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HOMOGENIZED MODEL FOR FLOW IN PARTIALLY FRACTURED MEDIA

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Abstract. We derive rigorously a homogenized model for the displacement of one compressible miscible fluid by another in a partially fractured porous reservoir. We denote by ϵ the characteristic size of the heterogeneity in the medium. A function α characterizes the cracking degree of the rock. Our starting point is an adapted microscopic model which is scaled by appropriate powers of ϵ. We then study its limit as $\epsilon \to 0$. Because of the partially fractured character of the medium, the equation expressing the conservation of total mass in the flow is of degenerate parabolic type. The homogenization process for this equation is thus nonstandard. To overcome this difficulty, we adapt two-scale convergence techniques, convexity arguments and classical compactness tools. The homogenized model contains both single porosity and double porosity characteristics.

1. Introduction and main result

We consider the displacement of a two-component mixture through a highly contrasted porous medium, with fractures and matrix blocks. Assuming that the matrix blocks are disconnected, one usually models this type of setting using the concept of double porosity introduced by Barenblatt et al [5]. The fractured part is responsible for the macro-scale transport and the matrix part can store a concentration longer than is to be expected in a single porous material. The less permeable part of the rock thus contributes as global sink or source terms for the transported solutes in the fracture (see for instance [4, 9]). By the way, the matrix of cells may also be connected so that some flow occurs directly within the cell matrix. We consider here such a *partially fissured medium*. Most commercial simulators have the added feature of including matrix-matrix connections. But there are very few theoretical derivations of a model for this phenomenon. The uncertainties relating to the size of the physical structure and the fluid content of the reservoir make understanding fluid flow through homogenization a pragmatic approach.

The present paper is an extension of the works [14, 13, 11]. In these latter references, the degree of interconnection between matrix and fractured part of the

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medium, was characterized by a constant averaged parameter. In the present paper, a function α describes the interconnection. This function may be zero in the *totally fractured part* Ω_0 of the domain Ω, and some technical challenges are encountered there. The required estimates for the homogenization process involve the degenerating function α and are technically more challenging than in [11]. The final homogenized equation are different in Ω_0 and in $\Omega \setminus \Omega_0$. From a more physical viewpoint, this contribution aims to give a model more adapted to the local geometry of a natural domain. The interconnection function α is considered as a first order property of the medium, comparable, for instance, with the fracture intensity function.

We begin by recalling the equations describing the transport of two miscible species in a slightly compressible flow through a homogeneous porous medium, see [6, 20, 15] for details. The unknowns of the problem are the pressure p and the concentration c of one of the two species of the mixture. Denoting by ϕ the porosity of the rock and by k its permeability, the mass conservation principles during the displacement are expressed by the equations

$$\phi \partial_t p + \operatorname{div}(\underline{v}) = q_s, \quad \underline{v} = -\frac{k}{\mu(c)} \nabla p, \tag{1.1}$$

$$\phi \partial_t c + \underline{v} \cdot \nabla c - \operatorname{div}(\mathcal{D}(\underline{v}) \nabla c) = q_s(\hat{c} - c). \tag{1.2}$$

The average velocity of the flow $\underline{v}$ is given by the Darcy law in (1.1). We neglect the gravitational terms. The viscosity μ is a nonlinear function depending on the concentration. For instance, in the Koval model [17], μ is defined for $c \in (0,1)$ by $\mu(c) = \mu(0)(1 + (M^{1/4} - 1)c)^{-4}$, the constant $M = \mu(0)/\mu(1)$ being the mobility ratio. Analogous to Fick's law the dispersive flux is considered proportional to the concentration gradient and the dispersion tensor is

$$\mathcal{D}(\underline{v}) = \phi_f \big(D_m Id + D_p(\underline{v}) \big) = \phi_f \big(D_m Id + |\underline{v}| \left(\alpha_l \mathcal{E}(\underline{v}) + \alpha_t (Id - \mathcal{E}(\underline{v})) \right) \big), \tag{1.3}$$

where $\mathcal{E}(\underline{v})_{ij} = \underline{v}_i \underline{v}_j / |\underline{v}|^2$, α_l and α_t are the longitudinal and transverse dispersion constants and D_m is the molecular diffusion. For the usual rates of flow, these real numbers are such that $\alpha_l \geq \alpha_t \geq D_m > 0$. The terms containing q_s are the injection and production terms.

We now aim to study a similar flow in a partially fractured porous medium. We thus consider a domain $\Omega \subset \mathbb{R}^3$ with a periodic structure, controlled by a parameter $\epsilon > 0$ which represents the size of each block of the matrix. The $\mathcal{C}^1$ boundary of Ω is Γ and ν is the corresponding exterior normal. The standard period ($\epsilon = 1$) is a cell Y consisting of a matrix block Y_m of external $\mathcal{C}^1$ boundary ∂Y_m and of a fracture domain Y_f. We assume that $|Y| = 1$. The ϵ-reservoir consists of copies ϵY covering Ω. The two subdomains of Ω are defined by

$$\Omega_f^\epsilon = \Omega \cap \big\{ \cup_{\xi \in \mathcal{A}} \epsilon (Y_f + \xi) \big\}, \quad \Omega_m^\epsilon = \Omega \cap \big\{ \cup_{\xi \in \mathcal{A}} \epsilon (Y_m + \xi) \big\},$$

where $\mathcal{A}$ is an appropriate infinite lattice. The fracture-matrix interface is denoted by $\Gamma_{fm}^\epsilon = \partial \Omega_f^\epsilon \cap \partial \Omega_m^\epsilon \cap \Omega$ and ν_{fm} is the corresponding unit normal pointing out Ω_f^ϵ. See [14] and [1] for some illustrations of admissible structures. To homogenize the reservoir, we shall let tend to zero the size ϵ of the cells.

Following [14], we assume that the flow is made of two parts. The first component accounts for the global diffusion in the fracture system. The second one corresponds to high frequency spatial variations which lead to local storage in the matrix. A function $\alpha \in \mathcal{C}^1(\overline{\Omega})$ characterizes the interconnection between fractures and matrix. It is assumed such that

$$0 \le \alpha(x) < 1,\ \alpha(x) + \beta(x) = 1,$$
$$\alpha(x) = 0 \quad \text{if and only if } x \in \Omega_0,$$

where Ω_0 is a bounded open subset of Ω. It may be given by experimental data on samples of porous media and by stochastic reconstruction (see [7] and the references therein). The function α describing the *interconnection intensity* is obviously linked with the commonly used concepts of fracture intensity and fracture-size distribution (see for instance [18]). Note that in [14, 13] and [11], the cracking degree was characterized by a constant $\alpha \in (0, 1)$.

We thus adapt System (1.1)-(1.2) to such a decomposition of the flow. We also scale the equations for the rapidly varying part by appropriate powers of ϵ to conserve the flow between the matrix and the fractures as $\epsilon \to 0$ (*cf* [4, 14]). The complete derivation of the microscopic model is justified in [11]. Denoting by $J = (0, T)$, $T > 0$, the time interval of interest, we consider:

$$\phi_f^\epsilon \partial_t f_1^\epsilon + \underline{v}_f^\epsilon \cdot \nabla f_1^\epsilon - \operatorname{div}(\mathcal{D}(\underline{v}_f^\epsilon)\nabla f_1^\epsilon) = q_s(\hat{f}_1 - f_1^\epsilon) \quad \text{in } \Omega_f^\epsilon \times J, \tag{1.4}$$

$$\phi_f^\epsilon \partial_t p_f^\epsilon + \operatorname{div}(\underline{v}_f^\epsilon) = q_s,\ \underline{v}_f^\epsilon = -\frac{k_f^\epsilon}{\mu(f_1^\epsilon)}\nabla p_f^\epsilon \quad \text{in } \Omega_f^\epsilon \times J, \tag{1.5}$$

$$\phi^\epsilon \partial_t C_1^\epsilon + \underline{\mathcal{V}}^\epsilon \cdot \nabla C_1^\epsilon - \operatorname{div}(\mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon)\nabla C_1^\epsilon) = q_s(\hat{C}_1 - C_1^\epsilon) \quad \text{in } \Omega_m^\epsilon \times J, \tag{1.6}$$

$$\phi^\epsilon \partial_t c_1^\epsilon + \underline{\mathcal{V}}^\epsilon \cdot \nabla c_1^\epsilon - \operatorname{div}(\mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon)\nabla c_1^\epsilon) = q_s(\hat{c}_1 - c_1^\epsilon) \quad \text{in } \Omega_m^\epsilon \times J, \tag{1.7}$$

$$\phi^\epsilon \partial_t c_2^\epsilon + \underline{\mathcal{V}}^\epsilon \cdot \nabla c_2^\epsilon - \operatorname{div}(\mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon)\nabla c_2^\epsilon) = q_s(\hat{c}_2 - c_2^\epsilon) \quad \text{in } \Omega_m^\epsilon \times J, \tag{1.8}$$

$$\phi^\epsilon \partial_t p^\epsilon + \operatorname{div}(\underline{\mathcal{V}}^\epsilon) = q_s, \quad \underline{\mathcal{V}}^\epsilon = \underline{\mathcal{V}}_s^\epsilon + \epsilon \underline{\mathcal{V}}_h^\epsilon, \quad \text{in } \Omega_m^\epsilon \times J, \tag{1.9}$$

$$\underline{\mathcal{V}}_s^\epsilon = -\alpha(c_1^\epsilon + c_2^\epsilon)\frac{k^\epsilon}{\mu(m_1^\epsilon)}\nabla p^\epsilon,\ \underline{\mathcal{V}}_h^\epsilon = -(1 - \alpha(c_1^\epsilon + c_2^\epsilon))\frac{\epsilon k^\epsilon}{\mu(m_1^\epsilon)}\nabla p^\epsilon, \tag{1.10}$$

where $m_1^\epsilon = \alpha c_1^\epsilon + \beta C_1^\epsilon$. The flow in the fractures is described by (1.4)-(1.5). The matrix behavior is described by (1.6)-(1.10). In particular, (1.6) governs the slowly varying component while (1.7)-(1.8) governs the high frequency varying ones. We note that the former system becomes of double degenerate type as $\epsilon \to 0$. Indeed, in the subset Ω_0, the parabolic character of (1.6)-(1.9) is only ensured by the term ϵ^2. Moreover Eq. (1.9) is also of degenerate parabolic type in $\Omega \setminus \Omega_0$ since we can solely state that $(c_1^\epsilon + c_2^\epsilon)(x, t) \ge 0$ in $\Omega \times J$.

The model is completed by the following boundary and initial conditions. We begin by the transmission relations across the interface $\Gamma_{fm}^\epsilon \times J$.

$$\beta \mathcal{D}(\underline{v}_f^\epsilon)\nabla f_1^\epsilon \cdot \nu_{fm} = \mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon)\nabla C_1^\epsilon \cdot \nu_{fm}, \tag{1.11}$$

$$\alpha \mathcal{D}(\underline{v}_f^\epsilon)\nabla f_1^\epsilon \cdot \nu_{fm} = \mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon)\nabla c_1^\epsilon \cdot \nu_{fm}, \tag{1.12}$$

$$\alpha \mathcal{D}(\underline{v}_f^\epsilon)\nabla(1 - f_1^\epsilon) \cdot \nu_{fm} = -\alpha \mathcal{D}(\underline{v}_f^\epsilon)\nabla f_1^\epsilon \cdot \nu_{fm} = \mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon)\nabla c_2^\epsilon \cdot \nu_{fm}, \tag{1.13}$$

$$f_1^\epsilon = \alpha c_1^\epsilon + \beta C_1^\epsilon, \quad \alpha(c_1^\epsilon + c_2^\epsilon) = \alpha, \tag{1.14}$$

$$\underline{v}_f^\epsilon \cdot \nu_{fm} = \underline{\mathcal{V}}^\epsilon \cdot \nu_{fm}, \quad p_f^\epsilon = p^\epsilon. \tag{1.15}$$

We add a zero flux condition out of the full domain Ω

$$\mathcal{D}(\underline{v}_f^\epsilon)\nabla f_1^\epsilon \cdot \nu = 0 \text{ on } \partial\Omega_f^\epsilon \cap \Gamma, \tag{1.16}$$

$$\mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon)\nabla C_1^\epsilon \cdot \nu = \mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon)\nabla c_1^\epsilon \cdot \nu = \mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon)\nabla c_2^\epsilon \cdot \nu = 0 \text{ on } \partial\Omega_m^\epsilon \cap \Gamma, \tag{1.17}$$

$$\underline{v}_f^\epsilon \cdot \nu = 0 \text{ on } \partial\Omega_f^\epsilon \cap \Gamma, \quad \underline{\mathcal{V}}^\epsilon \cdot \nu = 0 \text{ on } \partial\Omega_m^\epsilon \cap \Gamma, \tag{1.18}$$

and the following initial conditions in Ω

$$(f_1^\epsilon(x,0), C_1^\epsilon(x,0), c_1^\epsilon(x,0), c_2^\epsilon(x,0)) = (f_1^o(x), C_1^o(x), c_1^o(x), c_2^o(x)), \tag{1.19}$$

$$p_f^\epsilon(x,0) = \chi_f^\epsilon(x) p^o(x), \quad p^\epsilon(x,0) = \chi_m^\epsilon(x) p^o(x). \tag{1.20}$$

Let us now enumerate the assumptions. The source term q_s is a nonnegative function of $L^q(\Omega \times J)$, $q > 2$, and

$$\alpha \hat{c}_1 + \beta \hat{C}_1 = \hat{f}_1, \quad 0 \le \hat{f}_1 \le 1, \quad \hat{c}_1 + \hat{c}_2 = 1.$$

As we assume a periodic structure in the reservoir, the porosities $(\phi_f^\epsilon(x), \phi^\epsilon(x)) = (\phi_f(\frac{x}{\epsilon}), \phi(\frac{x}{\epsilon}))$ and the permeabilities $(k_f^\epsilon(x), k^\epsilon(x)) = (k_f(\frac{x}{\epsilon}), k(\frac{x}{\epsilon}))$ of the fracture and of the matrix are periodic of period $(\epsilon Y_f, \epsilon Y_m)$. These quantities are assumed to be smooth and bounded, but globally they are discontinuous across Γ_{fm}^ϵ. We assume moreover

$$0 < \phi_- \le \phi_f(x),\ \phi(x) \le \phi_-^{-1}, \quad k_-|\xi|^2 \le k_f(x)\xi \cdot \xi, \quad k(x)\xi \cdot \xi \le k_-^{-1}|\xi|^2,$$

$k_- > 0$, a.e. in Ω, for all $\xi \in \mathbb{R}^3$. The viscosity $\mu \in W^{1,\infty}(\Omega \times (0,1))$ is such that

$$0 < \mu_- \le \mu(x,c) \le \mu_+ \ \forall c \in (0,1), \quad \mu(x,c) = \mu \in \mathbb{R}_+^* \quad \text{in } \Omega_0.$$

For sake of simplicity we have assumed that the viscosity is constant in Ω_0. We then can pass to the limit in Ω_0 without introducing a dilation operator (see [11] Section 4 for the details). The tensor $\mathcal{D}$ is already defined in (1.3). The tensor $\mathcal{D}^\epsilon$ has a similar structure but its diffusive part $(\alpha + \beta\epsilon^2) D_m Id$ contains the proportions of slowly and rapidly varying flows in the matrix. The main property of these tensors is

$$\begin{gathered} \mathcal{D}(\underline{v}_f^\epsilon)\xi \cdot \xi \ge \phi_-(D_m + \alpha_t|\underline{v}_f^\epsilon|)|\xi|^2, \quad \forall \xi \in \mathbb{R}^3, \\ \mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon)\xi \cdot \xi \ge \phi_-(D_m(\alpha + \beta\epsilon^2) + \alpha_t|\underline{\mathcal{V}}_s^\epsilon + \epsilon^2 \underline{\mathcal{V}}_h^\epsilon|)|\xi|^2, \quad \forall \xi \in \mathbb{R}^3. \end{gathered} \tag{1.21}$$

We assume that p^o belongs to $H^1(\Omega)$, and that $(f_1^o, C_1^o, c_1^o, c_2^o) \in (L^\infty(\Omega))^4$ satisfies

$$0 \le f_1^o(x) \le 1 \text{ a.e. in } \Omega, \tag{1.22}$$

$$\gamma_- \le c_1^o(x) \le \gamma_+,\ (\gamma_-, \gamma_+) \in \mathbb{R}^2, \text{ a.e. in } \Omega, \tag{1.23}$$

$$\alpha c_1^o(x) + \beta C_1^o(x) = \chi_m^\epsilon f_1^o(x),\ 0 \le c_1^o(x) + c_2^o(x) \le 1 \text{ a.e. in } \Omega_m^\epsilon. \tag{1.24}$$

The main result of the paper is the following.

Theorem 1.1. *As the scaling parameter ϵ tends to zero, the microscopic model (1.4)-(1.20) converges to the following macroscopic model. The homogenized pressure problem is*

$$\big(\overline{\phi_f}^{Y_f} + \chi_{\Omega\setminus\Omega_0}\overline{\phi}^{Y_m}\big)\,\partial_t p_f - \operatorname{div}\Big(\frac{\overline{K}_\alpha^H}{\mu(f_1)}\nabla p_f\Big) = q_s - \chi_{\Omega_0}\int_{Y_m}\phi\,\partial_t p^0\,dy \quad \text{in } \Omega \times J,$$

$$\phi(y)\partial_t p^0 + \operatorname*{div}_y(\underline{\mathcal{V}}^0) = q_s,\ \underline{\mathcal{V}}^0 = -\frac{k(y)}{\mu(f_1)} \quad \text{in } \Omega_0 \times Y_m \times J,$$

$$p_f(x,t) = p^0(x,y,t) \quad \text{if } y \in \Gamma_{fm},\ (x,t) \in \Omega \times J,$$

$$\overline{K}_\alpha^H \nabla p_f \cdot \nu = 0 \text{ on } \partial\Omega \times J,\ p_f(x,0) = p^0(x,y,0) = p^o(x) \quad \text{in } \Omega \times Y_m.$$

The homogenized concentrations problem is in $\Omega \times J$:

$$\overline{\phi_f}^{Y_f}\partial_t f_1 + \chi_{\Omega\setminus\Omega_0}\frac{1}{\beta}\overline{\phi}^{Y_m}\partial_t C_1 + \chi_{\Omega_0}\frac{1}{\beta}\int_{Y_m}\phi(y)\,\partial_t C_1^0 dy$$

$$
\begin{aligned}
&- \frac{K^H_{Y_f}}{\mu(f_1)}\nabla p_f\cdot\nabla f_1 - \chi_{\Omega\setminus\Omega_0}\frac{K^H_{Y_m}}{\beta\mu(f_1)}\nabla p_f\cdot\nabla C_1 - \frac{\chi_{\Omega_0}}{\beta}\Big(\int_{Y_m}\frac{k(y)}{\mu}\nabla_y p^0\cdot\nabla_y C_1^0\,dy\Big)\\
&- \operatorname{div}(\mathcal{D}^H_f(\nabla p_f)\nabla f_1) - \chi_{\Omega\setminus\Omega_0}\frac{1}{\beta}\operatorname{div}(\mathcal{D}^H_m(\nabla p_f)\nabla C_1)\\
&= q_s|Y_f|\,(\hat f_1 - f_1) + \frac{1}{\beta}q_s\Big(\hat C_1 - \chi_{\Omega\setminus\Omega_0}|Y_m|\,C_1 - \chi_{\Omega_0}\int_{Y_m}C_1^0(\cdot,y,\cdot)\,dy\Big),
\end{aligned}
$$

$$
\begin{aligned}
&\overline{\phi}^{Y_m}\chi_{\Omega\setminus\Omega_0}\partial_t c_1 + \chi_{\Omega_0}\int_{Y_m}\phi(y)\,\partial_t c_1^0 dy - \frac{\alpha}{\beta}\overline{\phi}^{Y_m}\chi_{\Omega\setminus\Omega_0}\partial_t C_1\\
&- \frac{\alpha}{\beta}\chi_{\Omega_0}\int_{Y_m}\phi(y)\,\partial_t C_1^0 dy - \chi_{\Omega\setminus\Omega_0}\frac{K^H_{Y_m}}{\mu(f_1)}\nabla p_f\cdot\big(\nabla c_1 - \frac{\alpha}{\beta}\nabla C_1\big)\\
&- \chi_{\Omega_0}\int_{Y_m}\frac{k(y)}{\mu}\nabla_y p^0\cdot\big(\nabla_y c_1^0 - \frac{\alpha}{\beta}\nabla_y C_1^0\big)\,dy\\
&- \chi_{\Omega\setminus\Omega_0}\operatorname{div}(\mathcal{D}^H_m(\nabla p_f)\nabla c_1) + \chi_{\Omega\setminus\Omega_0}\frac{\alpha}{\beta}\operatorname{div}(\mathcal{D}^H_m(\nabla p_f)\nabla C_1)\\
&= q_s\Big(\hat c_1 - \chi_{\Omega\setminus\Omega_0}|Y_m|\,c_1 - \chi_{\Omega_0}\int_{Y_m}c_1^0(\cdot,y,\cdot)\,dy\Big)\\
&\quad - \frac{\alpha}{\beta}\,q_s\Big(\hat C_1 - \chi_{\Omega\setminus\Omega_0}|Y_m|\,C_1 - \chi_{\Omega_0}\int_{Y_m}C_1^0(\cdot,y,\cdot)\,dy\Big),
\end{aligned}
$$

and in $\Omega_0\times Y_m\times J$

$$
\begin{gathered}
\phi(y)\partial_t C_1^0 - \frac{k(y)}{\mu}\nabla_y p^0\cdot\nabla_y C_1^0 - \operatorname*{div}_y\big(\mathcal{D}(\frac{k(y)}{\mu}\nabla_y p^0)\nabla_y C_1^0\big) = q_s\,(\hat C_1 - C_1^0),\\
\phi(y)\partial_t c_1^0 - \frac{k(y)}{\mu}\nabla_y p^0\cdot\nabla_y c_1^0 - \operatorname*{div}_y\big(\mathcal{D}(\frac{k(y)}{\mu}\nabla_y p^0)\nabla_y c_1^0\big) = q_s\,(\hat c_1 - c_1^0),
\end{gathered}
$$

completed by

$$
\begin{gathered}
\big(\mathcal{D}^H_f(\nabla p_f)\nabla f_1 - \frac{1}{\beta}\mathcal{D}^H_m(\nabla p_f)\nabla C_1\big)\cdot\nu\big|_{\Gamma\times J} = 0,\\
\big(\mathcal{D}^H_m(\nabla p_f)\nabla c_1 - \frac{\alpha}{\beta}\mathcal{D}^H_m(\nabla p_f)\nabla C_1\big)\cdot\nu\big|_{\Gamma\times J} = 0,\\
f_1\big|_{t=0} = f_1^o,\quad c_1\big|_{t=0} = c_1^0\big|_{t=0} = c_1^o,\quad C_1\big|_{t=0} = C_1^0\big|_{t=0} = C_1^o,\\
f_1 = \alpha c_1 + \beta C_1\ \text{a.e. in}\ \Omega\times J,\quad f_1 = \beta C_1^0\ \text{a.e. in}\ \Omega_0\times\Gamma_{fm}\times J.
\end{gathered}
$$

The homogenized quantities $\overline{K}^H_\alpha$, $\mathcal{D}^H_f$ *and* $\mathcal{D}^H_m$ *are defined in (3.3), (3.17) and (3.18) below.*

The homogenization process then leads to a macroscopic model containing both single porosity and double porosity characteristics. We show that the double porosity part of the model almost disappears as soon as a direct flow occurs in the matrix (see the equations in $\Omega\setminus\Omega_0$). It emphasizes in particular the role of the dispersion tensor which models all the velocities heterogeneity at the microscopic level. It is characteristic of a miscible flow (see [3] and the references therein). The result is thus quite different of the one obtained in [14, 13]. Nevertheless, even in $\Omega\setminus\Omega_0$, the model captures the interactions between the matrix and the fractured part. Indeed, the homogenized permeability and diffusion tensors strongly depend on the transmission function α. One could compare this effects with some models where the permeability is concentration dependent: propagation in clays (see [16] and the references therein) or blood flow in micro vessels (see [22] and the references therein) for instance. And in the subset Ω_0 where no direct transmission occurs, the model is of double porosity type.

This paper is organized as follows. Section 2 is devoted to the analysis of the microscopic model. We derive uniform estimates for the solutions. Convergence results are stated using two-scale convergence techniques, convexity arguments and classical compactness tools. In Section 3, we pass to the limit $\epsilon \to 0$ and we get the homogenized model described in Theorem 1.1.

2. Analysis of the microscopic model

The existence of weak solutions for the problem (1.4)-(1.20) is proved in [11]. The proof is of course inspired by the statement of the existence of solutions for Problem (1.1)-(1.2) in a homogeneous porous medium (see [10]). But the decomposition of the flow in the matrix part of the domain induces additional difficulties. Appropriate concentrations spaces for the problem are introduced following [13]: H^ϵ is the Hilbert space $H^\epsilon = L^2(\Omega_f^\epsilon) \times L^2(\Omega_m^\epsilon) \times L^2(\Omega_m^\epsilon)$ with the inner product

$$\begin{aligned}&\big([u_f, u_m, U_m], [\psi_f, \psi_m, \Psi_m]\big)_{H^\epsilon}\\&= \int_{\Omega_f^\epsilon} u_f(x)\,\psi_f(x)\,dx + \int_{\Omega_m^\epsilon} u_m(x)\,\psi_m(x)\,dx + \int_{\Omega_m^\epsilon} U_m(x)\,\Psi_m(x)\,dx,\end{aligned}$$

and V^ϵ is the Banach space

$$\begin{aligned}V^\epsilon = H^\epsilon \cap \big\{&(u_f, u_m, U_m) \in H^1(\Omega_f^\epsilon) \times H^1(\Omega_m^\epsilon) \times H^1(\Omega_m^\epsilon);\\&\gamma_f^\epsilon u_f = \alpha\gamma_m^\epsilon u_m + \beta\gamma_m^\epsilon U_m \text{ on } \Gamma_{fm}^\epsilon\big\}\end{aligned}$$

endowed with the norm

$$\begin{aligned}\|(u_f, u_m, U_m)\|_{V^\epsilon} &= \|\chi_f^\epsilon u_f\|_{L^2(\Omega)} + \|\chi_m^\epsilon u_m\|_{L^2(\Omega)} + \|\chi_m^\epsilon U_m\|_{L^2(\Omega)}\\&\quad + \|\chi_f^\epsilon \nabla u_f\|_{(L^2(\Omega))^3} + \|\chi_m^\epsilon \nabla u_m\|_{(L^2(\Omega))^3} + \|\chi_m^\epsilon \nabla U_m\|_{(L^2(\Omega))^3},\end{aligned}$$

where $\gamma_j^\epsilon : H^1(\Omega_j^\epsilon) \to L^2(\partial\Omega_j^\epsilon)$ is the usual trace map and χ_j^ϵ is the characteristic function associated with Ω_j^ϵ, $j = f, m$. We also introduce the Banach space V_c^ϵ

$$\begin{aligned}V_c^\epsilon = L^2(\Omega_m^\epsilon) \times L^2(\Omega_m^\epsilon) \cap \big\{&(u_1, u_2) \in H^1(\Omega_m^\epsilon) \times H^1(\Omega_m^\epsilon);\\&\alpha = \alpha\gamma_m^\epsilon(u_1 + u_2) \text{ on } \Gamma_{fm}^\epsilon\big\}\end{aligned}$$

endowed with the norm

$$\|(u_1, u_2)\|_{V_c^\epsilon} = \|\chi_m^\epsilon u_1\|_{L^2(\Omega)} + \|\chi_m^\epsilon u_2\|_{L^2(\Omega)} + \|\chi_m^\epsilon \nabla u_1\|_{(L^2(\Omega))^3} + \|\chi_m^\epsilon \nabla u_2\|_{(L^2(\Omega))^3}.$$

We note that for any fixed $\epsilon > 0$, the problem is of parabolic type. Then, adapting the proof of [10] to the present piecewise structure, one can state the following existence result (see [11] for a detailed proof).

Theorem 2.1. *Let $0 < \epsilon < 1$. There exists a solution $(p_f^\epsilon, p^\epsilon, f_1^\epsilon, c_1^\epsilon, C_1^\epsilon, c_2^\epsilon)$ of Problem* (1.4)-(1.20) *in the following sense.*
(i) The pressure part $(p_f^\epsilon, p^\epsilon)$ belongs to $L^2(J; H^1(\Omega_f^\epsilon)) \times L^2(J; H^1(\Omega_m^\epsilon))$ and is a weak solution of (1.5), (1.9)-(1.10), (1.15), (1.18) *and* (1.20). *Indeed, for any*

function $\psi \in \mathcal{C}^1(J; H^1(\Omega))$ *such that* $\psi\,|_{t=T}= 0$,

$$\begin{aligned}
&-\int_{\Omega\times J} (\chi_f^\epsilon \phi_f^\epsilon p_f^\epsilon + \chi_m^\epsilon \phi^\epsilon p^\epsilon)\partial_t \psi \\
&+\int_{\Omega\times J} \big(\chi_f^\epsilon \frac{k_f^\epsilon}{\mu(f_1^\epsilon)}\nabla p_f^\epsilon + \chi_m^\epsilon(\alpha(c_1^\epsilon + c_2^\epsilon)(1-\epsilon^2)+\epsilon^2)\frac{k^\epsilon}{\mu(m_1^\epsilon)}\nabla p_\epsilon\big)\cdot\nabla\psi \\
&= -\int_\Omega (\chi_f^\epsilon \phi_f^\epsilon + \chi_m^\epsilon \phi^\epsilon)p^o \psi(x,0) + \int_{\Omega\times J} q_s \psi.
\end{aligned} \tag{2.1}$$

(ii) The concentration part $(f_1^\epsilon, c_1^\epsilon, C_1^\epsilon, c_2^\epsilon)$ *is such that* $(f_1^\epsilon, c_1^\epsilon, C_1^\epsilon) \in L^2(J; V^\epsilon) \cap H^1(J; (V^\epsilon)')$ *and* $(c_1^\epsilon, c_2^\epsilon) \in L^2(J; V\epsilon_c) \cap H^1(J; (V_c^\epsilon)')$. *It satisfies for any test functions* $(d_f, d_1, D_1) \in L^2(J; V^\epsilon)$ *and* $d_2 \in L^2(J; H^1(\Omega_m^\epsilon))$ *the following relations.*

$$\begin{aligned}
&\int_{\Omega_f^\epsilon\times J} \phi_f^\epsilon \partial_t f_1^\epsilon\, d_f + \int_{\Omega_m^\epsilon\times J} \phi^\epsilon \partial_t c_1^\epsilon\, d_1 + \int_{\Omega_m^\epsilon\times J} \phi^\epsilon \partial_t C_1^\epsilon D_1 + \int_{\Omega_f^\epsilon\times J} (\underline{v}_f^\epsilon \cdot \nabla f_1^\epsilon)\, d_f \\
&+\int_{\Omega_m^\epsilon\times J} \underline{\mathcal{V}}^\epsilon \cdot (d_1 \nabla c_1^\epsilon + D_1 \nabla C_1^\epsilon) + \int_{\Omega_f^\epsilon\times J} \mathcal{D}(\underline{v}_f^\epsilon)\nabla f_1^\epsilon \cdot \nabla d_f \\
&+\int_{\Omega_m^\epsilon\times J} \mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon)\nabla c_1^\epsilon \cdot \nabla d_1 + \int_{\Omega_m^\epsilon\times J} \mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon)\nabla C_1^\epsilon \cdot \nabla D_1 \\
&= \int_{\Omega_f^\epsilon\times J} q_s\,(\hat{f}_1 - f_1^\epsilon)\, d_f + \int_{\Omega_m^\epsilon\times J} q_s\,(\hat{c}_1 - c_1^\epsilon)\, d_1 + \int_{\Omega_m^\epsilon\times J} q_s\,(\hat{C}_1 - C_1^\epsilon)\, D_1,
\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}
&\int_{\Omega_m^\epsilon\times J} \phi^\epsilon \partial_t c_2^\epsilon\, d_2 + \int_{\Omega_m^\epsilon\times J} (\underline{\mathcal{V}}^\epsilon \cdot \nabla c_2^\epsilon)\, d_2 + \int_{\Omega_m^\epsilon\times J} \mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon)\nabla c_2^\epsilon \cdot \nabla d_2 \\
&-\int_{\partial\Omega_m^\epsilon\times J} (\mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon)\nabla c_2^\epsilon \cdot \nu_m)\,\gamma_m^\epsilon d_2 \\
&= \int_{\Omega_m^\epsilon\times J} q_s\,(1 - c_2^\epsilon)\, d_2.
\end{aligned} \tag{2.3}$$

Furthermore, the following maximum principles hold:

$$0 \le f_1^\epsilon(x,t) \le \hat{f}_1 \quad a.e.\ in\ \Omega_f^\epsilon \times J, \tag{2.4}$$

$$0 \le m_1^\epsilon(x,t) \le \hat{f}_1,\ 0 \le c_1^\epsilon(x,t) + c_2^\epsilon(x,t) \le 1 \quad a.e.\ in\ \Omega_m^\epsilon \times J, \tag{2.5}$$

$$\gamma_- \le c_1^\epsilon(x,t) \le \gamma_+ \quad a.e.\ in\ \Omega_m^\epsilon \times J. \tag{2.6}$$

We now state some uniform estimates for the solutions of the microscopic system. We begin by stating the following properties of the pressure solutions of the problem (1.5), (1.9)–(1.10), (1.15), (1.18), (1.20). One of the main difficulties of the homogenization problem appears in the following lemma. Indeed, letting ϵ to 0, Equation (1.9) is of degenerate parabolic type because one can only ensure that $c_1^\epsilon + c_2^\epsilon \ge 0$. It is a main difference with our former work in [11].

Lemma 2.2. *The pressure satisfies the following uniform estimates*

$$\begin{gathered}
\|p_f^\epsilon\|_{L^\infty(J;L^q(\Omega_f^\epsilon))} + \|p_f^\epsilon\|_{L^2(J;H^1(\Omega_f^\epsilon))} \le C, \\
\|\underline{v}_f^\epsilon\|_{(L^2(J;L^2(\Omega_f^\epsilon)))^2} \le C, \\
\|p^\epsilon\|_{L^\infty(J;L^q(\Omega_m^\epsilon))} \le C,
\end{gathered}$$

$$
\begin{gathered}
\|\alpha^{1/2}(c_1^\epsilon + c_2^\epsilon)^{1/2}\nabla p^\epsilon\|_{(L^2(J;L^2(\Omega_m^\epsilon)))^3} + \|\epsilon\nabla p^\epsilon\|_{(L^2(J;L^2(\Omega_m^\epsilon)))^3} \le C,\\
\|\underline{\mathcal{V}}_s^\epsilon\|_{(L^2(J;L^2(\Omega_m^\epsilon)))^3} \le C,\\
\|\underline{\mathcal{V}}_h^\epsilon\|_{(L^2(J;L^2(\Omega_m^\epsilon)))^3} \le C.
\end{gathered}
$$

Furthermore the time derivative $(\chi_f^\epsilon\phi_f^\epsilon\partial_t p_f^\epsilon + \chi_m^\epsilon\phi^\epsilon\partial_t p^\epsilon)$ is uniformly bounded in $L^2(J;(H^1(\Omega))')$.

Proof. The estimates are derived from integration by parts. We multiply (1.5) by p_f^ϵ and integrate over $\Omega_f^\epsilon \times J$. We multiply (1.9) by p^ϵ and integrate over $\Omega_m^\epsilon \times J$. Summing up the resulting relations, we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_{\Omega_f^\epsilon}\phi_f^\epsilon\,|p_f^\epsilon|^2\,dx + \frac{1}{2}\int_{\Omega_m^\epsilon}\phi^\epsilon\,|p^\epsilon|^2\,dx + \int_{\Omega_f^\epsilon\times J}\frac{k_f^\epsilon}{\mu(f_1^\epsilon)}\nabla p_f^\epsilon\cdot\nabla p_f^\epsilon\,dxdt\\
&+\int_{\Omega_m^\epsilon\times J}\big(\alpha(c_1^\epsilon+c_2^\epsilon)(1-\epsilon^2)+\epsilon^2\big)\,\frac{k^\epsilon}{\mu(m_1^\epsilon)}\nabla p^\epsilon\cdot\nabla p^\epsilon\,dxdt\\
&=\frac{1}{2}\int_\Omega(\chi_f^\epsilon\phi_f^\epsilon(x)+\chi_m^\epsilon\phi^\epsilon(x))\,|p^o(x)|^2\,dx + \int_{\Omega\times J} q_s\,(\chi_f^\epsilon p_f^\epsilon + \chi_m^\epsilon p^\epsilon)\,dxdt.
\end{aligned}
$$

Applying the Cauchy-Schwarz and Young inequalities with the properties of ϕ_f^ϵ, ϕ^ϵ, k_f^ϵ, k^ϵ and μ in the latter relation, we get

$$
\begin{aligned}
&\frac{\phi_-}{2}\int_{\Omega_f^\epsilon}|p_f^\epsilon|^2\,dx + \frac{\phi_-}{2}\int_{\Omega_m^\epsilon}|p^\epsilon|^2\,dx + \frac{k_-}{\mu_+}\int_{\Omega_f^\epsilon\times J}|\nabla p_f^\epsilon|^2\,dxdt\\
&+\frac{k_-}{\mu_+}\int_{\Omega_m^\epsilon\times J}\big(\alpha(c_1^\epsilon+c_2^\epsilon)\,|\nabla p^\epsilon|^2 + \epsilon^2\,(1-\alpha(c_1^\epsilon+c_2^\epsilon))\,|\nabla p^\epsilon|^2\big)\,dxdt\\
&\le C\big(\|p^o\|_{L^2(\Omega)},\|q_s\|_{L^2(\Omega\times J)}\big) + \int_{\Omega_f^\epsilon\times J}|p_f^\epsilon|^2\,dxdt + \int_{\Omega_m^\epsilon\times J}|p^\epsilon|^2\,dxdt.
\end{aligned}
$$

Using the Gronwall lemma, we prove the desired estimates, but in L^2 instead of L^q. The result on the time derivatives then follows straightforward from (1.5), (1.9)-(1.10). It remains to show that the pressure is uniformly bounded in $L^\infty(J;L^q(\Omega))$. Let $\eta > 0$. We multiply Eq. (1.5) (respectively (1.9)) by $qp_f^\epsilon({p_f^\epsilon}^2+\eta)^{q/2-1}$ (resp. $qp^\epsilon({p^\epsilon}^2+\eta)^{q/2-1}$) and we integrate by parts over Ω_f^ϵ (resp. Ω_m^ϵ). We obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega\big(\chi_f^\epsilon({p_f^\epsilon}^2+\eta)^{q/2}+\chi_m^\epsilon({p^\epsilon}^2+\eta)^{q/2}\big) + \int_{\Omega_f^\epsilon}\frac{k_f^\epsilon}{\mu(f_1^\epsilon)}({p_f^\epsilon}^2+\eta)^{q/2-1}\nabla p_f^\epsilon\cdot\nabla p_f^\epsilon\\
&+\int_{\Omega_f^\epsilon}\frac{k_f^\epsilon}{\mu(f_1^\epsilon)}q{p_f^\epsilon}^2({p_f^\epsilon}^2+\eta)^{q/2-2}\nabla p_f^\epsilon\cdot\nabla p_f^\epsilon\\
&+\int_{\Omega_m^\epsilon}(\alpha(c_1^\epsilon+c_2^\epsilon)(1-\epsilon^2)+\epsilon^2)\frac{k^\epsilon}{\mu(m_1^\epsilon)}({p^\epsilon}^2+\eta)^{q/2-1}\nabla p^\epsilon\cdot\nabla p^\epsilon\\
&+\int_{\Omega_m^\epsilon}(\alpha(c_1^\epsilon+c_2^\epsilon)(1-\epsilon^2)+\epsilon^2)\frac{k^\epsilon}{\mu(m_1^\epsilon)}q{p^\epsilon}^2({p^\epsilon}^2+\eta)^{q/2-2}\nabla p^\epsilon\cdot\nabla p^\epsilon\\
&=\int_\Omega q_s q\big(\chi_f^\epsilon p_f^\epsilon({p_f^\epsilon}^2+\eta)^{q/2-1}+\chi_m^\epsilon p^\epsilon({p^\epsilon}^2+\eta)^{q/2-1}\big).
\end{aligned}
$$

The four last terms of the left hand side of the latter relation are nonnegative. The right hand side term is estimated as follows using the Hölder inequality.

$$
\begin{aligned}
&\Big|\int_\Omega q_s q\big(\chi_f^\epsilon p_f^\epsilon ({p_f^\epsilon}^2+\eta)^{q/2-1}+\chi_m^\epsilon p^\epsilon (p^{\epsilon 2}+\eta)^{q/2-1}\big)\Big|\\
&\le C\int_\Omega |q_s|\,\big(\chi_f^\epsilon ({p_f^\epsilon}^2+\eta)^{(q-1)/2}+\chi_m^\epsilon (p^{\epsilon 2}+\eta)^{(q-1)/2}\big)\\
&\le C\Big(\int_\Omega \chi_f^\epsilon ({p_f^\epsilon}^2+\eta)^{q/2}+\chi_m^\epsilon (p^{\epsilon 2}+\eta)^{q/2}\Big)^{(q-1)/q}\Big(\int_\Omega |q_s|^q\Big)^{1/q}\\
&\le C\Big(\int_\Omega \chi_f^\epsilon ({p_f^\epsilon}^2+\eta)^{q/2}+\chi_m^\epsilon (p^{\epsilon 2}+\eta)^{q/2}\Big)^{(q-1)/q}.
\end{aligned}
$$

We conclude with the Gronwall lemma that $\chi_f^\epsilon ({p_f^\epsilon}^2+\eta)^{1/2}+\chi_m^\epsilon (p^{\epsilon 2}+\eta)^{1/2}$ is uniformly bounded in $L^\infty(J;L^q(\Omega))$. It follows that $\chi_f^\epsilon p_f^\epsilon+\chi_m^\epsilon p^\epsilon$ is also uniformly bounded in $L^\infty(J;L^q(\Omega))$. $\square$

We now establish the following results concerning the concentrations functions $(f_1^\epsilon, C_1^\epsilon, c_1^\epsilon, c_2^\epsilon)$.

Lemma 2.3. (i) *The functions $(f_1^\epsilon, C_1^\epsilon, c_1^\epsilon, c_2^\epsilon)$ are uniformly bounded in the space $L^\infty(J;L^2(\Omega_f^\epsilon))\times(L^\infty(J;L^2(\Omega_f^\epsilon)))^3$ and are such that*

$$
\begin{gathered}
0\le f_1^\epsilon(x,t)\le \hat f_1\le 1 \quad \text{almost everywhere in } \Omega_f^\epsilon\times J,\\
0\le \alpha c_1^\epsilon(x,t)+\beta C_1^\epsilon(x,t)\le \hat f_1\le 1 \quad \text{almost everywhere in } \Omega_m^\epsilon\times J\\
0\le c_1^\epsilon(x,t)+c_2^\epsilon(x,t)\le 1 \quad \text{almost everywhere in } \Omega_m^\epsilon\times J,\\
\gamma_-\le c_1^\epsilon(x,t)\le \gamma_+ \quad \text{almost everywhere in } \Omega_m^\epsilon\times J;
\end{gathered}
$$

(ii) *the sequence $((D_m^{1/2}+\alpha_t^{1/2}|\underline{v}_f^\epsilon|^{1/2})\nabla f_1^\epsilon)$ is uniformly bounded in $(L^2(\Omega_f^\epsilon\times J))^3$;*

(iii) *for $i=1,2$, the diffusive terms $\alpha^{1/2}(1+(c_1^\epsilon+c_2^\epsilon)^{1/2}|\nabla p^\epsilon|^{1/2})\nabla c_i^\epsilon$ and $\epsilon(1+|\epsilon\nabla p^\epsilon|^{1/2})\nabla c_i^\epsilon$ are uniformly bounded in $(L^2(\Omega_m^\epsilon\times J))^3$. The same estimates hold for C_1^ϵ.*

Proof. The maximum principles of *(i)* are a direct consequence of the construction of the solution $(f_1^\epsilon, C_1^\epsilon, c_1^\epsilon, c_2^\epsilon)$ in Theorem 2.1. We write the variational formulation (2.2) with the test function $(d_f,d_1,D_1)=(f_1^\epsilon,c_1^\epsilon,C_1^\epsilon)$. We get

$$
\begin{aligned}
&\frac12\int_{\Omega_f^\epsilon}\phi_f^\epsilon|f_1^\epsilon|^2\,dx+\frac12\int_{\Omega_m^\epsilon}\phi^\epsilon(|c_1^\epsilon|^2+|C_1^\epsilon|^2)\,dx+\int_{\Omega_f^\epsilon\times J}\mathcal{D}(\underline{v}_f^\epsilon)\nabla f_1^\epsilon\cdot\nabla f_1^\epsilon\,dx\,dt\\
&+\int_{\Omega_m^\epsilon\times J}(\mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon)\nabla c_1^\epsilon\cdot\nabla c_1^\epsilon+\mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon)\nabla C_1^\epsilon\cdot\nabla C_1^\epsilon)\,dx\,dt\\
&+\int_{\Omega_f^\epsilon\times J}(\underline{v}_f^\epsilon\cdot\nabla f_1^\epsilon)\,f_1^\epsilon\,dx\,dt+\int_{\Omega_m^\epsilon\times J}\underline{\mathcal{V}}^\epsilon\cdot(c_1^\epsilon\,\nabla c_1^\epsilon+C_1^\epsilon\,\nabla C_1^\epsilon)\,dx\,dt\\
&+\int_{\Omega\times J}q_s\,(\chi_f^\epsilon|f_1^\epsilon|^2+\chi_m^\epsilon\,(|c_1^\epsilon|^2+|C_1^\epsilon|^2))\,dx\,dt \qquad (2.7)\\
&=\int_{\Omega\times J}q_s\hat f_1 f_1^\epsilon\,dx\,dt+\int_{\Omega_m^\epsilon\times J}q_s(\hat c_1 c_1^\epsilon+\hat C_1 C_1^\epsilon)\,dx\,dt\\
&\quad+\frac12\int_\Omega\big(\phi_f^\epsilon|f_1^o|^2+\phi^\epsilon(|c_1^o|^2+|C_1^o|^2)\big)dx.
\end{aligned}
$$

The convective terms in (2.7) are estimated as follows using the Cauchy-Schwarz and Young inequalities. In the fractured part, we write

$$\Big|\int_{\Omega_f^\epsilon\times J}(\underline{v}_f^\epsilon\cdot\nabla f_1^\epsilon)\,f_1^\epsilon\,dx\Big|\le\int_{\Omega_f^\epsilon\times J}\frac{\alpha_t}{2}|\underline{v}_f^\epsilon|\,|\nabla f_1^\epsilon|^2\,dx+C\|f_1^\epsilon\|_\infty^2\int_{\Omega_f^\epsilon}|\underline{v}_f^\epsilon|\,dx,$$

where $0\le f_1^\epsilon(x,t)\le 1$ a.e. in $\Omega_f^\epsilon\times J$ and $\underline{v}_f^\epsilon$ is uniformly bounded in $(L^1(\Omega_f^\epsilon\times J))^3$ thanks to Lemma 2.2. In the matrix part, we get firstly

$$\begin{aligned}\Big|\int_{\Omega_m^\epsilon\times J}\underline{\mathcal{V}}^\epsilon\cdot(c_1^\epsilon\,\nabla c_1^\epsilon+C_1^\epsilon\,\nabla C_1^\epsilon)\Big|&\le\int_{\Omega_m^\epsilon\times J}\frac{\alpha_t}{2}|\underline{\mathcal{V}}_s^\epsilon+\epsilon^2\underline{\mathcal{V}}_h^\epsilon|\,(|\nabla c_1^\epsilon|^2+|\nabla C_1^\epsilon|^2)\\&\quad+C\int_{\Omega_f^\epsilon}(|\underline{\mathcal{V}}_s^\epsilon|+|\underline{\mathcal{V}}_h^\epsilon|)\,(|c_1^\epsilon|^2+|C_1^\epsilon|^2)\,dx.\end{aligned}$$

The second term of the right-hand side of the latter relation is treated as follows using Lemma 2.2.

$$\begin{aligned}&\int_{\Omega_f^\epsilon}|\underline{\mathcal{V}}_s^\epsilon+\underline{\mathcal{V}}_h^\epsilon|\,(|c_1^\epsilon|^2+|C_1^\epsilon|^2)\\&\le\frac{k_+}{\mu_-}\int_{\Omega_f^\epsilon}\big(\alpha(c_1^\epsilon+c_2^\epsilon)(1-\epsilon)+\epsilon\big)|\nabla p^\epsilon|\,\big(|c_1^\epsilon|^2+|C_1^\epsilon|^2\big)\\&\le C\Big(\int_{\Omega_f^\epsilon}\big(\alpha^2(c_1^\epsilon+c_2^\epsilon)^2+\epsilon^2(1-\alpha(c_1^\epsilon+c_2^\epsilon))^2|\nabla p^\epsilon|^2\Big)^{1/2}\\&\times\big(\|c_1^\epsilon\|_{L^\infty(\Omega_m^\epsilon)}^2+\|C_1^\epsilon\|_{L^\infty(\Omega_m^\epsilon)}^2\big)\\&\le\frac{C}{\delta}\big(\|c_1^\epsilon\|_{L^\infty(\Omega_m^\epsilon)}^2+\|C_1^\epsilon\|_{L^\infty(\Omega_m^\epsilon)}^2\big)+\phi_-\delta\int_{\Omega_f^\epsilon}(\alpha+\epsilon^2)D_m\big(|\nabla c_1^\epsilon|^2+|\nabla C_1^\epsilon|^2\big),\end{aligned}$$

for any $\delta>0$. The last term in the left-hand side of (2.7) is nonnegative. Using the latter estimates, the Cauchy-Schwarz and Young inequalities for the right-hand side source terms and the basic properties (1.21) of the tensors $\mathcal{D}$ and $\mathcal{D}^\epsilon$, it follows from (2.7) that

$$\begin{aligned}&\frac{\phi_-}{2}\int_\Omega(\chi_f^\epsilon|f_1^\epsilon|^2+\chi_m^\epsilon(|c_1^\epsilon|^2+|C_1^\epsilon|^2))\,dx+\phi_-\int_{\Omega_f^\epsilon\times J}(D_m+\frac{\alpha_t}{2}|\underline{v}_f^\epsilon|)\,|\nabla f_1^\epsilon|^2\,dxdt\\&+\phi_-\int_{\Omega_m^\epsilon\times J}((\alpha+\epsilon^2)(1-\delta)D_m+\frac{\alpha_t}{2}|\underline{\mathcal{V}}_s^\epsilon+\epsilon^2\underline{\mathcal{V}}_h^\epsilon|)\,\big(|\nabla c_1^\epsilon|^2+|\nabla C_1^\epsilon|^2\big)\,dxdt\\&\le\frac{C}{\delta}+C\int_{\Omega_f^\epsilon\times J}|f_1^\epsilon|^2\,dxdt.\end{aligned}$$

We choose $0<\delta<1$. We use the Gronwall lemma to infer from the latter relation that $\sqrt{\alpha+\beta\epsilon^2}\nabla c_1^\epsilon$ and $|\alpha(c_1^\epsilon+c_2^\epsilon)+\epsilon^3(1-\alpha(c_1^\epsilon+c_2^\epsilon))\nabla p^\epsilon|^{1/2}\nabla c_1^\epsilon$ are uniformly bounded in $(L^2(\Omega\times J))^3$. The estimates for f_1^ϵ, c_1^ϵ and C_1^ϵ follow. Once we know the estimate for c_1^ϵ, we obtain similar ones for c_2^ϵ by multiplying (1.7) by c_1^ϵ, (1.8) by c_2^ϵ, integrating over Ω_m^ϵ and summing up the results to kill the terms on Γ_{fm}^ϵ. Our claim is proved. □

We now have sufficient estimates to state the first convergence result. The proof of the homogenization process will be carried out by using the two-scale convergence introduced by G.Nguetseng in [19] and developed by Allaire in [2]. The basic definition and properties of this concept follow.

Proposition 2.4. *A sequence of functions* (v^ϵ) *bounded in* $L^2(\Omega\times J)$ *two-scale converges to a limit* $v^o(x,y,t)$ *belonging to* $L^2(\Omega\times Y\times J)$, $v^\epsilon \overset{2}{\rightharpoonup} v^o$, *if*

$$\lim_{\epsilon\to 0}\int_{\Omega\times J} v^\epsilon(x,t)\,\Psi(x,x/\epsilon,t)\,dxdt = \int_{\Omega\times J}\int_Y v^o(x,y,t)\,\Psi(x,y,t)\,dxdydt,$$

for any test function $\Psi(x,y,t)$, *Y-periodic in the second variable, satisfying*

$$\lim_{\epsilon\to 0}\int_{\Omega\times J}|\Psi(x,x/\epsilon,t)|^2\,dxdt = \int_{\Omega\times J}\int_Y |\Psi(x,y,t)|^2\,dxdydt.$$

(i) From each bounded sequence (v^ϵ) *in* $L^2(\Omega\times J)$ *one can extract a subsequence which two-scale converges.*
(ii) Let (v^ϵ) *be a bounded sequence in* $L^2(J;H^1(\Omega))$ *which converges weakly to* v *in* $L^2(J;H^1(\Omega))$. *Then* $v^\epsilon \overset{2}{\rightharpoonup} v$ *and there exists a function* $v^1\in L^2(\Omega\times J;H^1_{per}(Y))$ *such that, up to a subsequence,* $\nabla v^\epsilon \overset{2}{\rightharpoonup} \nabla v(x,t)+\nabla_y v^1(x,y,t)$.
(iii) Let (v^ϵ) *be a bounded sequence in* $L^2(\Omega\times J)$ *with* $(\epsilon\nabla v^\epsilon)$ *bounded in* $(L^2(\Omega\times J))^3$. *Then, there exists a function* $v^o\in L^2(\Omega\times J;H^1_{per}(Y))$ *such that, up to a subsequence,* $v^\epsilon \overset{2}{\rightharpoonup} v^o$ *and* $\epsilon\nabla v^\epsilon \overset{2}{\rightharpoonup} \nabla_y v^o(x,y,t)$.

Before applying these results, we have to extend some functions to the whole domain Ω. We begin by defining a global pressure θ^ϵ by

$$\theta^\epsilon = \chi_f^\epsilon p_f^\epsilon + \chi_m^\epsilon p^\epsilon.$$

We have assumed that the connected sets Ω_f^ϵ and Ω_m^ϵ have the admissible structure to apply the results of [1]. We thus claim that, for $j=f,m$, there exists three constants $k_i^j = k_i^j(Y_j)>0$, $i=1,2,3$, and a linear and continuous extension operator $\Pi_j^\epsilon : H^1(\Omega_j^\epsilon)\to H^1_{loc}(\Omega)$ such that $\Pi_j^\epsilon v = v$ a.e. in Ω_j^ϵ and

$$\int_{\Omega(\epsilon k_1)}|\Pi_j^\epsilon v|^2\,dx \le k_2\int_{\Omega_j^\epsilon}|v|^2\,dx,\quad \int_{\Omega(\epsilon k_1)}|\nabla(\Pi_j^\epsilon v)|^2\,dx \le k_3\int_{\Omega_j^\epsilon}|\nabla v|^2\,dx$$

for all $v\in H^1(\Omega_j^\epsilon)$, with $\Omega(\epsilon k_1)=\{x\in\Omega \mid \operatorname{dist}(x,\Gamma)>\epsilon k_1\}$. To avoid dealing with boundary layers, we make the following additional assumption on the structure of the domain Ω:

$$\Omega_m^\epsilon = \Omega(\epsilon k_1)\cap\big\{\cup_{k\in\mathbb{Z}^3}\epsilon\,(Y_m+k)\big\}\quad\text{and}\quad \Omega_f^\epsilon=\Omega\setminus\overline{\Omega_m^\epsilon}.$$

We then define the extension C^ϵ of

$$c^\epsilon = c_1^\epsilon + c_2^\epsilon$$

by

$$C^\epsilon = \Pi_m^\epsilon c^\epsilon$$

and the extension of p_f^ϵ by $P_f^\epsilon = \Pi_f^\epsilon p_f^\epsilon$.

Now, in view of Lemmas 2.2 and 2.3, there exist functions $p_f\in L^2(J;H^1(\Omega))$, $p_f^1\in L^2(\Omega\times J;H^1_{per}(Y))$, $p^0\in L^2(\Omega\times J;H^1_{per}(Y))$, $(f_1,C_1,c_1,c_2)\in (L^\infty(J;L^2(\Omega))\cap L^2(J;H^1(\Omega)))^4$, $(c_1^0,C_1^0)\in (L^2(\Omega\times J;H^1_{per}(Y)))^2$ and $(f_1^1,C_1^1,c_1^1,c_2^1)\in (L^2(\Omega\times J;H^1_{per}(Y)))^4$ such that, up to extracted subsequences, as $\varepsilon\to 0$,

$$\begin{gathered}\theta^\epsilon = \chi_f^\epsilon p_f^\epsilon+\chi_m^\epsilon p^\epsilon \overset{2}{\rightharpoonup} p^0(x,y,t) = \chi_f(y)p_f(x,t)+\chi_m(y)p^0(x,y,t),\\ P_f^\epsilon \rightharpoonup p_f \text{ weakly in } L^2(\Omega\times J),\\ P_f^\epsilon \overset{2}{\rightharpoonup} p_f,\quad \nabla P_f^\epsilon \overset{2}{\rightharpoonup} \nabla p_f(x,t)+\nabla_y p_f^1(x,y,t),\end{gathered}$$

$$\epsilon\nabla\theta^\epsilon \xrightarrow{2} \chi_m(y)\nabla_y p^0(x,y,t),$$
$$\alpha^{1/2}C^\epsilon \xrightarrow{2} \alpha^{1/2}(c_1+c_2), \quad \alpha^{1/2}\nabla C^\epsilon \xrightarrow{2} \alpha^{1/2}\nabla(c_1+c_2)+\nabla_y(c_1^1+c_2^1),$$
$$\chi_f^\epsilon f_1^\epsilon \xrightarrow{2} \chi_f(y)f_1, \quad \chi_f^\epsilon\nabla f_1^\epsilon \xrightarrow{2} \chi_f(y)(\nabla f_1(x,t)+\nabla_y f_1^1(x,y,t)),$$
$$\chi_m^\epsilon\alpha^{1/2}C_1^\epsilon \xrightarrow{2} \chi_m(y)\alpha^{1/2}C_1,$$
$$\chi_m^\epsilon\alpha^{1/2}\nabla C_1^\epsilon \xrightarrow{2} \chi_m(y)\alpha^{1/2}(\nabla C_1(x,t)+\nabla_y C_1^1(x,y,t)),$$
$$\chi_m^\epsilon C_1^\epsilon \xrightarrow{2} \chi_m(y)C_1^0(x,y,t),\ \chi_m^\epsilon\epsilon\nabla C_1^\epsilon \xrightarrow{2} \chi_m(y)\nabla_y C_1^0(x,y,t),$$
$$\chi_m^\epsilon\alpha^{1/2}c_i^\epsilon \xrightarrow{2} \chi_m(y)\alpha^{1/2}c_i, \quad \chi_m^\epsilon\alpha^{1/2}\nabla c_i^\epsilon \xrightarrow{2} \chi_m(y)\alpha^{1/2}(\nabla c_i(x,t)+\nabla_y c_i^1(x,y,t)),$$
$$\chi_m^\epsilon c_i^\epsilon \xrightarrow{2} \chi_m(y)c_i^0(x,y,t),\ \chi_m^\epsilon\epsilon\nabla c_i^\epsilon \xrightarrow{2} \chi_m(y)\nabla_y c_i^0(x,y,t), \quad i=1,2.$$

We note that

$$C_1=\int_Y C_1^0(\cdot,y,\cdot)\,dy,\ \chi_{\Omega\setminus\Omega_0}C_1=\chi_{\Omega\setminus\Omega_0}\frac{1}{Y_m}\int_{Y_m} C_1^0(\cdot,y,\cdot)\,dy,$$
$$c_1=\int_Y c_1^0(\cdot,y,\cdot)\,dy,\ \chi_{\Omega\setminus\Omega_0}c_1=\chi_{\Omega\setminus\Omega_0}\frac{1}{Y_m}\int_{Y_m} c_1^0(\cdot,y,\cdot)\,dy.$$

We also assert that

$$\Phi^\epsilon=\chi_f^\epsilon\phi_f^\epsilon+\chi_m^\epsilon\phi^\epsilon \xrightarrow{2} \Phi(y)=\chi_f(y)\phi_f(y)+\chi_m(y)\phi(y),$$
$$K^\epsilon=\chi_f^\epsilon k_f^\epsilon+\chi_m^\epsilon k^\epsilon \xrightarrow{2} K(y)=\chi_f(y)k_f(y)+\chi_m(y)k(y),$$

and that Φ^ϵ and K^ϵ are admissible test functions for the two-scale convergence.

Furthermore, some two-scale limits are linked across the interface Γ_{fm}. We claim the following results.

Lemma 2.5. *The two limit pressures are equal on the matrix-fracture interface:*

$$p_f(x,t)=p^0(x,s,t) \text{ for } s\in\Gamma_{fm},\ (x,t)\in\Omega\times J.$$

Proof. We recall that $\theta^\epsilon=\chi_f^\epsilon p_f^\epsilon+\chi_m^\epsilon p^\epsilon\in L^2(J;H^1(\Omega))$ satisfies $\gamma_f^\epsilon\theta^\epsilon=\gamma_f^\epsilon p_f^\epsilon=\gamma_m^\epsilon p^\epsilon=\gamma_m^\epsilon\theta^\epsilon$ and $\epsilon\nabla\theta^\epsilon=\epsilon\chi_f^\epsilon\nabla p_f^\epsilon+\epsilon\chi_m^\epsilon\nabla p^\epsilon\in(L^2(\Omega\times J))^3$ for any fixed $\epsilon>0$. We know that $\theta^\epsilon \xrightarrow{2} \chi_f(y)p_f(x,t)+\chi_m(y)p^0(x,y,t)$ and $\epsilon\nabla\theta^\epsilon\xrightarrow{2}\chi_m(y)\nabla_y p^0(x,y,t)$. For any $\underline{\Psi}\in(\mathcal{C}_o^\infty(\Omega;\mathcal{C}_{per}^\infty(Y)))^3$ we write

$$\int_\Omega \epsilon\nabla\theta^\epsilon\cdot\underline{\Psi}(x,\frac{x}{\epsilon})\,dx=-\int_\Omega\theta^\epsilon\Big(\epsilon\operatorname*{div}_x\underline{\Psi}(x,\frac{x}{\epsilon})+\operatorname*{div}_y\underline{\Psi}(x,\frac{x}{\epsilon})\Big)\,dx.$$

We take the two-scale limits on both sides. We get

$$\begin{aligned}
&\int_\Omega\int_Y \chi_m(y)\nabla_y p^0\cdot\underline{\Psi}\,dx\\
&=-\int_\Omega\int_Y(\chi_f(y)p_f(x,t)+\chi_m(y)p^0(x,y,t))\operatorname*{div}_y\underline{\Psi}(x,y)\,dxdy\\
&=-\int_\Omega\int_{\partial Y_f} p_f(x,t)\underline{\Psi}(x,s)\cdot\nu_f\,dxds-\int_\Omega\int_{\partial Y_m} p^0(x,s,t)\underline{\Psi}(x,s)\cdot\nu_m\,dxds\\
&\quad+\int_\Omega\int_Y\chi_m(y)\nabla_y p^0\cdot\underline{\Psi}\,dx.
\end{aligned}$$

This proves that $p_f(x,t)=p^0(x,s,t)$ for $s\in\partial Y_f\cap\partial Y_m=\Gamma_{fm}$. $\square$

We add the following result linking the limit concentrations f_1 and $m_1 = \alpha c_1 + \beta C_1$. This lemma was already stated in [9] when α is a constant parameter. By the way, we detail its proof for the convenience of the reader.

Lemma 2.6. *The concentrations $f_1(x,t)$ and $m_1(x,t) = \alpha c_1(x,t) + \beta C_1(x,t)$ are equal almost everywhere in $(\Omega \setminus \Omega_0) \times J$ and $f_1(x,t) = \beta C_1^0(x,s,t)$ for a.e. $(x,t) \in \Omega_0 \times J$, $s \in \Gamma_{fm}$. Furthermore $\alpha(x)(c_1 + c_2)(x,t) = \alpha(x)$ almost everywhere in $\Omega \times J$.*

Proof. Let $d^\epsilon = \chi_f^\epsilon f_1^\epsilon + \chi_m^\epsilon m_1^\epsilon \in L^2(J; H^1(\Omega))$. It satisfies $\gamma_f^\epsilon d^\epsilon = \gamma_f^\epsilon f_1^\epsilon = \gamma_m^\epsilon m_1^\epsilon = \gamma_m^\epsilon d^\epsilon$ and $\epsilon \nabla d^\epsilon = \epsilon \chi_f^\epsilon \nabla f_1^\epsilon + \epsilon \chi_m^\epsilon \nabla m_1^\epsilon \in (L^2(\Omega \times J))^3$. We know that $d^\epsilon \overset{2}{\rightharpoonup} \chi_f(y) f_1 + \chi_{\Omega\setminus\Omega_0} \chi_m(y) m_1 + \chi_{\Omega_0} \chi_m(y) \beta C_1^0$ and $\epsilon \nabla d_\epsilon \overset{2}{\rightharpoonup} \chi_{\Omega_0} \chi_m(y) \beta \nabla_y C_1^0$. For any $\underline{\Psi} \in (\mathcal{C}_o^\infty(\Omega; \mathcal{C}_{per}^\infty(Y)))^3$ we write

$$\int_{\Omega \times J} \epsilon \nabla d^\epsilon \cdot \underline{\Psi}(x, \frac{x}{\epsilon})\, dxdt = -\int_{\Omega \times J} d^\epsilon \Big(\epsilon \operatorname*{div}_x \underline{\Psi}(x, \frac{x}{\epsilon}) + \operatorname*{div}_y \underline{\Psi}(x, \frac{x}{\epsilon}) \Big)\, dxdt.$$

We take the two-scale limits on both sides. We get

$$\begin{aligned}
&\int_{\Omega \times J} \int_{Y_m} \chi_{\Omega_0} \beta \nabla_y C_1^0 \cdot \underline{\Psi}\, dxdydt \\
&= -\int_{\Omega \times J} \int_Y (\chi_f(y) f_1(x,t) + \chi_m(y) \chi_{\Omega\setminus\Omega_0}(x) m_1(x,t) \\
&\quad + \chi_m(y) \chi_{\Omega_0}(x) \beta(x) C_1^0(x,y,t)) \operatorname*{div}_y \underline{\Psi}(x,y)\, dxdydt \\
&= -\int_{\Omega \times J} \int_{\partial Y_f} f_1(x,t) \underline{\Psi}(x,s) \cdot \nu_f\, dxdsdt - \int_{\Omega \times J} \int_{\partial Y_m} \big(\chi_{\Omega\setminus\Omega_0}(x) m_1(x,t) \\
&\quad + \chi_{\Omega_0}(x) \beta(x) C_1^0(x,s,t) \big) \underline{\Psi}(x,s) \cdot \nu_m\, dxdsdt + \int_{\Omega \times J} \int_{Y_m} \chi_{\Omega_0} \beta \nabla_y C_1^0 \cdot \underline{\Psi}\, dxdydt.
\end{aligned}$$

This proves that $f_1(x,t) = \chi_{\Omega\setminus\Omega_0}(x) m_1(x,t) + \chi_{\Omega_0}(x) \beta(x) C_1^0(x,s,t)$ for $s \in \partial Y_f \cap \partial Y_m = \Gamma_{fm}$ and thus $f_1(x,t) = m_1(x,t)$ a.e. in $(\Omega \setminus \Omega_0) \times J$ and $f_1(x,t) = \beta(x) C_1^0(x,s,t)$ a.e. $(x,t) \in \Omega_0 \times J$, $s \in \Gamma_{fm}$. The same computations for $d_c^\epsilon = \chi_f^\epsilon \alpha + \chi_m^\epsilon \alpha (c_1^\epsilon + c_2^\epsilon)$ show that $\alpha(x)(c_1 + c_2)(x,t) = \alpha(x)$ a.e. in $\Omega \times J$. $\square$

We then claim and prove the following compactness result, of course penalized by the degeneracy of function α in the set Ω_0.

Lemma 2.7. *The sequences $(\chi_f^\epsilon \chi_{\Omega\setminus\Omega_0} f_1^\epsilon)$, $(\chi_m^\epsilon \chi_{\Omega\setminus\Omega_0} m_1^\epsilon)$ and $(\chi_m^\epsilon \alpha^{1/2}(c_1^\epsilon + c_2^\epsilon))$ are sequentially compact in $L^2(\Omega \times J)$.*

Proof. We begin by writing the problem satisfied by m_1^ϵ in $\Omega_m^\epsilon \times J$.

$$\begin{aligned}
&\phi^\epsilon \partial_t m_1^\epsilon + \mathcal{V}^\epsilon \cdot \nabla m_1^\epsilon - \operatorname{div}(\mathcal{D}^\epsilon(\mathcal{V}^\epsilon) \nabla m_1^\epsilon) - (c_1^\epsilon - C_1^\epsilon) \mathcal{V}^\epsilon \cdot \nabla \alpha \\
&+ \mathcal{D}^\epsilon(\mathcal{V}^\epsilon) \nabla c_1^\epsilon \cdot \nabla \alpha - \mathcal{D}^\epsilon(\mathcal{V}^\epsilon) \nabla C_1^\epsilon \cdot \nabla \alpha + \operatorname{div}((c_1^\epsilon - C_1^\epsilon) \mathcal{D}^\epsilon(\mathcal{V}^\epsilon) \nabla \alpha) = q_s(\hat{f}_1 - m_1^\epsilon),
\end{aligned} \tag{2.8}$$

$$\begin{gathered}
\mathcal{D}^\epsilon(\mathcal{V}^\epsilon) \nabla m_1^\epsilon \cdot \nu_{fm} = (\alpha^2 + \beta^2) \mathcal{D}(\underline{v}_f^\epsilon) \nabla f_1^\epsilon \cdot \nu_{fm} + (c_1^\epsilon - C_1^\epsilon) \mathcal{D}^\epsilon(\mathcal{V}^\epsilon) \nabla \alpha \cdot \nu_{fm}, \\
m_1^\epsilon = f_1^\epsilon \quad \text{on } \Gamma_{fm} \times J,
\end{gathered} \tag{2.9}$$

$$\mathcal{D}^\epsilon(\mathcal{V}^\epsilon) \nabla m_1^\epsilon \cdot \nu = 0 \quad \text{on } (\partial \Omega_m^\epsilon \cap \Gamma) \times J, \tag{2.10}$$

$$m_1^\epsilon(x,0) = \alpha c_1^o(x) + \beta C_1^o(x) = \chi_m^\epsilon(x) f_1^o(x) \quad \text{in } \Omega_m^\epsilon. \tag{2.11}$$

On the one hand, let $\psi \in L^4(J;H^2(\Omega))$. We multiply (1.4) by $(\alpha^2+\beta^2)\chi_f^\epsilon\psi$ and (2.8) by $\chi_m^\epsilon\psi$. We integrate over $\Omega\times J$ and sum up the results. We get

$$
\begin{aligned}
&\langle(\chi_f^\epsilon\phi_f^\epsilon+\chi_m^\epsilon\phi^\epsilon)(\chi_f^\epsilon(\alpha^2+\beta^2)f_1^\epsilon+\chi_m^\epsilon m_1^\epsilon),\psi\rangle_{L^2(J;(H^2(\Omega))')\times L^2(J;H^2(\Omega))}\\
&=\int_{\Omega\times J}(\chi_f^\epsilon(\alpha^2+\beta^2)\underline{v}_f^\epsilon\cdot\nabla f_1^\epsilon+\chi_m^\epsilon\mathcal{V}^\epsilon\cdot\nabla m_1^\epsilon)\,\psi\,dxdt\\
&\quad+\int_{\Omega\times J}\chi_m^\epsilon(c_1^\epsilon-C_1^\epsilon)\,(\mathcal{V}^\epsilon\cdot\nabla\alpha)\,\psi\,dxdt+\int_{\Omega\times J}\chi_m^\epsilon(c_1^\epsilon-C_1^\epsilon)\,\mathcal{D}^\epsilon\nabla\alpha\cdot\nabla\psi\,dxdt\\
&\quad-\int_{\Omega\times J}(\chi_f^\epsilon(\alpha^2+\beta^2)\mathcal{D}\nabla f_1^\epsilon+\chi_m^\epsilon\mathcal{D}^\epsilon\nabla m_1^\epsilon)\cdot\nabla\psi\,dxdt\\
&\quad-\int_{\Omega\times J}2(2\alpha-1)\chi_f^\epsilon\mathcal{D}\nabla f_1^\epsilon\cdot\nabla\alpha\,dxdt\\
&\quad-\int_{\Omega\times J}\chi_m^\epsilon\big(\mathcal{D}^\epsilon\nabla c_1^\epsilon\cdot\nabla\alpha-\mathcal{D}^\epsilon\nabla C_1^\epsilon\cdot\nabla\alpha\big)\,\psi\,dxdt\\
&\quad+\int_{\Omega\times J}q_s\big((\alpha^2+\beta^2+1)\hat{f}_1-(\chi_f^\epsilon(\alpha^2+\beta^2)f_1^\epsilon+\chi_m^\epsilon m_1^\epsilon)\big)\,\psi\,dxdt.
\end{aligned}
$$

We recall that $\alpha\in\mathcal{C}^1(\overline{\Omega})$. Moreover, in view of the previous lemmas, we have

$$
\begin{aligned}
&\Big|\int_{\Omega\times J}(\chi_f^\epsilon(\alpha^2+\beta^2)\underline{v}_f^\epsilon\cdot\nabla f_1^\epsilon+\chi_m^\epsilon\mathcal{V}^\epsilon\cdot\nabla m_1^\epsilon)\,\psi\,dxdt\Big|\\
&\le C\||\underline{v}_f^\epsilon|^{1/2}\nabla f_1^\epsilon\|_{(L^2(\Omega_f^\epsilon\times J)^3}\||\underline{v}_f^\epsilon|^{1/2}\|_{L^4(\Omega_f^\epsilon\times J)}\|\psi\|_{L^4(\Omega_f^\epsilon\times J)}\\
&\quad+C\||\mathcal{V}^\epsilon|^{1/2}\nabla m_1^\epsilon\|_{(L^2(\Omega_m^\epsilon\times J)^3}\||\mathcal{V}^\epsilon|^{1/2}\|_{L^4(\Omega_m^\epsilon\times J)}\|\psi\|_{L^4(\Omega_m^\epsilon\times J)}\\
&\le C\|\psi\|_{L^4(J;H^2(\Omega))},
\end{aligned}
$$

$$
\begin{aligned}
&\Big|\int_{\Omega\times J}(\chi_f^\epsilon(\alpha^2+\beta^2)\mathcal{D}\nabla f_1^\epsilon+\chi_m^\epsilon\mathcal{D}^\epsilon\nabla m_1^\epsilon)\cdot\nabla\psi\,dxdt\Big|\\
&\le C\|\nabla\psi\|_{(L^4(\Omega\times J))^3}\\
&\le C\|\psi\|_{L^4(J;H^2(\Omega))},
\end{aligned}
$$

$$
\begin{aligned}
&\Big|\int_{\Omega\times J}q_s\big((\alpha^2+\beta^2+1)\hat{f}_1-(\chi_f^\epsilon(\alpha^2+\beta^2)f_1^\epsilon+\chi_m^\epsilon m_1^\epsilon)\big)\,\psi\,dxdt\Big|\\
&\le C(\|f_1^\epsilon\|_\infty,\|m_1^\epsilon\|_\infty)\|q_s\|_{L^2(\Omega\times J)}\|\psi\|_{L^2(\Omega\times J)}\\
&\le C\|\psi\|_{L^4(J;H^2(\Omega))}.
\end{aligned}
$$

We infer from the latter computations that the sequence $\partial_t(\phi_f^\epsilon(\alpha^2+\beta^2)\chi_f^\epsilon f_1^\epsilon+\phi^\epsilon\chi_m^\epsilon m_1^\epsilon)$ is uniformly bounded in $L^{4/3}(J;(H^2(\Omega))')$. Since $(\phi_f^\epsilon(\alpha^2+\beta^2)\chi_f^\epsilon f_1^\epsilon+\phi^\epsilon\chi_m^\epsilon m_1^\epsilon)$ is uniformly bounded in $L^\infty(\Omega\times J)$, a standard argument of Aubin's type proves that $(\phi_f^\epsilon(\alpha^2+\beta^2)\chi_f^\epsilon f_1^\epsilon+\phi^\epsilon\chi_m^\epsilon m_1^\epsilon)$ lies in a compact subset of $\mathcal{C}(J;(H^1(\Omega))')$. Therefore, there is $\xi\in L^2(J;(H^1(\Omega))')$, such that, up to an extracted subsequence,

$$
\phi_f^\epsilon(\alpha^2+\beta^2)\chi_f^\epsilon f_1^\epsilon+\phi^\epsilon\chi_m^\epsilon m_1^\epsilon\to\xi\quad\text{in }\mathcal{C}(J;(H^1(\Omega))')\text{ as }\epsilon\to0.
$$

Two-scale convergence arguments show that

$$
\xi=(\alpha^2+\beta^2)\Big(\int_{Y_f}\phi_f(y)dy\Big)f_1+\chi_{\Omega\setminus\Omega_0}\Big(\int_{Y_m}\phi(y)dy\Big)m_1+\chi_{\Omega_0}\int_{Y_m}\phi(y)\beta C_1^0(\cdot,y,\cdot)dy,
$$

where $m_1=\alpha c_1+\beta C_1=f_1$ a.e. in $\omega\setminus\Omega_0$ by Lemma 2.6.

On the other hand, the sequence $\alpha^{1/2}(\chi_f^\epsilon f_1^\epsilon + \chi_m^\epsilon m_1^\epsilon)$ is uniformly bounded in the space $L^2(J;H^1(\Omega))$. We thus can pass to the limit in the product $\langle \phi_f^\epsilon(\alpha^2 + \beta^2)\chi_f^\epsilon f_1^\epsilon + \phi^\epsilon \chi_m^\epsilon m_1^\epsilon, \alpha^{1/2}(\chi_f^\epsilon f_1^\epsilon + \chi_m^\epsilon m_1^\epsilon)\rangle_{(H^1(\Omega))'\times H^1(\Omega)}$ as follows.

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\Big(\big\langle \chi_f^\epsilon(\alpha^2+\beta^2)\phi_f^\epsilon f_1^\epsilon + \chi_m^\epsilon \phi^\epsilon m_1^\epsilon, \alpha^{1/2}\chi_f^\epsilon f_1^\epsilon\big\rangle + \big\langle \chi_f^\epsilon(\alpha^2+\beta^2)\phi_f^\epsilon f_1^\epsilon \\
&+ \chi_m^\epsilon \phi^\epsilon m_1^\epsilon, \alpha^{1/2}\chi_m^\epsilon m_1^\epsilon\big\rangle\Big) \\
&= \Big\langle \Big((\alpha^2+\beta^2)\int_{Y_f}\phi_f(y)dy + \int_{Y_m}\phi(y)dy\Big)f_1, \alpha^{1/2}|Y_f|f_1\Big\rangle \\
&\quad + \Big\langle \Big((\alpha^2+\beta^2)\int_{Y_f}\phi_f(y)dy + \int_{Y_m}\phi(y)dy\Big)f_1, \alpha^{1/2}|Y_m|m_1\Big\rangle \\
&= \Big\langle \Big((\alpha^2+\beta^2)\int_{Y_f}\phi_f(y)dy + \int_{Y_m}\phi(y)dy\Big)f_1, \alpha^{1/2}f_1\Big\rangle.
\end{aligned}
$$

As a consequence we have

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\big\langle \big((\alpha^2+\beta^2)\chi_f^\epsilon\phi_f^\epsilon + \chi_m^\epsilon\phi^\epsilon\big)(\chi_f^\epsilon f_1^\epsilon + \chi_m^\epsilon m_1^\epsilon - f_1), \alpha^{1/2}(\chi_f^\epsilon f_1^\epsilon + \chi_m^\epsilon m_1^\epsilon - f_1)\big\rangle \\
&= \lim_{\epsilon\to 0}\Big(\big\langle \big((\alpha^2+\beta^2)\chi_f^\epsilon\phi_f^\epsilon + \chi_m^\epsilon\phi^\epsilon\big)(\chi_f^\epsilon f_1^\epsilon + \chi_m^\epsilon m_1^\epsilon), \alpha^{1/2}(\chi_f^\epsilon f_1^\epsilon + \chi_m^\epsilon m_1^\epsilon)\big\rangle \\
&\quad - 2\big\langle \big((\alpha^2+\beta^2)\chi_f^\epsilon\phi_f^\epsilon + \chi_m^\epsilon\phi^\epsilon\big)(\chi_f^\epsilon f_1^\epsilon + \chi_m^\epsilon m_1^\epsilon), \alpha^{1/2}f_1\big\rangle \\
&\quad + \big\langle \big((\alpha^2+\beta^2)\chi_f^\epsilon\phi_f^\epsilon + \chi_m^\epsilon\phi^\epsilon\big)f_1, \alpha^{1/2}f_1\big\rangle\Big) = 0.
\end{aligned}
$$

Since $\alpha^2+\beta^2 > 0$, $\phi_f^\epsilon, \phi^\epsilon \geq \phi_- > 0$, this shows that $\alpha^{1/2}(\chi_f^\epsilon f_1^\epsilon + \chi_m^\epsilon m_1^\epsilon)$ strongly converges to $\alpha^{1/2}f_1$ in $L^2(\Omega\times J)$.

The compactness result for $\alpha^{1/2}(c_1^\epsilon + c_2^\epsilon)$ is proved using similar calculations. Note in particular that the problem satisfied by $c_1^\epsilon + c_2^\epsilon$ in $\Omega_m^\epsilon \times J$ is

$$
\phi^\epsilon\partial_t(c_1^\epsilon + c_2^\epsilon) + \mathcal{V}^\epsilon\cdot\nabla(c_1^\epsilon + c_2^\epsilon) - \operatorname{div}(\mathcal{D}^\epsilon\nabla(c_1^\epsilon + c_2^\epsilon)) = q_s(1 - c_1^\epsilon - c_2^\epsilon), \tag{2.12}
$$

$$
\mathcal{D}^\epsilon\nabla(c_1^\epsilon + c_2^\epsilon)\cdot\nu = 0 \quad \text{on } \partial\Omega_m^\epsilon\times J, \tag{2.13}
$$

$$
(c_1^\epsilon + c_2^\epsilon)(x,0) = c_1^o(x) + c_2^o(x) \quad \text{in } \Omega_m^\epsilon. \tag{2.14}
$$

This completes the proof. □

We now aim to state some compactness result for the pressure. But as emphasized in Lemma 1, we have no direct estimate for the pressure gradient in the matrix part. Our solely estimate is weighted by $(c_1^\epsilon + c_2^\epsilon)$. We thus begin by the following result for the weight function.

Lemma 2.8. *For any real number a such that $0 < a < 1$, the sequence $\alpha^{1/2}(c^\epsilon + \epsilon^2)^{(a-1)/2}\nabla c^\epsilon$, $c^\epsilon = c_1^\epsilon + c_2^\epsilon$, is uniformly bounded in $(L^2(\Omega_m^\epsilon\times J))^3$.*

Proof. The function c^ϵ is solution of Problem (2.12)-(2.14). We also already know that c^ϵ is uniformly bounded in $L^\infty(\Omega_m^\epsilon\times J)$. Let $0<a<1$. We multiply (2.12) by $(c^\epsilon+\epsilon^2)^a$ and we integrate by parts over $\Omega_m^\epsilon\times(0,t)$, $t\in(0,T)$. We obtain

$$
\begin{aligned}
&\frac{1}{1+a}\int_{\Omega_m^\epsilon}(c^\epsilon(x,t)+\epsilon^2)^{1+a}\,dx - \frac{1}{1+a}\int_{\Omega_m^\epsilon}(c_1^o(x)+c_2^o(x)+\epsilon^2)^{1+a}\,dx \\
&+ \int_{\Omega_m^\epsilon\times(0,t)}(\mathcal{V}^\epsilon\cdot\nabla c^\epsilon)\,(c^\epsilon+\epsilon^2)^a\,dxdt + \int_{\Omega_m^\epsilon\times(0,t)}\frac{a}{(c^\epsilon+\epsilon^2)^{1-a}}\mathcal{D}^\epsilon\nabla c^\epsilon\cdot\nabla c^\epsilon\,dxdt
\end{aligned}
$$

$$= \int_{\Omega_m^\epsilon\times(0,t)} q_s(1-c^\epsilon)\,(c^\epsilon+\epsilon^2)^a\,dxdt. \tag{2.15}$$

We write

$$\begin{aligned}&\Big|\int_{\Omega_m^\epsilon\times(0,t)} (\mathcal{V}^\epsilon\cdot\nabla c^\epsilon)\,(c^\epsilon+\epsilon^2)^a\,dxdt\\ &\le \||\mathcal{V}^\epsilon|^{1/2}\nabla c^\epsilon\|_{(L^2(\Omega_m^\epsilon\times J))^3}\||\mathcal{V}^\epsilon|^{1/2}\|_{L^4(\Omega_m^\epsilon\times J)}\|(c^\epsilon+\epsilon^2)^a\|_\infty \le C,\end{aligned}$$

$$\Big|\int_{\Omega_m^\epsilon} (c^\epsilon(x,t)+\epsilon^2)^{1+a}\,dx\Big| \le C\|(c^\epsilon+\epsilon^2)^{1+a}\|_\infty \le C,$$

$$\Big|\int_{\Omega_m^\epsilon\times(0,t)} q_s(1-c^\epsilon)\,(c^\epsilon+\epsilon^2)^a\,dxdt\Big| \le C\|q_s\|_{L^2(\Omega\times J}\|1-c^\epsilon\|_\infty\|(c^\epsilon+\epsilon^2)^a\|_\infty \le C.$$

The result of the lemma then follows from (2.15). $\square$

We now claim the following weighted compactness result for the pressure in the matrix part of the domain.

Lemma 2.9. *The following strong convergence holds true.*

$$\sqrt{\alpha C^\epsilon}\theta^\epsilon \to \sqrt{\alpha(c_1+c_2)}p \quad \textit{in } L^2(\Omega\times J),$$

where p is the weak limit of θ^ϵ in $L^2(\Omega\times J)$:

$$p(x,t) = \int_Y p^0(x,y,t)\,dy.$$

Proof. The following lines being quite technical, we assume for sake of simplicity that the global pressure θ^ϵ is nonnegative. In the general case, one would perform the same computations as below replacing θ^ϵ by the nonnegative function $\sqrt{{\theta^\epsilon}^2+\eta}$, $\eta>0$. We define the auxiliary function P^ϵ by

$$P^\epsilon = \frac{\theta^\epsilon}{\theta^\epsilon+1}.$$

We note that (P^ϵ) is a bounded sequence of $L^\infty(\Omega\times J)$. Let us denote by P its weak limit. We have introduced the function P^ϵ in view to apply the convexity results for limits of bounded sequences in L^∞ of [23]. We also note in view of Lemma 2.2 and Proposition 2.4 *(ii)* that the two-scale limit of the sequence $(\alpha C^\epsilon\theta^\epsilon)$ does not depend on the microscopic variable y. The same holds true for the sequence $(\alpha C^\epsilon P^\epsilon)$. Furthermore, by Lemma 2.7, $(\alpha^{1/2}C^\epsilon)$ is sequentially compact in $L^2(\Omega\times J)$ and we have denoted by $\alpha^{1/2}c = \alpha^{1/2}(c_1+c_2)$ its limit. We then assert that

$$\Phi^\epsilon\alpha C^\epsilon P^\epsilon \rightharpoonup \overline{\Phi}\,\alpha cP \text{ in } L^2(\Omega\times J). \tag{2.16}$$

Choosing $\psi/(p^\epsilon+1)^2$ as test function in the variational formulation (2.1), one easily checks that $\Phi^\epsilon\partial_t P^\epsilon$ is uniformly bounded in $L^1(J;(W^{1,3}(\Omega))')$. Using a classical argument of Aubin's type (see [21]), we conclude that $\Phi^\epsilon P^\epsilon$ is sequentially compact in $L^2(J;(H^1(\Omega))')$.

We thus can pass to the limit $\epsilon\to0$ in the duality product

$$\langle\Phi^\epsilon P^\epsilon, \alpha C^\epsilon P^\epsilon/(P^\epsilon+1)\rangle_{(H^1(\Omega))'\times H^1(\Omega)} \to \langle\overline{\Phi^\epsilon P^\epsilon}, \alpha c\,\overline{P^\epsilon/(P^\epsilon+1)}\rangle$$

where $\overline{f^\epsilon}$ denotes the *ad hoc* limit of the sequence f^ϵ. In view of (2.16), the latter convergence reads

$$\alpha\Phi^\epsilon \frac{P^{\epsilon 2}}{P^\epsilon+1}C^\epsilon \rightharpoonup \alpha\overline{\Phi}P\overline{\frac{P^\epsilon}{P^\epsilon+1}}\, c \quad \text{in } L^2(\Omega\times J). \tag{2.17}$$

Since $\Phi^\epsilon(P^\epsilon+1)$ is also sequentially compact in $L^2(J;(H^1(\Omega))')$, we compute

$$\langle\Phi^\epsilon(P^\epsilon+1), \alpha C^\epsilon P^\epsilon/(P^\epsilon+1)\rangle_{(H^1(\Omega))'\times H^1(\Omega)} \to \langle\overline{\Phi^\epsilon(P^\epsilon+1)}, \alpha c\,\overline{P^\epsilon/(P^\epsilon+1)}\rangle,$$

which means with (2.16)

$$\alpha\Phi^\epsilon \frac{P^\epsilon(P^\epsilon+1)}{P^\epsilon+1}C^\epsilon \rightharpoonup \alpha\overline{\Phi}c(P+1)\overline{P^\epsilon/(P^\epsilon+1)}.$$

But $\Phi^\epsilon\frac{P^\epsilon(P^\epsilon+1)}{P^\epsilon+1}\alpha^{1/2}C^\epsilon = \Phi^\epsilon P^\epsilon\alpha^{1/2}C^\epsilon \rightharpoonup \overline{\Phi}P\alpha^{1/2}c$. We thus infer from the latter relation that

$$\alpha c\,\overline{\frac{P^\epsilon}{P^\epsilon+1}} = \alpha c\,\frac{P}{P+1}. \tag{2.18}$$

Inserting (2.18) in (2.17) yields

$$\alpha\Phi^\epsilon\frac{P^{\epsilon 2}}{P^\epsilon+1}C^\epsilon \rightharpoonup \alpha\overline{\Phi}\frac{P^2}{P+1}\, c \quad \text{in } L^2(\Omega\times J). \tag{2.19}$$

Now we note that $(\alpha C^\epsilon/\sqrt{P^\epsilon+1})$ is uniformly bounded in $L^2(J;H^1(\Omega))$. We then pass to the limit in the following duality product

$$\langle\Phi^\epsilon P^\epsilon, \alpha C^\epsilon/\sqrt{P^\epsilon+1}\rangle_{(H^1(\Omega))'\times H^1(\Omega)} \to \langle\overline{\Phi^\epsilon P^\epsilon}, \alpha c\,\overline{1/\sqrt{P^\epsilon+1}}\rangle,$$

that is with (2.16)

$$\alpha\Phi^\epsilon P^\epsilon\frac{C^\epsilon}{\sqrt{P^\epsilon+1}} \rightharpoonup \alpha\overline{\Phi}Pc\overline{\frac{1}{\sqrt{P^\epsilon+1}}}. \tag{2.20}$$

Using the strong convergence of $\alpha^{1/2}C^\epsilon$ to $\alpha^{1/2}c$ in $L^2(\Omega\times J)$, we also have

$$\alpha\Phi^\epsilon P^\epsilon\frac{C^\epsilon}{\sqrt{P^\epsilon+1}} \rightharpoonup \alpha\overline{\Phi}c\overline{\frac{P^\epsilon}{\sqrt{P^\epsilon+1}}}. \tag{2.21}$$

Since we manipulate here bounded sequences in $L^\infty(\Omega\times J)$, we can use convexity arguments of Tartar [23] to claim that

$$\begin{gathered}
\frac{1}{\sqrt{P+1}} \le \overline{\frac{1}{\sqrt{P^\epsilon+1}}} \text{because of the concavity of } x\mapsto\frac{1}{\sqrt{x+1}} \text{ in } \mathbb{R}_+,\\
\overline{\frac{P^\epsilon}{\sqrt{P^\epsilon+1}}} \le \frac{P}{\sqrt{P+1}} \text{because of the convexity of } x\mapsto\frac{x}{\sqrt{x+1}} \text{ in } \mathbb{R}_+.
\end{gathered}$$

The two latter relations with (2.20) and (2.21) give

$$\alpha\Phi^\epsilon C^\epsilon\frac{P^\epsilon}{\sqrt{P^\epsilon+1}} \rightharpoonup \alpha\overline{\Phi}c\frac{P}{\sqrt{P+1}} \quad \text{in } L^2(\Omega\times J). \tag{2.22}$$

Bearing in mind that $\Phi^\epsilon \ge \phi_- > 0$, we infer from (2.19) and (2.22) that

$$\alpha C^\epsilon\frac{P^\epsilon}{\sqrt{P^\epsilon+1}} \to \alpha c\frac{P}{\sqrt{P+1}}$$

a.e. in $\Omega\times J$ and strongly in $L^p(\Omega\times J)$, for all $1\le p<\infty$. It follows that $(\alpha P^{\epsilon 2}/(P^\epsilon+1))C^\epsilon(P^\epsilon+1) \rightharpoonup \alpha P^2 c$, and then

$$\sqrt{\alpha C^\epsilon}P^\epsilon \to \sqrt{\alpha c}P \text{ a.e. in } \Omega\times J \text{ and strongly in } L^p(\Omega\times J),\ \forall 1\le p<\infty.$$

We note that all the latter computations can be performed using the function

$$P^{\epsilon\prime} = \frac{\theta^{\epsilon 2}}{\theta^\epsilon + 1}$$

instead of P^ϵ. It leads to

$$\sqrt{\alpha C^\epsilon} P^{\epsilon\prime} \to \sqrt{\alpha c} P' \text{ a.e. in } \Omega \times J \text{ and strongly in } L^p(\Omega \times J),\ \forall 1 \le p < \infty.$$

Let us go back to the problem of the limit behavior of the global pressure θ^ϵ. The two latter convergence results read

$$\sqrt{\alpha C^\epsilon}\frac{\theta^\epsilon}{\theta^\epsilon + 1} \to \sqrt{\alpha c}\overline{\frac{\theta^\epsilon}{\theta^\epsilon + 1}}, \quad \sqrt{\alpha C^\epsilon}\frac{\theta^{\epsilon 2}}{\theta^\epsilon + 1} \to \sqrt{\alpha c}\overline{\frac{\theta^{\epsilon 2}}{\theta^\epsilon + 1}}. \tag{2.23}$$

Similar computations than those performed with P^ϵ in the latter lines allow to assert that

$$\sqrt{\alpha c}\overline{\frac{\theta^\epsilon}{\theta^\epsilon + 1}} = \sqrt{\alpha c}\frac{p}{p+1},$$

where p is the weak limit in $L^2(\Omega \times J)$ of θ^ϵ. It then follows from the first convergence in (2.23) that $\alpha C^\epsilon \theta^{\epsilon 2}/(\theta^\epsilon + 1)^2 \to \alpha c p^2/(p+1)^2$ a.e. in $\Omega \times J$. Multiplying the latter relation by $\theta^\epsilon + 1$, we conclude that

$$\alpha C^\epsilon \frac{\theta^{\epsilon 2}}{\theta^\epsilon + 1} \rightharpoonup \alpha c \frac{p^2}{p+1}.$$

This convergence together with the second one in (2.23) proves that

$$\sqrt{\alpha C^\epsilon}\frac{\theta^{\epsilon 2}}{\theta^\epsilon + 1} \to \sqrt{\alpha c}\frac{p^2}{p+1} \text{ a.e. in } \Omega \times J.$$

Since $\alpha^{1/2} C^\epsilon \to \alpha^{1/2} c$ and thus $\sqrt{\alpha C^\epsilon}(\theta^\epsilon + 1) \rightharpoonup \sqrt{\alpha c}(p+1)$, it follows that

$$\sqrt{\alpha C^\epsilon}\frac{\theta^{\epsilon 2}}{\theta^\epsilon + 1}\sqrt{\alpha C^\epsilon}(\theta^\epsilon + 1) = \alpha C^\epsilon \theta^{\epsilon 2} \rightharpoonup \alpha c p^2.$$

We conclude that

$$\sqrt{\alpha C^\epsilon}\theta^\epsilon \to \sqrt{\alpha c} p \text{ strongly in } L^2(\Omega \times J).$$

Lemma 2.9 is proved. □

Our last preliminary lemma before to pass to the limit in the pressure problem gives the two-scale limit of the weighted pressure.

Lemma 2.10. *There exists some function $\xi_2 \in L^{2q/(q+2)}(\Omega \times J; H^1_{per}(Y))$ such that*

$$\alpha C^\epsilon \nabla \theta^\epsilon \overset{2}{\rightharpoonup} \alpha(c_1 + c_2)\nabla p + \nabla_y \xi_2.$$

Proof. Let $\underline{\Psi} \in (\mathcal{D}(\Omega \times J; \mathcal{C}^\infty_{per}(Y)))^3$ such that $\operatorname{div}_y \underline{\Psi} = 0$. We set $\underline{\Psi}^\epsilon(x,t) = \underline{\Psi}(x, x/\epsilon, t)$. The sequence $((\alpha C^\epsilon + \epsilon^2)\nabla\theta^\epsilon)$ being uniformly bounded in $(L^2(\Omega \times J))^3$, it admits a two-scale limit. Let us denote it by $\underline{\xi}$. Since $\epsilon\nabla\theta^\epsilon$ is uniformly bounded in $(L^2(\Omega \times J))^3$, we also have

$$\alpha C^\epsilon \nabla\theta^\epsilon \overset{2}{\rightharpoonup} \underline{\xi}.$$

By definition of ξ we have

$$\lim_{\epsilon\to 0}\int_{\Omega\times J}(\alpha C^\epsilon + \epsilon^2)\nabla\theta^\epsilon \cdot \underline{\Psi}^\epsilon \, dxdt = \int_{\Omega\times J}\int_Y \underline{\xi}\cdot\underline{\Psi}\, dxdt. \tag{2.24}$$

We also have

$$
\begin{aligned}
&\int_{\Omega\times J}(\alpha C^\epsilon+\epsilon^2)\nabla\theta^\epsilon\cdot\underline{\Psi}^\epsilon\,dxdt\\
&=\int_{\Omega\times J}(\alpha C^\epsilon+\epsilon^2)^{1/2}(\alpha C^\epsilon+\epsilon^2)^{1/2}\nabla\theta^\epsilon\cdot\underline{\Psi}^\epsilon\,dxdt\\
&=-2\int_{\Omega\times J}(\alpha C^\epsilon+\epsilon^2)^{1/2}\theta^\epsilon\nabla\big((\alpha C^\epsilon+\epsilon^2)^{1/2}\big)\cdot\underline{\Psi}^\epsilon\,dxdt\\
&\quad-\int_{\Omega\times J}(\alpha C^\epsilon+\epsilon^2)\theta^\epsilon\,\operatorname*{div}_x\underline{\Psi}^\epsilon\,dxdt
\end{aligned}
$$

and then

$$
\begin{aligned}
&\lim_{\epsilon\to0}\int_{\Omega\times J}(\alpha C^\epsilon+\epsilon^2)\nabla\theta^\epsilon\cdot\underline{\Psi}^\epsilon\,dxdt\\
&=-2\int_{\Omega\times J}\int_Y\alpha(x)^{1/2}(c_1+c_2)^{1/2}p(x,t)\\
&\quad\times\nabla\big(\alpha^{1/2}(c_1+c_2)^{1/2}+\nabla_yC_{sqrt}\big)\underline{\Psi}(x,y,t)\,dxdydt\\
&\quad-\int_{\Omega\times J}\int_Y\alpha(c_1+c_2)\,p(x,t)\,\operatorname*{div}_x\underline{\Psi}(x,y,t)\,dxdydt,
\end{aligned}
\tag{2.25}
$$

where $C_{sqrt}\in L^2(\Omega\times J;H^1_{per}(Y))$ is such that

$$
\nabla(\alpha C^\epsilon)^{1/2}\xrightarrow{2}\nabla\big(\alpha(c_1+c_2)\big)^{1/2}+\nabla_yC_{sqrt}.
$$

It follows from (2.24)-(2.25) that

$$
\begin{aligned}
\underline{\xi}(x,y,t)&=-p(x,t)\,\nabla\big(\alpha(x)(c_1(x,t)+c_2(x,t))\big)\\
&\quad-2\alpha(x)^{1/2}(c_1(x,t)+c_2(x,t))^{1/2}p(x,t)\,\nabla_yC_{sqrt}(x,y,t)\\
&\quad+\nabla\big(\alpha(x)(c_1(x,t)+c_2(x,t))p(x,t)\big)+\nabla_y\xi_1(x,y,t)
\end{aligned}
$$

for some function $\xi_1\in L^2(\Omega\times J;H^1_{per}(Y))$. Defining the function $\xi_2\in L^{2q/(q+2)}(\Omega\times J;H^1_{per}(Y))$ by $\xi_2(x,y,t)=-2\alpha(x)^{1/2}(c_1(x,t)+c_2(x,t))^{1/2}p(x,t)C_{sqrt}(x,y,t)+\xi_1$, we have

$$
\underline{\xi}=\alpha(c_1+c_2)\nabla p+\nabla_y\xi_2.
$$

This completes the proof. □

We now have the main tools to pass to the limit $\epsilon\to0$ in the microscopic problem.

3. Derivation of the homogenized problem

We begin by studying the limit behavior of the pressure problem. We multiply Equation (1.5) by a test function in the form $\Psi(x,t)+\epsilon\Psi_{1,f}(x,x/\epsilon,t)$, with $\Psi\in\mathcal{D}(\Omega\times J)$ and $\Psi_{1,f}\in\mathcal{D}(\Omega\times J;C^\infty_{per}(Y))$. We also multiply Equation (1.9) by $\Psi(x,t)+\epsilon\Psi_{1,m}(x,x/\epsilon,t)+\psi(x,x/\epsilon,t)$, where $\Psi_{1,m}\in\mathcal{D}(\Omega\times J;C^\infty_{per}(Y))$ is such that $\Psi_{1,m}(x,y,t)=\Psi_{1,f}(x,y,t)$ if $y\in\Gamma_{fm}$ and $\psi\in\mathcal{D}(\Omega\times J;\mathcal{C}^\infty_{per}(Y)$ with support in $\Omega_0\times Y_m\times J$. Integrating over $\Omega\times J$, we obtain

$$
\begin{aligned}
&\int_{\Omega\times J}\chi^\epsilon_f\phi^\epsilon_f(x)\,\partial_tp^\epsilon_f\big(\Psi(x,t)+\epsilon\,\Psi_{1,f}(x,x/\epsilon,t)\big)\\
&+\int_{\Omega\times J}\chi^\epsilon_m\phi^\epsilon(x)\,\partial_tp^\epsilon\big(\Psi(x,t)+\epsilon\,\Psi_{1,m}(x,x/\epsilon,t)+\psi(x,x/\epsilon,t)\big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_{\Omega\times J} \chi_f^\epsilon \frac{k_f^\epsilon(x)}{\mu(f_1^\epsilon)} \nabla p_f^\epsilon \cdot \big(\nabla\Psi + \epsilon\nabla_x\Psi_{1,f}^\epsilon + \nabla_y\Psi_{1,f}^\epsilon\big)\\
&+ \int_{\Omega\times J} \chi_m^\epsilon \frac{k^\epsilon(x)}{\mu(m_1^\epsilon)} \big(\alpha(c_1^\epsilon + c_2^\epsilon)(1-\epsilon^2) + \epsilon^2\big)\nabla p^\epsilon \cdot \big(\nabla\Psi + \epsilon\nabla_x\Psi_{1,m}^\epsilon + \nabla_y\Psi_{1,m}^\epsilon\\
&+ \nabla_x\psi^\epsilon + \frac{1}{\epsilon}\nabla_y\psi^\epsilon\big)\\
&= \int_{\Omega\times J} q_s\,(\Psi + \epsilon\Psi_{1,f}^\epsilon + \epsilon\Psi_{1,m}^\epsilon + \psi).
\end{aligned}
$$

Letting $\epsilon \to 0$, we get

$$
\begin{aligned}
&\int_{\Omega\times J}\int_{Y_f} \phi_f(y)\,\partial_t p_f\,\Psi(x,t) + \int_{\Omega\times J}\int_{Y_m} \phi(y)\,\partial_t p^0\,(\Psi(x,t) + \psi(x,y,t))\\
&+ \int_{\Omega\times J}\int_{Y_f} \frac{k_f(y)}{\mu(f_1)}(\nabla p_f + \nabla_y p_f^1)\cdot(\nabla\Psi + \nabla_y\Psi_{1,f})\\
&+ \int_{\Omega\times J}\int_{Y_m} \frac{k(y)}{\mu(f_1)}(\alpha(c_1+c_2)\nabla p + \nabla_y\xi_2)\cdot(\nabla\Psi + \nabla_y\Psi_{1,m})\\
&+ \int_{\Omega_0\times J}\int_{Y_m} \frac{k(y)}{\mu}\nabla_y p^0\cdot\nabla_y\psi\\
&= \int_{\Omega\times J\times Y} q_s\,(\Psi+\psi).
\end{aligned}
\tag{3.1}
$$

By density arguments we conclude that the corresponding two-scale homogenized system in $\Omega \times J$ is:

$$
\begin{gathered}
\begin{aligned}
&(\overline{\phi_f}^{Y_f} + \chi_{\Omega\setminus\Omega_0}\overline{\phi}^{Y_m})\partial_t p_f + \chi_{\Omega_0}\int_{Y_m}\phi(y)\partial_t p^0\,dy\\
&- \operatorname{div}\Big(\frac{1}{\mu(f_1)}\int_{Y_f} k_f(y)(\nabla p_f + \nabla_y p_f^1)dy\Big)\\
&- \operatorname{div}\Big(\frac{1}{\mu(f_1)}\int_{Y_m} k(y)(\alpha\nabla p_f + \nabla_y\xi_2)dy\Big) = q_s,
\end{aligned}\\[2ex]
-\operatorname*{div}_y\Big(\frac{k_f(y)}{\mu(f_1)}(\nabla p_f + \nabla_y p_f^1)\Big) = 0 \quad \text{in } Y_f,\\
-\operatorname*{div}_y\Big(\frac{k(y)}{\mu(f_1)}(\alpha\nabla p_f + \nabla_y\xi_2)\Big) = 0 \quad \text{in } Y_m,\\
k_f(y)(\nabla p_f + \nabla_y p_f^1)\cdot\nu_y = 0 \quad \text{on } \Gamma_{fm}, \quad k_f(y)(\nabla p_f + \nabla_y p_f^1)\cdot\nu = 0 \quad \text{on } \Gamma,\\
k(y)(\alpha\nabla p_f + \nabla_y\xi_2)\cdot\nu_y = 0 \quad \text{on } \Gamma_{fm}, \quad k(y)(\alpha\nabla p_f + \nabla_y\xi_2)\cdot\nu = 0 \quad \text{on } \Gamma,\\
\phi(y)\partial_t p^0 + \operatorname*{div}_y(\underline{\mathcal{V}}^0) = q_s,\ \underline{\mathcal{V}}^0 = -\frac{k(y)}{\mu}\nabla_y p^0 \quad \text{in } \Omega_0\times Y_m\times J,\\
p_f(x,0) = p^0(x,y,0) = p^o(x) \quad \text{in } \Omega\times Y_m,\\
p_f(x,t) = p^0(x,y,t) \text{ if } y\in\Gamma_{fm}, \quad \alpha p_f = \alpha p \text{ in } \Omega\times J.
\end{gathered}
$$

Let us add some justifications of the latter relation. We have already proved in Lemma 2.5 that $p^0(x,y,t) = p_f(x,t)$ if $y \in \Gamma_{fm}$. We thus assert that $(c_1 + c_2)(x,t)p^0(x,y,t) = (c_1+c_2)(x,t)p_f(x,t)$ if $y \in \Gamma_{fm}$. We also recall that $\alpha(c_1+c_2) =$

α a.e. in $\Omega \times J$. Because of Lemma 2.7,

$$\alpha C^\epsilon \theta^\epsilon \stackrel{2}{\rightharpoonup} \alpha(c_1 + c_2)p^0 = \alpha p^0,$$

and because of Lemma 2.9,

$$\alpha C^\epsilon \theta^\epsilon \stackrel{2}{\rightharpoonup} \alpha(c_1 + c_2)p = \alpha p.$$

It follows that $\alpha(x)p^0(x, y, t) = \alpha(x)p(x, t) = \alpha(x)p_f(x, t)$ if $y \in \Gamma_{fm}$, that is $p_f = p$ a.e. in $(\Omega \setminus \Omega_0) \times J$.

Now we eliminate the function p_f^1 in the former system. We use the solution $(v^i)_{1\le i\le 3}$ of the cell problem (3.2) below.

$$\begin{gathered} -\operatorname{div}_y\big((\chi_f(y)k_f(y) + \chi_m(y)k(y))(\nabla_y v^i(y) + e^i)\big) = 0 \quad \text{in } Y, \\ \int_Y v^i(y)\,dy = 0, \quad y \mapsto v^i(y)\ Y\text{-periodic}, \end{gathered} \tag{3.2}$$

where e^j is the unit vector in the j-th direction. We define the homogenized permeability tensor $\overline{K}_\alpha^H$ by

$$\overline{K}_{\alpha\, ij}^H = \int_Y \big(\chi_f(y)k_f(y) + \chi_m(y)\alpha k(y)\big)(\nabla_y v^i(y) + e^i) \cdot (\nabla_y v^j(y) + e^j)dy, \tag{3.3}$$

$1 \le i, j \le 3$. Through the relations $p_f^1(x, y, t) = \chi_f(y) \sum_{i=1}^3 \partial_{x_i} p_f(x, t)\, v^i(y)$ and $\xi_2(x, y, t) = \chi_m(y)\alpha(x) \sum_{i=1}^3 \partial_{x_i} p_f(x, t)\, v^i(y)$ we recover the following homogenized system.

Proposition 3.1. *The homogenized pressure problem is*

$$\big(\overline{\phi_f}^{Y_f} + \chi_{\Omega\setminus\Omega_0}\overline{\phi}^{Y_m}\big)\,\partial_t p_f - \operatorname{div}\Big(\frac{\overline{K}_\alpha^H}{\mu(f_1)}\nabla p_f\Big) = q_s - \chi_{\Omega_0}\int_{Y_m} \phi\, \partial_t p^0\, dy \quad \text{in } \Omega \times J, \tag{3.4}$$

$$\phi(y)\partial_t p^0 + \operatorname*{div}_y(\underline{\mathcal{V}}^0) = q_s,\ \underline{\mathcal{V}}^0 = -\frac{k(y)}{\mu} \quad \text{in } \Omega_0 \times Y_m \times J, \tag{3.5}$$

$$p_f(x, t) = p^0(x, y, t) \text{ if } y \in \Gamma_{fm},\ (x, t) \in \Omega \times J, \tag{3.6}$$

$$\overline{K}_\alpha^H \nabla p_f \cdot \nu = 0 \text{ on } \partial\Omega \times J,\ p_f(x, 0) = p^0(x, y, 0) = p^o(x) \text{ in } \Omega \times Y_m, \tag{3.7}$$

where the homogenized porosity is defined by

$$\overline{\phi_f}^{Y_f} = \int_{Y_f} \phi_f(y)dy,\ \overline{\phi}^{Y_m} = \int_{Y_m} \phi(y)dy$$

and the homogenized permeability tensor $\overline{K}_\alpha^H$ *is defined in* (3.3).

In Ω_0, the matrix plays the role of a source and produces the additional right-hand side source-like term which is characteristic of a double porosity model (see [5]). In $\Omega\setminus\Omega_0$, the model is of single porosity type. But the interconnection function α influences the homogenized permeability $\overline{K}_\alpha^H$.

We now have to state some strong convergence for the Darcy velocities in order to pass to the limit in the non linear terms of the concentrations equations. We claim and prove the following result.

Lemma 3.2. *We have the following strong two-scale convergences.*

$$\chi_f^\epsilon \frac{k_f^\epsilon}{\mu(f_1^\epsilon)}\nabla p_f^\epsilon \xrightarrow{2} \chi_f(y)\frac{k_f(y)}{\mu(f_1)}(\nabla p_f+\nabla_y p_f^1),$$
$$\chi_{\Omega\setminus\Omega_0}\chi_m^\epsilon \alpha(c_1^\epsilon+c_2^\epsilon)\frac{k^\epsilon}{\mu(m_1^\epsilon)}\nabla p^\epsilon \xrightarrow{2} \chi_{\Omega\setminus\Omega_0}\chi_m(y)\frac{k(y)}{\mu(f_1)}(\alpha\nabla p_f+\nabla_y\xi_2),$$
$$\chi_{\Omega_0}\chi_m^\epsilon \frac{\epsilon k^\epsilon}{\mu(m_1^\epsilon)}\nabla p^\epsilon \xrightarrow{2} \chi_{\Omega_0}\chi_m(y)\frac{k(y)}{\mu}\nabla_y p^0.$$

Proof. We first prove the following convergence result.

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_{\Omega\times J}\frac{K_\alpha^\epsilon}{\mu(\xi^\epsilon)}\nabla\theta^\epsilon\cdot\nabla\theta^\epsilon\,dxdt\\
&=\int_{\Omega\times J}\int_{Y_f}\frac{k_f(y)}{\mu(f_1)}(\nabla p_f+\nabla_y p_f^1)\cdot(\nabla p_f+\nabla_y p_f^1)\,dxdydt\\
&\quad+\int_{(\Omega\setminus\Omega_0)\times J}\int_{Y_m}\frac{k(y)}{\mu(f_1)}(\alpha\nabla p_f+\nabla_y\xi_2)\cdot(\alpha\nabla p_f+\nabla_y\xi_2)\,dxdydt\\
&\quad+\int_{\Omega_0\times J}\int_{Y_m}\frac{k(y)}{\mu}\nabla_y p^0\cdot\nabla_y p^0\,dxdydt,
\end{aligned}\tag{3.8}$$

where $\xi^\epsilon=\chi_f^\epsilon f_1^\epsilon+\chi_m^\epsilon m_1^\epsilon$ and $K_\alpha^\epsilon=\chi_f^\epsilon k_f^\epsilon+\chi_m^\epsilon\big(\alpha(c_1^\epsilon+c_2^\epsilon)(1-\epsilon^2)+\epsilon^2\big)k^\epsilon$. Setting $\Omega_t=\Omega\times(0,t)$, we consider the following energy equation for $t\in J$.

$$\frac12\int_\Omega\Phi^\epsilon(\theta^\epsilon(x,t)^2-p^o(x)^2)dx+\int_{\Omega_t}\frac{K_\alpha^\epsilon}{\mu(\xi^\epsilon)}\nabla\theta^\epsilon\cdot\nabla\theta^\epsilon\,dxds=\int_{\Omega_t}q_s\,\theta^\epsilon\,dxds.$$

In view of the two-scale convergence of θ^ϵ we have

$$\begin{aligned}
&\lim_{\epsilon\to0}\int_{\Omega_t}q_s\,\theta^\epsilon\,dx\,ds\\
&=\int_{\Omega_t}\int_Y q_s\big(\chi_f(y)p_f(x,t)+\chi_m(y)(\chi_{\Omega_0}p^0(x,y,t)+\chi_{\Omega\setminus\Omega_0}p_f(x,t)\big)\,dxdyds\\
&=\int_{\Omega_t}q_s\big(|Y_f|p_f+\chi_{\Omega\setminus\Omega_0}|Y_m|p_f+\chi_{\Omega_0}\Big(\int_{Y_m}p^0dy\Big)\big)\,dxds\\
&=\int_{\Omega_t}q_s\big(p_f+\chi_{\Omega_0}\Big(\int_{Y_m}(p^0-p_f)dy\Big)\big)\,dxds.
\end{aligned}$$

Then we write the variational formulation (3.1) with $\Psi=p_f$, $\psi=\chi_{\Omega_0}(p^0-p_f)$, $\Psi_{1,f}=p_f^1$ and $\Psi_{1,m}=\xi_2$. Bearing in mind that $p_f=p^0$ a.e. in $(\Omega\setminus\Omega_0)\times J$, we assert that

$$\begin{aligned}
&\lim_{\epsilon\to0}\Big(\frac12\int_\Omega\Phi^\epsilon\theta^\epsilon(x,t)^2\,dx+\int_{\Omega_t}\frac{K_\alpha^\epsilon}{\mu(\xi^\epsilon)}\nabla\theta^\epsilon\cdot\nabla\theta^\epsilon\,dxds\Big)\\
&=\frac12\Big(\int_\Omega\int_{Y_f}\phi_f p_f^2(x,t)\,dxdy+\int_\Omega\int_{Y_m}\phi\big(\chi_{\Omega\setminus\Omega_0}p_f^2(x,t)+\chi_{\Omega_0}p^0(x,y,t)^2\big)\,dxdy\Big)\\
&\quad+\int_{\Omega_t}\int_{Y_f}\frac{k_f(y)}{\mu(f_1)}(\nabla p_f+\nabla_y p_f^1)\cdot(\nabla p_f+\nabla_y f_f^1)\,dxdyds\\
&\quad+\int_{(\Omega\setminus\Omega_0)\times(0,t)}\int_{Y_m}\frac{k(y)}{\mu(f_1)}(\alpha\nabla p_f+\nabla_y\xi_2)\cdot(\nabla p_f+\nabla_y\xi_2)\,dxdyds
\end{aligned}$$

$$+\int_{\Omega_0\times(0,t)}\int_{Y_m}\frac{k(y)}{\mu}\ \nabla_y p^0\cdot\nabla_y p^0\,dxdyds.$$

The limit of each term in the left-hand side of the last relation is larger than the corresponding two-scale limit in the right-hand side. Thus equality holds for each contribution and (3.8) is proved. Now we recall the two-scale convergences

$$\begin{gathered}
\chi_f^\epsilon(K_\alpha^\epsilon/\mu(\xi^\epsilon))\nabla\theta^\epsilon \xrightarrow{2} \chi_f(y)(k_f(y)/\mu(f_1))(\nabla p_f+\nabla_y p_f^1),\\
\chi_{\Omega\setminus\Omega_0}\chi_m^\epsilon(K_\alpha^\epsilon/\mu(\xi^\epsilon))\nabla\theta^\epsilon \xrightarrow{2} \chi_{\Omega\setminus\Omega_0}\chi_m(y)(k(y)/\mu(f_1))(\alpha\nabla p_f+\nabla_y\xi_2)\\
\chi_{\Omega_0}\chi_m^\epsilon(K_\alpha^\epsilon/\mu(\xi^\epsilon))\nabla\theta^\epsilon \xrightarrow{2} \chi_{\Omega_0}\chi_m(y)(k(y)/\mu)\nabla_y p^0.
\end{gathered}$$

It thus follows from (3.8) that

$$\begin{gathered}
\lim_{\epsilon\to0}\int_{\Omega\times J}\chi_f^\epsilon\frac{K_\alpha^\epsilon}{\mu(\xi^\epsilon)}\nabla\theta^\epsilon\cdot\nabla\theta^\epsilon\,dxdt\\
=\int_{\Omega\times J}\int_{Y_f}\frac{k_f(y)}{\mu(f_1)}(\nabla p_f+\nabla_y p_f^1)\cdot(\nabla p_f+\nabla_y p_f^1)\,dxdydt,\\
\lim_{\epsilon\to0}\int_{\Omega\times J}\chi_{\Omega\setminus\Omega_0}\chi_m^\epsilon\frac{K_\alpha^\epsilon}{\mu(\xi^\epsilon)}\nabla\theta^\epsilon\cdot\nabla\theta^\epsilon\,dxdt\\
=\int_{(\Omega\setminus\Omega_0)\times J}\int_{Y_m}\frac{k(y)}{\mu(f_1)}(\alpha^{1/2}\nabla p_f+\nabla_y\xi_2)\cdot(\alpha^{1/2}\nabla p_f+\nabla_y\xi_2)\,dxdydt,\\
\lim_{\epsilon\to0}\int_{\Omega\times J}\chi_{\Omega_0}\chi_m^\epsilon\frac{K_\alpha^\epsilon}{\mu(\xi^\epsilon)}\nabla\theta^\epsilon\cdot\nabla\theta^\epsilon\,dxdt=\int_{\Omega_0\times J}\int_{Y_m}\frac{k(y)}{\mu}\nabla_y p^0\cdot\nabla_y p^0\,dxdydt.
\end{gathered}$$

Bearing in mind that K^ϵ is a symmetric definite positive tensor and that it is considered as an admissible test function for the two-scale convergence, the latter relations are sufficient to assert the result of the lemma. $\square$

Let us now turn to the concentration problem. Let $(\psi_f,\psi_1,\Psi_1)\in(\mathcal{D}(\Omega\times J))^3$, $(\psi_f^1,\psi_1^1,\Psi_1^1)\in(\mathcal{D}(\Omega\times J;\mathcal{C}_{per}^\infty(Y)))^3$, $(\psi,\Psi)\in(\mathcal{D}(\Omega\times J;\mathcal{C}_{per}^\infty(Y)))^2$ with supports in $\Omega_0\times Y_m\times J$, such that

$$\begin{gathered}
\psi_f(x,t)=\alpha(x)\psi_1(x,t)+\beta(x)\Psi_1(x,t)\quad\text{in }\Omega\times J,\\
\epsilon\psi_f^1(x,y,t)=\alpha(x)\big(\epsilon\psi_1^1(x,y,t)+\psi(x,y,t)\big)+\beta(x)\big(\epsilon\Psi_1^1(x,y,t)+\Psi(x,y,t)\big)\\
\text{if }(x,t)\in\Omega\times J,\ y\in\Gamma_{fm}.
\end{gathered}$$

We write the variational formulation (2.2) with $d_f=\psi_f+\epsilon\psi_f^{1,\epsilon}$, $d_1=\psi_1+\epsilon\psi_1^{1,\epsilon}+\psi^\epsilon$ and $D_1=\Psi_1+\epsilon\Psi_1^{1,\epsilon}+\Psi^\epsilon$. We obtain

$$\begin{aligned}
&\int_{\Omega_f^\epsilon\times J}\phi_f^\epsilon\partial_t f_1^\epsilon\,(\psi_f+\epsilon\psi_f^{1,\epsilon})+\int_{\Omega_m^\epsilon\times J}\phi^\epsilon\partial_t c_1^\epsilon\,(\psi_1+\epsilon\psi_1^{1,\epsilon}+\psi^\epsilon)\\
&+\int_{\Omega_m^\epsilon\times J}\phi^\epsilon\partial_t C_1^\epsilon\,(\Psi_1+\epsilon\Psi_1^{1,\epsilon}+\Psi^\epsilon)+\int_{\Omega_f^\epsilon\times J}\Big(-\frac{k_f^\epsilon}{\mu(f_1^\epsilon)}\nabla p_f^\epsilon\Big)\cdot\nabla f_1^\epsilon\,(\psi_f+\epsilon\psi_f^{1,\epsilon})\\
&+\int_{\Omega_m^\epsilon\times J}\Big(-\frac{k^\epsilon}{\mu(m_1^\epsilon)}\big(\alpha(c_1^\epsilon+c_2^\epsilon)(1-\epsilon^2)+\epsilon^2\big)\nabla p^\epsilon\Big)\cdot\nabla c_1^\epsilon\,(\psi_1+\epsilon\psi_1^{1,\epsilon}+\psi^\epsilon)\\
&+\int_{\Omega_m^\epsilon\times J}\Big(-\frac{k^\epsilon}{\mu(m_1^\epsilon)}\big(\alpha(c_1^\epsilon+c_2^\epsilon)(1-\epsilon^2)+\epsilon^2\big)\nabla p^\epsilon\Big)\cdot\nabla C_1^\epsilon\,(\Psi_1+\epsilon\Psi_1^{1,\epsilon}+\Psi^\epsilon)
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\Omega_f^\epsilon\times J}\mathcal{D}\Big(\frac{k_f^\epsilon}{\mu(f_1^\epsilon)}\nabla p_f^\epsilon\Big)\nabla f_1^\epsilon\cdot\big(\nabla\psi_f+\epsilon\nabla_x\psi_f^{1,\epsilon}+\nabla_y\psi_f^{1,\epsilon}\big)\\
&+\int_{\Omega_m^\epsilon\times J}\mathcal{D}^\epsilon\Big(\frac{k^\epsilon}{\mu(m_1^\epsilon)}\big(\alpha(c_1^\epsilon+c_2^\epsilon)(1-\epsilon^2)+\epsilon^2\big)\nabla p^\epsilon\Big)\nabla c_1^\epsilon\cdot\big(\nabla\psi_1+\epsilon\nabla_x\psi_1^{1,\epsilon}\\
&+\nabla_y\psi_1^{1,\epsilon}+\nabla_x\psi^\epsilon+\frac{1}{\epsilon}\nabla_y\psi^\epsilon\big)+\int_{\Omega_m^\epsilon\times J}\mathcal{D}^\epsilon\Big(\frac{k^\epsilon}{\mu(m_1^\epsilon)}\big(\alpha(c_1^\epsilon+c_2^\epsilon)(1-\epsilon^2)+\epsilon^2\big)\\
&\nabla p^\epsilon\Big)\nabla C_1^\epsilon\cdot\big(\nabla\Psi_1+\epsilon\nabla_x\Psi_1^{1,\epsilon}+\nabla_y\Psi_1^{1,\epsilon}+\nabla_x\Psi^\epsilon+\frac{1}{\epsilon}\nabla_y\Psi^\epsilon\big)\\
&=\int_{\Omega_f^\epsilon\times J}q_s(\hat f_1-f_1^\epsilon)\,(\psi_f+\epsilon\psi_f^{1,\epsilon})+\int_{\Omega_m^\epsilon\times J}q_s(\hat c_1-c_1^\epsilon)\,(\psi_1+\epsilon\psi_1^{1,\epsilon}+\psi^\epsilon)\\
&\quad+\int_{\Omega_m^\epsilon\times J}q_s(\hat C_1-C_1^\epsilon)\,(\Psi_1+\epsilon\Psi_1^{1,\epsilon}+\Psi^\epsilon).
\end{aligned}
$$

Letting ϵ to 0, we get

$$
\begin{aligned}
&\int_{\Omega\times J}\Big(\int_{Y_f}\phi_f(y)dy\Big)\,\partial_t f_1\,\psi_f+\int_{\Omega\times J}\chi_{\Omega\setminus\Omega_0}\Big(\int_{Y_m}\phi(y)dy\Big)\,(\partial_t c_1\,\psi_1+\partial_t C_1\,\Psi_1)\\
&+\int_{\Omega\times J}\chi_{\Omega_0}\int_{Y_m}\phi(y)\big(\partial_t c_1(\psi_1+\psi)+\partial_t C_1(\Psi_1+\Psi)\big)\\
&-\int_{\Omega\times J}\int_{Y_f}\frac{k_f}{\mu(f_1)}(\nabla p_f+\nabla_y p_f^1)\cdot(\nabla f_1+\nabla_y f_1^1)\,\psi_f\\
&-\int_{\Omega\times J}\chi_{\Omega\setminus\Omega_0}\int_{Y_m}\frac{k}{\mu(f_1)}(\alpha\nabla p+\nabla_y\xi_2)\cdot\big((\nabla c_1+\nabla_y c_1^1)\psi_1+(\nabla C_1+\nabla_y C_1^1)\Psi_1\big)\\
&-\int_{\Omega\times J}\chi_{\Omega_0}\int_{Y_m}\frac{k}{\mu}\nabla_y p^0\cdot\big(\nabla_y c_1^0\,(\psi_1+\psi)+\nabla_y C_1^0\,(\Psi_1+\Psi)\big)\\
&+\int_{\Omega\times J}\int_{Y_f}\mathcal{D}\Big(\frac{k_f}{\mu(f_1)}(\nabla p_f+\nabla_y p_f^1)\Big)(\nabla f_1+\nabla_y f_1^1)\cdot\big(\nabla\psi_f+\nabla_y\psi_f^1\big)\\
&+\int_{\Omega\times J}\chi_{\Omega\setminus\Omega_0}\int_{Y_m}\mathcal{D}\Big(\frac{k}{\mu(f_1)}\big(\alpha\nabla p_f+\nabla_y\xi_2\big)\Big)(\nabla c_1+\nabla_y c_1^1)\cdot\big(\nabla\psi_1+\nabla_y\psi_1^1\big)\\
&+\int_{\Omega\times J}\chi_{\Omega\setminus\Omega_0}\int_{Y_m}\mathcal{D}\Big(\frac{k}{\mu(f_1)}\big(\alpha\nabla p_f+\nabla_y\xi_2\big)\Big)(\nabla C_1+\nabla_y C_1^1)\cdot\big(\nabla\Psi_1+\nabla_y\Psi_1^1\big)\\
&+\int_{\Omega\times J}\chi_{\Omega_0}\int_{Y_m}\mathcal{D}\Big(\frac{k}{\mu}\nabla_y p^0\Big)\big(\nabla_y c_1^0\cdot\nabla_y\psi+\nabla_y C_1^0\cdot\nabla_y\Psi\big)\\
&=\int_{\Omega\times J}|Y_f|\,q_s(\hat f_1-f_1)\,\psi_f\\
&\quad+\int_{\Omega\times J}\int_{Y_m}q_s\big((\hat c_1-c_1^0)\,(\psi_1+\psi)+(\hat C_1-C_1^0)\,(\Psi_1+\Psi)\big).
\end{aligned}
$$

Choosing $\psi_f\neq 0$, $\Psi_1=\psi_f/\beta$ and the other test functions equal to zero, we obtain for instance the equation satisfies by f_1 in $\Omega\times J$. We finally obtain the following homogenized problem in $\Omega\times J$.

$$
\overline{\phi_f}^{Y_f}\partial_t f_1+\chi_{\Omega\setminus\Omega_0}\frac{1}{\beta}\overline{\phi}^{Y_m}\partial_t C_1+\chi_{\Omega_0}\frac{1}{\beta}\int_{Y_m}\phi(y)\,\partial_t C_1^0 dy
$$

$$
\begin{aligned}
&-\frac{K^H_{Y_f}}{\mu(f_1)}\nabla p_f\cdot\nabla f_1-\chi_{\Omega\backslash\Omega_0}\frac{K^H_{Y_m}}{\beta\mu(f_1)}\nabla p_f\cdot\nabla C_1-\frac{\chi_{\Omega_0}}{\beta}\Big(\int_{Y_m}\frac{k(y)}{\mu}\nabla_y p^0\cdot\nabla_y C_1^0\,dy\Big)\\
&-\operatorname{div}(\mathcal{D}^H_f(\nabla p_f)\nabla f_1)-\chi_{\Omega\backslash\Omega_0}\frac{1}{\beta}\operatorname{div}(\mathcal{D}^H_m(\nabla p_f)\nabla C_1)=q_s|Y_f|\,(\hat f_1-f_1)\\
&\quad+\frac{1}{\beta}q_s\Big(\hat C_1-\chi_{\Omega\backslash\Omega_0}|Y_m|\,C_1-\chi_{\Omega_0}\int_{Y_m}C_1^0(\cdot,y,\cdot)\,dy\Big),
\end{aligned}\tag{3.9}
$$

$$
\begin{aligned}
&\overline{\phi}^{Y_m}\chi_{\Omega\backslash\Omega_0}\partial_t c_1+\chi_{\Omega_0}\int_{Y_m}\phi(y)\,\partial_t c_1^0dy-\frac{\alpha}{\beta}\overline{\phi}^{Y_m}\chi_{\Omega\backslash\Omega_0}\partial_t C_1\\
&-\frac{\alpha}{\beta}\chi_{\Omega_0}\int_{Y_m}\phi(y)\,\partial_t C_1^0dy-\chi_{\Omega\backslash\Omega_0}\frac{K^H_{Y_m}}{\mu(f_1)}\nabla p_f\cdot\big(\nabla c_1-\frac{\alpha}{\beta}\nabla C_1\big)\\
&-\chi_{\Omega_0}\int_{Y_m}\frac{k(y)}{\mu}\nabla_y p^0\cdot\big(\nabla_y c_1^0-\frac{\alpha}{\beta}\nabla_y C_1^0\big)\,dy\\
&-\chi_{\Omega\backslash\Omega_0}\operatorname{div}(\mathcal{D}^H_m(\nabla p_f)\nabla c_1)+\chi_{\Omega\backslash\Omega_0}\frac{\alpha}{\beta}\operatorname{div}(\mathcal{D}^H_m(\nabla p_f)\nabla C_1)\\
&=q_s\Big(\hat c_1-\chi_{\Omega\backslash\Omega_0}|Y_m|\,c_1-\chi_{\Omega_0}\int_{Y_m}c_1^0(\cdot,y,\cdot)\,dy\Big)\\
&\quad-\frac{\alpha}{\beta}\,q_s\Big(\hat C_1-\chi_{\Omega\backslash\Omega_0}|Y_m|\,C_1-\chi_{\Omega_0}\int_{Y_m}C_1^0(\cdot,y,\cdot)\,dy\Big),
\end{aligned}\tag{3.10}
$$

with the boundary conditions

$$
\big(\mathcal{D}^H_f(\nabla p_f)\nabla f_1-\frac{1}{\beta}\mathcal{D}^H_m(\nabla p_f)\nabla C_1\big)\cdot\nu\big|_{\Gamma\times J}=0,\tag{3.11}
$$

$$
\big(\mathcal{D}^H_m(\nabla p_f)\nabla c_1-\frac{\alpha}{\beta}\mathcal{D}^H_m(\nabla p_f)\nabla C_1\big)\cdot\nu\big|_{\Gamma\times J}=0.\tag{3.12}
$$

Functions C_1^0 and c_1^0 satisfy the following problem in $\Omega_0\times Y_m\times J$.

$$
\phi(y)\partial_t C_1^0-\frac{k(y)}{\mu}\nabla_y p^0\cdot\nabla_y C_1^0-\operatorname*{div}_y\big(\mathcal{D}(\frac{k(y)}{\mu}\nabla_y p^0)\nabla_y C_1^0\big)=q_s\,(\hat C_1-C_1^0),\tag{3.13}
$$

$$
\phi(y)\partial_t c_1^0-\frac{k(y)}{\mu}\nabla_y p^0\cdot\nabla_y c_1^0-\operatorname*{div}_y\big(\mathcal{D}(\frac{k(y)}{\mu}\nabla_y p^0)\nabla_y c_1^0\big)=q_s\,(\hat c_1-c_1^0),\tag{3.14}
$$

The homogenized tensors are defined as follows.

$$
K^H_{Y_f\,ij}=\int_{Y_f}k_f(y)(\nabla_y v^i(y)+e^i)\cdot(\nabla_y v^j(y)+e^j)dy,\tag{3.15}
$$

$$
K^H_{Y_m\,ij}=\int_{Y_m}\alpha k(y)(\nabla_y v^i(y)+e^i)\cdot(\nabla_y v^j(y)+e^j)dy,\tag{3.16}
$$

$$
\mathcal{D}^H_f(\nabla p_f)_{ij}=\int_{Y_f}\mathcal{D}(\underline{v}_o)\,\big(\nabla_y w^i(x,y)+e^i\big)\cdot\big(\nabla_y w^j(x,y)+e^j\big)dy,\tag{3.17}
$$

$$
\mathcal{D}^H_m(\nabla p_f)_{ij}=\int_{Y_m}\mathcal{D}(\underline{v}_o)\,\big(\nabla_y w^i(x,y)+e^i\big)\cdot\big(\nabla_y w^j(x,y)+e^j\big)dy,\tag{3.18}
$$

where $\underline{v}_o$ is defined by

$$
\underline{v}_o(x,y,t)=-\chi_f(y)\frac{k_f(y)}{\mu(f_1)}(\nabla p_f+\nabla_y p_f^1)-\chi_m(y)\frac{k(y)}{\mu(f_1)}(\alpha\nabla p_f+\nabla\xi_2),\tag{3.19}
$$

and $w^j(x,t,y)$ is the Y-periodic solution of the following cell-problem for $(x,t) \in \Omega \times J$:

$$\begin{gathered} -\operatorname*{div}_y\big(\mathcal{D}(\underline{v}_o)(\nabla_y w^j + e^j)\big) = 0 \quad \text{in } Y, \\ \int_Y w^j(x,t,y)\,dy = 0, \quad j = 1,2,3. \end{gathered} \tag{3.20}$$

It remains to add some initial conditions.

$$f_1\big|_{t=0} = f_1^o, \quad c_1\big|_{t=0} = c_1^0\big|_{t=0} = c_1^o \quad C_1\big|_{t=0} = C_1^0\big|_{t=0} = C_1^o \tag{3.21}$$

and to recall that

$$f_1 = \alpha c_1 + \beta C_1 \text{ a.e. in } \Omega \times J, \quad f_1 = \beta C_1^0 \text{ a.e. in } \Omega_0 \times \Gamma_{fm} \times J.$$

Note that (3.10) characterizes function $c_1 - \frac{\alpha}{\beta} C_1$. Noting that $C_1 = \frac{\beta}{\alpha^2+\beta^2}\big(f_1 - \alpha(c_1 - \frac{\alpha}{\beta}C_1)\big)$, Equation (3.9) then gives f_1.

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