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EXISTENCE OF SOLUTIONS FOR A RESONANT PROBLEM UNDER LANDESMAN-LAZER CONDITIONS

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Abstract. This article shows the existence of weak solutions in $W_0^1(\Omega)$ to a class of Dirichlet problems of the form

$$-\operatorname{div}(a(x,\nabla u)) = \lambda_1 |u|^{p-2}u + f(x,u) - h$$

in a bounded domain Ω of $\mathbb{R}^N$. Here a satisfies

$$|a(x,\xi)| \le c_0\big(h_0(x) + h_1(x)|\xi|^{p-1}\big)$$

for all $\xi \in \mathbb{R}^N$, a.e. $x \in \Omega$, $h_0 \in L^{\frac{p}{p-1}}(\Omega)$, $h_1 \in L^1_{\rm loc}(\Omega)$, $h_1(x) \ge 1$ for a.e. x in Ω; λ_1 is the first eigenvalue for $-\Delta_p$ on Ω with zero Dirichlet boundary condition and g, h satisfy some suitable conditions.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$. In the present paper we study the existence of weak solutions of the following Dirichlet problem

$$-\operatorname{div}(a(x,\nabla u)) = \lambda_1 |u|^{p-2}u + f(x,u) - h \tag{1.1}$$

where $|a(x,\xi)| \le c_0(h_0(x) + h_1(x)|\xi|^{p-1})$ for any ξ in $\mathbb{R}^N$ and a.e. $x \in \Omega$, $h_0(x) \ge 0$ and $h_1(x) \ge 1$ for any x in Ω. λ_1 is the first eigenvalue for $-\Delta_p$ on Ω with zero Dirichlet boundary condition. We define $X := W_0^{1,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\| = \Big(\int_\Omega |\nabla u|^p\,dx\Big)^{1/p}.$$

It is well-known that

$$\lambda_1 = \inf_{u\in W_0^{1,p}(\Omega)} \Big\{\int_\Omega |\nabla u|^p\,dx : \int_\Omega |u|^p\,dx = 1\Big\}.$$

Recall that λ_1 is simple and positive. Moreover, there exists a unique positive eigenfunction ϕ_1 whose norm in $W_0^{1,p}(\Omega)$ equals to one. Regarding the functions f, we assume that f is a bounded Carathéodory function. We also assume that $h \in L^{p'}(\Omega)$ where $p' = \frac{p}{p-1}$.

In the present paper, we study the case in which h_0 and h_1 belong to $L^{\frac{p}{p-1}}(\Omega)$ and $L^1_{\rm loc}(\Omega)$, respectively. The problem now may be non-uniform in sense that the functional associated to the problem may be infinity for some u in X. Hence, weak

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solutions of the problem must be found in some suitable subspace of X. To our knowledge, such problems were firstly studied by [9, 16, 15]. In order to state our main theorem, let us introduce our hypotheses on the structure of problem (1.1).

Assume that $N \geq 1$ and $p > 1$. Ω be a bounded domain in $\mathbb{R}^N$ having C^2 boundary $\partial\Omega$. Consider $a : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$, $a = a(x, \xi)$, as the continuous derivative with respect to ξ of the continuous function $A : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, $A = A(x, \xi)$, that is, $a(x, \xi) = \frac{\partial A(x,\xi)}{\partial \xi}$. Assume that there are a positive real number c_0 and two nonnegative measurable functions h_0, h_1 on Ω such that $h_1 \in L^1_{\mathrm{loc}}(\Omega)$, $h_0 \in L^{\frac{p}{p-1}}(\Omega)$, $h_1(x) \geq 1$ for a.e. x in Ω.

Suppose that a and A satisfy the hypotheses below

- (A1) $|a(x, \xi)| \leq c_0(h_0(x) + h_1(x)|\xi|^p - 1)$ for all $\xi \in \mathbb{R}^N$, a.e. $x \in \Omega$.
- (A2) There exists a constant $k_1 > 0$ such that
$$A\Big(x, \frac{\xi + \psi}{2}\Big) \leq \frac{1}{2}A(x, \xi) + \frac{1}{2}A(x, \psi) - k_1 h_1(x)|\xi - \psi|^p$$
for all x, ξ, ψ, that is, A is p-uniformly convex
- (A3) A is p-subhomogeneous, that is,
$$0 \leq a(x, \xi)\xi \leq pA(x, \xi)$$
for all $\xi \in \mathbb{R}^N$, a.e. $x \in \Omega$.
- (A4) There exists a constant $k_0 \geq 1/p$ such that
$$A(x, \xi) \geq k_0 h_1(x)|\xi|^p$$
for all $\xi \in \mathbb{R}^N$, a.e. $x \in \Omega$.
- (A5) $A(x, 0) = 0$ for all $x \in \Omega$.

A special case of (1.1) is the following equation involving the p-Laplacian operator

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda_1 |u|^{p-2}u + f(x, u) - h \tag{1.2}$$

We refer the reader to [9, 11, 12, 15, 16] for more examples. We suppose also that there exists

(H1)
$$\lim_{x\to-\infty} f(x, s) = f_{-\infty}(x) \quad , \quad \lim_{x\to+\infty} f(x, s) = f^{+\infty}(x)$$
for almost every $x \in \Omega$.

As is well-known, under (H1), problem (1.2) may not have solutions. In [3, 5, 4], the existence of solutions of (1.2) is shown provided that one of the following two conditions are satisfied

(H2)
$$\int_\Omega f^{+\infty}(x)\phi_1(x)\,dx < \int_\Omega h(x)\phi_1(x)\,dx < \int_\Omega f_{-\infty}(x)\phi_1(x)\,dx.$$

(H2')
$$\int_\Omega f_{-\infty}(x)\phi_1(x)\,dx < \int_\Omega h(x)\phi_1(x)\,dx < \int_\Omega f^{+\infty}(x)\phi_1(x)\,dx.$$

It should be noticed that though conditions (H2) and (H2') look rather similar, the existence proof in [4] is different in each case. Indeed, under (H2) the functional associated with the problem is coercive and achieves a minimum, whereas under (H2') the functional has the geometry of the saddle point theorem. In the present paper, we only consider the case (H2), the case (H2') is still an open question due to the fact that there are some difficulties in verifying geometric conditions of the saddle point theorem.

We also point out that in that papers, the property $pA(x,\xi) = a(x,\xi)\cdot\xi$, which may not hold under our assumptions by (A4), play an important role in the arguments. In this paper, we shall extend some results in [3, 5, 4] in two directions: one is from p-Laplacian operators to general elliptic operators in divergence form and the other is to the case on non-uniform problem.

Since problem (1.1) may be non-uniform, then we must consider the problem in a suitable subspace of X. In fact, we consider the following subspace of $W_0^{1,p}(\Omega)$

$$E = \Big\{u \in W_0^{1,p}(\Omega) : \int_\Omega h_1(x)|\nabla u|^p\,dx < +\infty\Big\}. \tag{1.3}$$

The space E can be endowed with the norm $\|u\|_E = (\int_\Omega h_1(x)|\nabla u|^p\,dx)^{1/p}$. As in [9], it is known that E is an infinite dimensional Banach space. We say that $u \in E$ is a weak solution for problem (1.1) if

$$\int_\Omega a(x,\nabla u)\nabla\phi\,dx - \lambda_1\int_\Omega |u|^{p-2}u\phi\,dx - \int_\Omega f(x,u)\phi\,dx + \int_\Omega h\phi\,dx = 0$$

for all $\phi \in E$. Letting

$$\begin{gathered}
F(x,t) = \int_0^t f(x,s)\,ds,\\
J(u) = \frac{\lambda_1}{p}\int_\Omega |u|^p\,dx + \int_\Omega F(x,u)\,dx - \int_\Omega hu\,dx,\\
\Lambda(u) = \int_\Omega A(x,\nabla u)\,dx,\\
I(u) = \Lambda(u) - J(u)
\end{gathered}$$

for all $u \in E$. The following remark plays an important role in our arguments.

Remark 1.1.

(i) $\|u\| \le \|u\|_E$ for all $u \in E$ since $h_1(x) \ge 1$.

(ii) By (A1), A satisfies the growth condition

$$|A(x,\xi)| \le c_0(h_0(x)|\xi| + h_1(x)|\xi|^p)$$

for all $\xi \in \mathbb{R}^N$, a.e. $x \in \Omega$.

(iii) By (ii) above and (A4), it is easy to see that

$$E = \big\{u \in W_0^{1,p}(\Omega) : \Lambda(u) < +\infty\big\} = \big\{u \in W_0^{1,p}(\Omega) : I(u) < +\infty\big\}.$$

(iv) $C_0^\infty(\Omega) \subset E$ since $|\nabla u|$ is in $C_c(\Omega)$ for any $u \in C_0^\infty(\Omega)$ and $h_1 \in L^1_{\rm loc}(\Omega)$.

(v) By (A4) and Poincaré inequality, we see that

$$\int_\Omega A(x,\nabla u)\,dx \ge \frac{1}{p}\int_\Omega |\nabla u|^p\,dx \ge \frac{\lambda_1}{p}\int_\Omega |u|^p\,dx,$$

for all $u \in W_0^{1,p}(\Omega)$.

Now we describe our main result.

Theorem 1.2. *Assume conditions* (A1)–(A5), (H1)–(H2) *are fulfilled. Then problem* (1.1) *has at least a weak solution in E.*

2. Auxiliary results

Due to the presence of h_1, the functional Λ may not belong to $C^1(E,\mathbb{R})$. This means that we cannot apply the Minimum Principle directly. In this situation, we need some modifications.

Definition 2.1. Let $\mathcal{F}$ be a map from a Banach space Y to $\mathbb{R}$. We say that $\mathcal{F}$ is weakly continuous differentiable on Y if and only if following two conditions are satisfied

(i) For any $u \in Y$ there exists a linear map $D\mathcal{F}(u)$ from Y to $\mathbb{R}$ such that
$$\lim_{t\to 0}\frac{\mathcal{F}(u+tv)-\mathcal{F}(u)}{t} = \langle D\mathcal{F}(u), v\rangle$$
for every $v \in Y$.

(ii) For any $v \in Y$, the map $u \mapsto \langle D\mathcal{F}(u), v\rangle$ is continuous on Y.

Denote by $C^1_w(Y)$ the set of weakly continuously differentiable functionals on Y. It is clear that $C^1(Y) \subset C^1_w(Y)$ where we denote by $C^1(Y)$ the set of all continuously Fréchet differentiable functionals on Y. Now let $\mathcal{F} \in C^1_w(Y)$, we put
$$\|D\mathcal{F}(u)\| = \sup\{|\langle D\mathcal{F}(u), h\rangle : |h \in Y, \|h\| = 1\}$$
for any $u \in Y$, where $\|D\mathcal{F}(u)\|$ may be $+\infty$.

Definition 2.2. We say that $\mathcal{F}$ satisfies the Palais-Smale condition if any sequence $\{u_n\} \subset Y$ for which $\mathcal{F}(u_n)$ is bounded and $\lim_{n\to\infty}\|D\mathcal{F}(u_n)\| = 0$ possesses a convergent subsequence.

The following theorem is our main ingredient.

Theorem 2.3 (The Minimum Principle, see [13])**.** *Let $\mathcal{F} \in C^1_w(Y)$ where Y is a Banach space. Assume that*

(i) *$\mathcal{F}$ is bounded from below, $c = \inf \mathcal{F}$,*

(ii) *$\mathcal{F}$ satisfies Palais-Smale condition.*

Then there exists $u_0 \in Y$ such that $\mathcal{F}(u_0) = c$.

The proof of Theorem 2.3 is similar to the proof of Theorem 3.1 in [6] where we need a modified Deformation Lemma which is proved in [16, Theorem 2.2]. For simplicity of notation, we shall denote $D\mathcal{F}(u)$ by $\mathcal{F}'(u)$. The following lemma concerns the smoothness of the functional Λ.

Lemma 2.4 ([9])**.**

(i) *If $\{u_n\}$ is a sequence weakly converging to u in X, denoted by $u_n \rightharpoonup u$, then $\Lambda(u) \le \liminf_{n\to\infty}\Lambda(u_n)$.*

(ii) *For all $u, z \in E$*
$$\Lambda\Big(\frac{u+z}{2}\Big) \le \frac{1}{2}\Lambda(u) + \frac{1}{2}\Lambda(z) - k_1\|u-z\|^p_E.$$

(iii) *Λ is continuous on E.*

(iv) *Λ is weakly continuously differentiable on E and*
$$\langle \Lambda'(u), v\rangle = \int_\Omega a(x,\nabla u)\nabla v\,dx$$
for all $u, v \in E$.

(v) $\Lambda(u)-\Lambda(v)\geq\langle\Lambda'(v),u-v\rangle$ *for all* $u,v\in E$.

The following lemma concerns the smoothness of the functional J. The proof is standard and simple, so we omit it.

Lemma 2.5.
(i) *If* $u_n\rightharpoonup u$ *in* X, *then* $\lim_{n\to\infty}J(u_n)=J(u)$.
(ii) J *is continuous on* E.
(iii) J *is weakly continuously differentiable on* E *and*

$$\langle J'(u),v\rangle=\lambda_1\int_\Omega|u|^{p-2}uv\,dx+\int_\Omega f(x,u)v\,dx-\int_\Omega hv\,dx$$

for all $u,v\in E$.

3. Proofs

We remark that the critical points of the functional I correspond to the weak solutions of (1.1). Throughout this paper, we sometimes denote by "const" a positive constant.

Lemma 3.1. *The functional* I *satisfies the Palais-Smale condition on* E *provided* (H2) *holds.*

Proof. Let $\{u_n\}$ be a sequence in E and β be a real number such that

$$|I(u_n)|\leq\beta\quad\text{for all } n \tag{3.1}$$

and

$$I'(u_n)\to 0\quad\text{in } E^\star. \tag{3.2}$$

We prove that $\{u_n\}$ is bounded in E. We assume by contradiction that $\|u_n\|_E\to\infty$ as $n\to\infty$.

Let $v_n=u_n/\|u_n\|_E$ for every n. Thus $\{v_n\}$ is bounded in E. By Remark 1.1(i), we deduce that $\{v_n\}$ is bounded in X. Since X is reflexive, then by passing to a subsequence, still denoted by $\{v_n\}$, we can assume that the sequence $\{v_n\}$ converges weakly to some v in X. Since the embedding $X\hookrightarrow L^p(\Omega)$ is compact then $\{v_n\}$ converges strongly to v in $L^p(\Omega)$.

Dividing (3.1) by $\|u_n\|_E^p$ together with Remark 1.1(v), we deduce that

$$\limsup_{n\to+\infty}\Big(\frac{1}{p}\int_\Omega|\nabla v_n|^p\,dx-\frac{\lambda_1}{p}\int_\Omega|v_n|^p\,dx-\int_\Omega\frac{F(x,u_n)}{\|u_n\|_E^p}\,dx+\int_\Omega h\frac{u_n}{\|u_n\|_E^p}\,dx\Big)\leq 0.$$

Since, by the hypotheses on p, f, h and $\{u_n\}$,

$$\limsup_{n\to+\infty}\Big(\int_\Omega\frac{F(x,u_n)}{\|u_n\|_E^p}\,dx+\int_\Omega h\frac{u_n}{\|u_n\|_E^p}\,dx\Big)=0,$$

while

$$\limsup_{n\to+\infty}\int_\Omega|v_n|^p\,dx=\int_\Omega|v|^p\,dx,$$

we have

$$\limsup_{n\to+\infty}\int_\Omega|\nabla v_n|^p\,dx\leq\lambda_1\int_\Omega|v|^p\,dx.$$

Using the weak lower semi-continuity of norm and Poincaré inequality, we get

$$\lambda_1\int_\Omega|v|^p\,dx\leq\int_\Omega|\nabla v|^p\,dx\leq\liminf_{n\to+\infty}\int_\Omega|\nabla v_n|^p\,dx$$

$$\le \limsup_{n\to+\infty} \int_\Omega |\nabla v_n|^p \, dx \le \lambda_1 \int_\Omega |v|^p \, dx.$$

Thus, these inequalities are indeed equalities. Besides, $\{v_n\}$ converges strongly to v in X and

$$\int_\Omega |\nabla v|^p \, dx = \lambda_1 \int_\Omega |v|^p \, dx.$$

This implies, by the definition of ϕ_1, that $v = \pm\phi_1$.

On the other hand, by means of (3.1), we deduce that

$$\begin{aligned} -\beta p \le -p \int_\Omega A(x, \nabla u_n) \, dx + \lambda_1 \int_\Omega |u_n|^p \, dx + p \int_\Omega F(x, u_n) \, dx \\ - p \int_\Omega h u_n \, dx \\ \le \beta p. \end{aligned} \tag{3.3}$$

In view of (3.2), there exists a sequence of positive real numbers $\{\varepsilon_n\}_n$ such that $\varepsilon_n \to 0$ as $n \to +\infty$ and

$$\begin{aligned} -\varepsilon_n \|u_n\|_E \le \int_\Omega a(x, \nabla u_n) \nabla u_n \, dx - \lambda_1 \int_\Omega |u_n|^p \, dx - \int_\Omega f(x, u_n) u_n \, dx + \\ \int_\Omega h u_n \, dx \\ \le \varepsilon_n \|u_n\|_E. \end{aligned} \tag{3.4}$$

Letting

$$g(x,s) = \begin{cases} \dfrac{F(x,s)}{s} & \text{if } s \ne 0, \\ f(x,0) & \text{if } s = 0. \end{cases}$$

We then consider the following two cases.

Case 1: Suppose that $v_n \to -\phi_1$. Letting $n \to +\infty$. Since $u_n(x) \to -\infty$, it follows that

$$\begin{gathered} f(x, u_n(x)) \to f_{-\infty}(x), \text{ a.e } x \in \Omega, \\ g(x, u_n(x)) \to f_{-\infty}(x), \text{ a.e } x \in \Omega. \end{gathered}$$

Therefore, the properties of f and F, the Lebesgue theorem then imply

$$\lim_{n\to+\infty} \int_\Omega (f(x, u_n) v_n - p g(x, u_n) v_n) \, dx = (p-1) \int_\Omega f_{-\infty}(x) \phi_1(x) \, dx. \tag{3.5}$$

On the other hand, by summing up (3.3) and (3.4), we get

$$\begin{aligned} &\int_\Omega (pF(x, u_n) - f(x, u_n) u_n) \, dx + (1-p) \int_\Omega h u_n \, dx \\ &\ge \int_\Omega (a(x, \nabla u_n) \nabla u_n - p A(x, \nabla u_n)) \, dx + \\ &\quad \int_\Omega (pF(x, u_n) - f(x, u_n) u_n) \, dx + (1-p) \int_\Omega h u_n \, dx \\ &\ge -\beta p - \varepsilon_n \|u_n\|_E, \end{aligned}$$

and after dividing by $\|u_n\|_E$, we obtain

$$\int_\Omega (p g(x, u_n) - f(x, u_n) v_n) \, dx + (1-p) \int_\Omega h v_n \, dx \ge -\frac{\beta p}{\|u_n\|_E} - \varepsilon_n. \tag{3.6}$$

Since $h \in L^{p'}$ and $\|v_n - (-\phi_1)\|_X \to 0$, we obtain

$$\lim_{n\to+\infty} \int_\Omega h v_n \, dx = -\int_\Omega h\phi_1 \, dx. \tag{3.7}$$

In (3.6), taking $\liminf$ to both sides together with (3.5) and (3.7), we deduce

$$(1-p)\int_\Omega f_{-\infty}(x)\phi_1(x)\, dx - (1-p)\int_\Omega h\phi_1 \, dx \geq 0,$$

which gives

$$(p-1)\int_\Omega h\phi_1(x)\, dx \geq (p-1)\int_\Omega f_{-\infty}(x)\phi_1(x)\, dx$$

which yields, since $p > 1$,

$$\int_\Omega h\phi_1(x)\, dx \geq \int_\Omega f_{-\infty}(x)\phi_1(x)\, dx$$

which contradicts (H2).

Case 2: Suppose that $v_n \to \phi_1$. Letting $n \to +\infty$. Since $u_n(x) \to \infty$,

$$\begin{gathered} f(x, u_n(x)) \to f^{+\infty}(x), \ \text{a.e } x \in \Omega, \\ g(x, u_n(x)) \to f^{+\infty}(x), \ \text{a.e } x \in \Omega. \end{gathered}$$

Therefore, by Lebesgue theorem,

$$\lim_{n\to+\infty} \int_\Omega (f(x,u_n)v_n - pg(x,u_n)v_n)\, dx = (1-p)\int_\Omega f^{+\infty}(x)\phi_1(x)\, dx. \tag{3.8}$$

On the other hand, by summing up (3.3) and (3.4), we get

$$\begin{aligned} &\int_\Omega (pF(x,u_n) - f(x,u_n)u_n)\, dx + (p-1)\int_\Omega h u_n \, dx \\ &\leq \int_\Omega (pA(x,\nabla u_n) - a(x,\nabla u_n)\nabla u_n)\, dx + \\ &\quad \int_\Omega \big(pF(x,u_n) - f(x,u_n)u_n\big)\, dx + (p-1)\int_\Omega h u_n \, dx \\ &\leq \beta p + \varepsilon_n \|u_n\|_E, \end{aligned}$$

and after dividing by $\|u_n\|_E$, we obtain

$$-\int_\Omega (pg(x,u_n) - f(x,u_n)v_n)\, dx + (p-1)\int_\Omega h v_n \, dx \leq -\frac{\beta p}{\|u_n\|_E} - \varepsilon_n. \tag{3.9}$$

Since $h \in L^{p'}$ and $\|v_n - \phi_1\|_X \to 0$,

$$\lim_{n\to+\infty} \int_\Omega h v_n \, dx = \int_\Omega h\phi_1 \, dx. \tag{3.10}$$

In (3.9), using (3.8), (3.10) and by taking $\limsup$ to both sides, we deduce

$$(p-1)\int_\Omega h\phi_1(x)\, dx \leq (p-1)\int_\Omega f^{+\infty}(x)\phi_1(x)\, dx$$

which yields, since $p > 1$,

$$\int_\Omega h\phi_1(x)\, dx \leq \int_\Omega f^{+\infty}(x)\phi_1(x)\, dx$$

which contradicts (H2).

From the two cases above, $\{u_n\}$ is bounded in E. By Remark 1.1(i), we deduce that $\{u_n\}$ is bounded in X. Since X is reflexive, then by passing to a subsequence, still denote by $\{u_n\}$, we can assume that the sequence $\{u_n\}$ converges weakly to some u in X. We shall prove that the sequence $\{u_n\}$ converges strongly to u in E.

We observe by Remark 1.1(iii) that $u \in E$. Hence $\{\|u_n - u\|_E\}$ is bounded. Since $\{\|I'(u_n - u)\|_{E^\star}\}$ converges to 0, then $\langle I'(u_n - u), u_n - u\rangle$ converges to 0.

By the hypotheses on f and h, we deduce that

$$\lim_{n\to+\infty}\int_\Omega |u_n|^{p-2}u_n(u_n-u)\,dx = 0,$$
$$\lim_{n\to+\infty}\int_\Omega f(x,u_n)(u_n-u)\,dx = 0,$$
$$\lim_{n\to+\infty}\int_\Omega h(u_n-u)\,dx = 0.$$

On the other hand,

$$\begin{aligned}&\langle J'(u_n), u_n - u\rangle\\&= \lambda_1\int_\Omega |u_n|^{p-2}u_n(u_n-u)\,dx + \int_\Omega f(x,u_n)(u_n-u)\,dx + \int_\Omega h(u_n-u)\,dx.\end{aligned}$$

Thus $\lim_{n\to\infty}\langle J'(u_n), u_n-u\rangle = 0$. This and the fact that

$$\langle \Lambda'(u_n), u_n-u\rangle = \langle I'(u_n), u_n-u\rangle + \langle J'(u_n), u_n-u\rangle$$

give

$$\lim_{n\to\infty}\langle \Lambda'(u_n), u_n-u\rangle = 0.$$

By using (v) in Lemma 2.4, we get

$$\Lambda(u) - \limsup_{n\to\infty}\Lambda(u_n) = \liminf_{n\to\infty}(\Lambda(u)-\Lambda(u_n)) \ge \lim_{n\to\infty}\langle \Lambda'(u_n), u-u_n\rangle = 0.$$

This and (i) in Lemma 2.4 give

$$\lim_{n\to\infty}\Lambda(u_n) = \Lambda(u).$$

Now if we assume by contradiction that $\|u_n-u\|_E$ does not converge to 0 then there exists $\varepsilon > 0$ and a subsequence $\{u_{n_m}\}$ of $\{u_n\}$ such that

$$\|u_{n_m} - u\|_E \ge \varepsilon.$$

By using relation (ii) in Lemma 2.4, we get

$$\frac{1}{2}\Lambda(u) + \frac{1}{2}\Lambda\big(u_{n_m}\big) - \Lambda\Big(\frac{u_{n_m}+u}{2}\Big) \ge k_1\|u_{n_m}-u\|_E^p \ge k_1\varepsilon^p.$$

Letting $m\to\infty$ we find that

$$\limsup_{m\to\infty}\Lambda\Big(\frac{u_{n_m}+u}{2}\Big) \le \Lambda(u) - k_1\varepsilon^p.$$

We also have $\frac{u_{n_m}+u}{2}$ converges weakly to u in E. Using (i) in Lemma 2.4 again, we get

$$\Lambda(u) \le \liminf_{m\to\infty}\Lambda\Big(\frac{u_{n_m}+u}{2}\Big).$$

That is a contradiction. Therefore $\{u_n\}$ converges strongly to u in E. $\square$

Lemma 3.2. *The functional I is coercive on E provided* (H2) *holds.*

Proof. We firstly note that, in the proof of the Palais-Smale condition, we have proved that if $I(u_n)$ is a sequence bounded from above with $\|u_n\|_E \to \infty$, then (up to a subsequence), $v_n = u_n/\|u_n\|_E \to \pm\phi_1$ in X. Using this fact, we will prove that I is coercive provided $(\mathbf{H_3})$ holds.

Indeed, if I is not coercive, it is possible to choose a sequence $\{u_n\} \subset E$ such that $\|u_n\|_E \to \infty$, $I(u_n) \le \text{const}$ and $v_n = u_n/\|u_n\|_E \to \pm\phi_1$ in X. By Remark 1.1 (v),

$$-\int_\Omega F(x,u_n)\,dx + \int_\Omega h u_n\,dx \le I(u_n). \tag{3.11}$$

Case 1: Assume that $v_n \to \phi_1$. Dividing (3.11) by $\|u_n\|_E$ we get

$$\begin{aligned}
-\int_\Omega f^{+\infty}\phi_1\,dx + \int_\Omega h\phi_1\,dx &= \lim_{n\to+\infty}\Big(-\int_\Omega \frac{F(x,u_n)}{\|u_n\|_E}\,dx + \int_\Omega h\frac{u_n}{\|u_n\|_E}\,dx\Big)\\
&\le \limsup_{n\to+\infty}\frac{I(u_n)}{\|u_n\|_E}\\
&\le \limsup_{n\to+\infty}\frac{\text{const}}{\|u_n\|_E} = 0,
\end{aligned}$$

which gives

$$-\int_\Omega f^{+\infty}\phi_1\,dx + \int_\Omega h\phi_1\,dx \le 0,$$

which contradicts (H2).

Case 2: Assume that $v_n \to -\phi_1$. Dividing (3.11) by $\|u_n\|_E$ we get

$$\begin{aligned}
\int_\Omega f_{-\infty}\phi_1\,dx - \int_\Omega h\phi_1\,dx &= \lim_{n\to+\infty}\Big(-\int_\Omega \frac{F(x,u_n)}{\|u_n\|_E}\,dx + \int_\Omega h\frac{u_n}{\|u_n\|_E}\,dx\Big)\\
&\le \limsup_{n\to+\infty}\frac{I(u_n)}{\|u_n\|_E}\\
&\le \limsup_{n\to+\infty}\frac{\text{const}}{\|u_n\|_E} = 0,
\end{aligned}$$

which gives

$$\int_\Omega f_{-\infty}\phi_1\,dx - \int_\Omega h\phi_1\,dx \le 0,$$

which contradicts (H2). ∎

Proof of Theorem 1.2. The coerciveness and the Palais-Smale condition are enough to prove that I attains its proper infimum in Banach space E (see Theorem 2.3), so that (1.1) has at least a weak solution in E. The proof is complete. ∎

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