, \quad t > 0,. \tag{4.1}$$

Assume that

$$\tilde{a}_j(t,r) \le \tilde{b}_j(r) \ \ (\tilde{b}_j(r) = \text{const};\ t \ge 0;\ j = 1, \dots, n, \tag{4.2}$$

and introduce the polynomial

$$\tilde{Q}(\lambda, r) = \lambda^n + \tilde{b}_1(r)\lambda^{n-1} + \tilde{b}_2(r)\lambda^{n-2} + \dots + \tilde{b}_n(r).$$

Then applying Theorem 1.1 to equation (4.1), we obtain the following result.

Corollary 4.1. *Under condition* (4.2)*, for a positive number r, let all the roots of polynomial $\tilde{Q}(\lambda, r)$ be real and non-negative. Then the Green function for* (1.1) *is positive. Moreover,*

$$\frac{\partial^k (e^{rt} G(t,s))}{\partial t^k} \ge 0, \quad t > s \ge 0;\ k = 1, \dots, n-1.$$

In particular, consider the equation

$$x'' + a_1(t)x' + a_2(t)x = 0 \tag{4.3}$$

assuming that

$$0 < m_k \le a_k(t) \le M_k, \quad t \ge 0;\ k = 1, 2, \tag{4.4}$$

where m_k and M_k are constant. Then

$$\tilde{a}_1(t,r) = -2r + a_1(t), \tilde{a}_2(t,r) = r^2 - a_1 r + a_2(t).$$

Hence

$$\tilde{a}_1(t,r) \le \tilde{b}_1(r) = -2r + M_1, \tilde{a}_2(t,r) \le \tilde{b}_2(r) = r^2 - m_1 r + M_2.$$

If $\tilde{b}_1(r) < 0, \tilde{b}_2(r) > 0$ and $\tilde{b}_1^2(r) \ge 4\tilde{b}_2(r)$, then $\tilde{Q}(z,r) = z^2 + \tilde{b}_1(r)z + \tilde{b}_2(r)$ has two non-negative roots. Let $m_1^2 > 4M_2$. Take $r \ge M_1$. Then

$$r \ge M_1 > m_1/2 + \sqrt{m_1^2/4 - M_2}$$

and therefore $\tilde{b}_2(r) > 0$. We also should have the inequality
$$(-2r + M_1)^2 \geq 4(r^2 - m_1 r + M_2).$$
Hence $M_1^2 - 4rM_1^2 \geq -4m_1 r + 4M_2$. So we get
$$M_1 \leq r \leq (M_1^2 - 4M_2)/4(M_1 - m_1). \tag{4.5}$$
Now the previous corollary implies the following result.

Corollary 4.2. *Let the conditions* (4.4)*,* $m_1^2 > 4M_2$ *and*
$$1 \leq \frac{(M_1^2 - 4M_2)}{4M_1(M_1 - m_1)}$$
hold, then the Green function $G(t,s)$ *for* (4.1) *is positive. Moreover,*
$$\frac{\partial(e^{rt}G(t,s))}{\partial t} \geq 0, \quad t > s \geq 0,$$
for any r *satisfying* (4.5)*. In particular for* $r = M_1$*.*

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