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NONEXISTENCE RESULTS FOR SEMILINEAR SYSTEMS IN UNBOUNDED DOMAINS

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Abstract. This paper concerns the non-existence of nontrivial solutions for the semi-linear system of gradient type

$$\lambda\frac{\partial^2 u_k}{\partial t^2}-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}(p_i(x)\frac{\partial u_k}{\partial x_i})+f_k(x,u_1,\dots,u_m)=0 \quad \text{in } \Omega,\ k=1,\dots,m$$

with Dirichlet, Neumann or Robin boundary conditions. The functions $f_k : \mathcal{D}\times\mathbb{R}^m\to\mathbb{R}$ $(k=1,\dots,m)$ are locally Lipschitz continuous and satisfy

$$2H(x,u_1,\dots,u_m)-\sum_{k=1}^{m}u_kf_k(x,u_1,\dots,u_m)\geq 0 \quad (\text{resp. } \leq 0)$$

for $\lambda>0$ (resp. $\lambda<0$). We establish the non-existence of nontrivial solutions using Pohozaev-type identities. Here $u_1,\dots,u_m$ are in $H^2(\Omega)\cap L^\infty(\Omega)$, $\Omega=\mathbb{R}\times\mathcal{D}$ with $\mathcal{D}=\prod_{i=1}^n(\alpha_i,\beta_i)$ and $H\in\mathcal{C}^1(\overline{\mathcal{D}}\times\mathbb{R}^m)$ such that $\frac{\partial H}{\partial u_k}=f_k$, $k=1,\dots,m$.

1. Introduction

In this paper we study the semi-linear system

$$\begin{gathered}
\lambda\frac{\partial^2 u_1}{\partial t^2}-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}(p_i(x)\frac{\partial u_1}{\partial x_i})+f_1(x,u_1,\dots,u_m)=0 \quad \text{in } \Omega,\\
\lambda\frac{\partial^2 u_2}{\partial t^2}-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}(p_i(x)\frac{\partial u_2}{\partial x_i})+f_2(x,u_1,\dots,u_m)=0 \quad \text{in } \Omega,\\
\dots\\
\lambda\frac{\partial^2 u_m}{\partial t^2}-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}(p_i(x)\frac{\partial u_m}{\partial x_i})+f_m(x,u_1,\dots,u_m)=0 \quad \text{in } \Omega,
\end{gathered} \tag{1.1}$$

under Dirichlet, Neumann or Robin boundary conditions. Here $\Omega=\mathbb{R}\times\mathcal{D}$ where $\mathcal{D}=\prod_{i=1}^n(\alpha_i,\beta_i)$, λ is a real parameter, $f_k:\mathcal{D}\times\mathbb{R}^m\to\mathbb{R}$ $(k=1,\dots,m)$ are locally Lipschitz continuous functions such that

$$f_k(x,0,\dots,0)=0 \quad \text{in } \mathcal{D},$$

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so that $(u_1,\dots,u_m)=0$ is a solution of (1.1) and $p_i:\overline{\mathcal{D}}\to\mathbb{R}$ $(i=1,\dots,n)$ are continuous functions satisfying

$$p_i(x)>0 \text{ or } p_i(x)<0 \quad \text{in } \mathcal{D}.$$

We assume that system (1.1) is of the gradient type; that is, there is a real-valued differentiable function $H(x,u_1,\dots,u_m)$ such that

$$\frac{\partial H}{\partial u_k}=f_k, \quad H(x,0,\dots,0)=0 \quad \text{for } x\in\mathcal{D}.$$

For $k=1,\dots,m$, u_k are in $H^2(\Omega)\cap L^\infty(\Omega)$ and satisfy

$$u_k(t,s)=0, \quad (t,s)\in\mathbb{R}\times\partial\mathcal{D} \tag{1.2}$$

(Dirichlet boundary condition), or

$$\frac{\partial u_k(t,s)}{\partial n}=0, \quad (t,s)\in\mathbb{R}\times\partial\mathcal{D} \tag{1.3}$$

(Neumann boundary condition), or

$$(u_k+\varepsilon\frac{\partial u_k}{\partial n})(t,s)=0, \quad (t,s)\in\mathbb{R}\times\partial\mathcal{D} \tag{1.4}$$

(Robin boundary condition), where ε is a positive real number. Throughout this paper we denote the boundary of Ω by

$$\partial\Omega=\mathbb{R}\times\partial\mathcal{D}=\Gamma_{\alpha_1}\cup\Gamma_{\beta_1}\cup\Gamma_{\alpha_2}\cup\Gamma_{\beta_2}\dots\cup\Gamma_{\alpha_n}\cup\Gamma_{\beta_n},$$

where

$$\Gamma_{\mu_s}=\{(t,x_1,\dots,x_{s-1},\mu_s,x_{s+1},\dots x_n),\ t\in\mathbb{R}, \quad 1\le s\le n\},$$

$(t,x)=(t,x_1,\dots,x_n)$, and

$$n(t,s)=(0,n_1(t,s),n_2(t,s),\dots,n_n(t,s))$$

is the outward normal to $\partial\Omega$ at the point (t,s). If $x\in\prod_{i=1}^n(\alpha_i,\beta_i)$, $l=1,2,\dots,n$ and $\tau\in\{\alpha_1,\beta_1,\alpha_2,\beta_2,\dots,\alpha_n,\beta_n\}$ one writes

$$x_l^\tau=(x_1,\dots,x_{l-1},\tau,x_{l+1},\dots,x_n), \quad dx_l^*=dx_1\dots dx_{l-1}dx_{l+1}\dots dx_n$$

and

$$\begin{aligned}&\int_{\alpha_1}^{\beta_1}\dots\int_{\alpha_{i-1}}^{\beta_{i-1}}\int_{\alpha_{i+1}}^{\beta_{i+1}}\dots\int_{\alpha_n}^{\beta_n} f_k(x,r_1,\dots,r_m)dx_1\dots dx_{i-1}dx_{i+1}\dots dx_n\\ &=\int_{\mathcal{D}_i^*} f_k(x,r_1,\dots,r_m)dx_i^* \quad \text{for all } k=1,\dots,m.\end{aligned}$$

The question of non-existence of nontrivial solutions for elliptic problems has been studied extensively in both bounded and unbounded domain (see [3],[4],[7]-[9] and their references). In particular, Amster et al. in [1] showed the non-solvability of the gradient elliptic system

$$\begin{gathered}-\Delta u_i=g_i(u) \quad \text{in } \Omega,\\ u_i=0 \quad \text{on } \partial\Omega,\ i=1,\dots,n,\end{gathered}$$

where Ω is a starshaped domain. A similar result was given for Hamiltonian systems by N. M. Chuong and T. D. Ke [2] in k-starshaped domain and by Khodja [6] in unbounded domain $\mathbb{R}^+\times\mathbb{R}$.

In the scalar case, when Ω is an unbounded domain, Haraux and Khodja [4] established that under assumptions

$$f(0)=0,$$
$$2F(u)-uf(u)\le 0,\quad u\neq 0$$

($F(u)=\int_0^u f(s)ds$), the problem

$$-\Delta u+f(u)=0\quad \text{in } \Omega,$$
$$(u \text{ or } \frac{\partial u}{\partial n})=0\quad \text{on } \partial\Omega,$$

has only a trivial solution in $H^2(\Omega)\cap L^\infty(\Omega)$, where $\Omega=J\times\omega$, $J\subset\mathbb{R}$ is an unbounded interval and ω a domain in $\mathbb{R}^N$. The case of Robin boundary conditions was treated by Khodja [5] and it was shown nonexistence results for the equation

$$\lambda\frac{\partial^2 u}{\partial t^2}-\frac{\partial}{\partial x}\big(p(x,y)\frac{\partial u}{\partial x}\big)-\frac{\partial}{\partial y}\big(q(x,y)\frac{\partial u}{\partial y}\big)+f(x,y,u)=0\quad \text{in } \Omega,$$

where $\Omega=\mathbb{R}\times]\alpha_1,\beta_1[\times]\alpha_2,\beta_2[$. In the above works, the integral identity of Pohozaev was adapted for each problem treated and applied to obtain the nonexistence results. The present study extends and complements these works. We shall prove the non-solvability results to the class of semi-linear system of gradient type (1.1) under Dirichlet, Neumann or Robin boundary conditions. By using a Pohozaev-type identity, our demonstration strategy will be to show that the function

$$\mathcal{E}(t)=\int_{\mathcal{D}}\big(\sum_{k=1}^m|u_k(t,x)|^2\big)dx$$

is convex in $\mathbb{R}$, and then, from the Maximum Principle, we obtain that any solution $(u_1,\dots,u_m)$ to the problems (1.1)-(1.2), (1.1)-(1.3) and (1.1)-(1.4) is trivial. We draw the attention of the reader to the use of the Pohozaev-type identity which, to the best of our knowledge, was not explored before in connection with gradient systems in an unbounded cylindrical-type domain.

This paper is organized as follows. In the next section, we give a Pohozaev-type identity adapted to the systems with Dirichlet, Neumann and Robin boundary conditions; section 3 gives our main results and some examples will be illustrated in section 4.

2. Integral identities

The proof of our main results which will appear in the next section use the following type of Pohozaev identity, adapted for systems.

Theorem 2.1. *Let $u_1,\dots,u_m$ in $H^2(\Omega)\cap L^\infty(\Omega)$ be a solution of problem* (1.1)–(1.4). *Then for each $t\in\mathbb{R}$ and $\varepsilon>0$, we have*

$$\begin{aligned}&\int_{\mathcal{D}}\Big[\frac{\lambda}{2}\sum_{k=1}^m|\frac{\partial u_k}{\partial t}|^2+\sum_{i=1}^n\frac{p_i(x)}{2}\Big(\sum_{k=1}^m|\frac{\partial u_k}{\partial x_i}|^2\Big)+H(x,u_1,\dots,u_m)\Big]dx\\&+\frac{1}{2\varepsilon}\sum_{i=1}^n\int_{\mathcal{D}_i^*}\Big[p_i(x_i^{\beta_i})\big(\sum_{k=1}^m|u_k|^2\big)(t,x_i^{\beta_i})+p_i(x_i^{\alpha_i})\big(\sum_{k=1}^m|u_k|^2\big)(t,x_i^{\alpha_i})\Big]dx_i^*=0.\end{aligned}\tag{2.1}$$

Proof. For $t \in \mathbb{R}$ we consider a function

$$\mathcal{K}(t) = \int_{\mathcal{D}} \big[\frac{\lambda}{2} \sum_{k=1}^{m} |\frac{\partial u_k}{\partial t}|^2 + \sum_{i=1}^{n} \frac{p_i(x)}{2} (\sum_{k=1}^{m} |\frac{\partial u_k}{\partial x_i}|^2) + H(x, u_1, \dots, u_m)\big] dx.$$

The hypothesis on u_k, f_k $(k = 1, \dots, m)$ and p_i $(i = 1, \dots, n)$ implies that $\mathcal{K}$ is absolutely continuous and thus differentiable almost everywhere on $\mathbb{R}$; we have

$$\begin{aligned} \frac{d\mathcal{K}(t)}{dt} = \int_{\mathcal{D}} \big[\lambda \sum_{k=1}^{m} \frac{\partial u_k}{\partial t} \frac{\partial^2 u_k}{\partial t^2} + \sum_{i=1}^{n} p_i(x) \Big(\sum_{k=1}^{m} \frac{\partial u_k}{\partial x_i} \frac{\partial^2 u_k}{\partial t \partial x_i} \Big) \\ + \sum_{k=1}^{m} \frac{\partial u_k}{\partial t} f_k(x, u_1, \dots, u_m) \big] dx. \end{aligned} \tag{2.2}$$

Fubini's theorem and an integration by part give

$$\begin{aligned} &\int_{\mathcal{D}} \sum_{i=1}^{n} p_i(x) \Big(\sum_{k=1}^{m} \frac{\partial u_k}{\partial x_i} \frac{\partial^2 u_k}{\partial t \partial x_i} \Big)(t, x) dx \\ &= -\int_{\mathcal{D}} \sum_{i=1}^{n} \big[\sum_{k=1}^{m} \frac{\partial}{\partial x_i} (p_i(x) \frac{\partial u_k}{\partial x_i}) \frac{\partial u_k}{\partial t} \big](t, x) dx \\ &\quad + \sum_{i=1}^{n} \int_{\mathcal{D}_i^*} \big[p_i(x_i^{\beta_i}) (\sum_{k=1}^{m} \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial t})(t, x_i^{\beta_i}) - p_i(x_i^{\alpha_i}) \Big(\sum_{k=1}^{m} \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial t} \Big)(t, x_i^{\alpha_i}) \big] dx_i^*. \end{aligned}$$

Replacing in (2.2) we find

$$\begin{aligned} &\frac{d\mathcal{K}(t)}{dt} \\ &= \sum_{k=1}^{m} \int_{\mathcal{D}} \Big[\lambda \frac{\partial^2 u_k}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (p_i(x) \frac{\partial u_k}{\partial x_i}) + f_k(x, u_1, \dots, u_m) \Big](t, x) \frac{\partial u_k}{\partial t} dx \\ &\quad + \sum_{i=1}^{n} \int_{\mathcal{D}_i^*} \Big[p_i(x_i^{\beta_i}) (\sum_{k=1}^{m} \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial t})(t, x_i^{\beta_i}) - p_i(x_i^{\alpha_i}) \Big(\sum_{k=1}^{m} \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial t} \Big)(t, x_i^{\alpha_i}) \Big] dx_i^*. \end{aligned}$$

Let us consider on $\partial\Omega$ the expression $u_k + \varepsilon \frac{\partial u_k}{\partial n} = 0$. For $k = 1, \dots, m$

$$u_k + \varepsilon \frac{\partial u_k}{\partial n} = 0 \Longleftrightarrow \begin{cases} (u_k - \varepsilon \frac{\partial u_k}{\partial x})(t, x_i^{\alpha_i}) = 0, \\ (u_k + \varepsilon \frac{\partial u_k}{\partial x})(t, x_i^{\beta_i}) = 0, \\ t \in \mathbb{R}, \alpha_i < x_i < \beta_i, i = 1, \dots n. \end{cases}$$

Then for $\varepsilon > 0$, one can write

$$\begin{aligned} &\sum_{i=1}^{n} \int_{\mathcal{D}_i^*} \Big[p_i(x_i^{\beta_i}) \Big(\sum_{k=1}^{m} \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial t} \Big)(t, x_i^{\beta_i}) - p_i(x_i^{\alpha_i}) (\sum_{k=1}^{m} \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial t} \Big)(t, x_i^{\alpha_i}) \Big] dx_i^* \\ &= \frac{-1}{\varepsilon} \sum_{i=1}^{n} \int_{\mathcal{D}_i^*} \Big[p_i(x_i^{\beta_i}) (\sum_{k=1}^{m} u_k \frac{\partial u_k}{\partial t})(t, x_i^{\beta_i}) + p_i(x_i^{\alpha_i}) \Big(\sum_{k=1}^{m} u_k \frac{\partial u_k}{\partial t} \Big)(t, x_i^{\alpha_i}) \Big] dx_i^* \\ &= \frac{-1}{2\varepsilon} \frac{d}{dt} (\sum_{i=1}^{n} \int_{\mathcal{D}_i^*} \Big[p_i(x_i^{\beta_i}) (\sum_{k=1}^{m} |u_k|^2)(t, x_i^{\beta_i}) + p_i(x_i^{\alpha_i}) (\sum_{k=1}^{m} |u_k|^2)(t, x_i^{\alpha_i}) \Big] dx_i^*). \end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{d}{dt}\Big(\mathcal{K}(t)+\frac{1}{2\varepsilon}\sum_{i=1}^{n}\int_{\mathcal{D}_i^*}\Big[p_i(x_i^{\beta_i})(\sum_{k=1}^{m}|u_k|^2)(t,x_i^{\beta_i})\\&+p_i(x_i^{\alpha_i})(\sum_{k=1}^{m}|u_k|^2)(t,x_i^{\alpha_i})\Big]dx_i^*\Big)=0.\end{aligned}$$

Integrating with respect to t, we obtain

$$\begin{aligned}&\mathcal{K}(t)+\frac{1}{2\varepsilon}\sum_{i=1}^{n}\int_{\mathcal{D}_i^*}\Big[p_i(x_i^{\beta_i})\big(\sum_{k=1}^{m}|u_k|^2\big)(t,x_i^{\beta_i})\\&+p_i(x_i^{\alpha_i})\big(\sum_{k=1}^{m}|u_k|^2\big)(t,x_i^{\alpha_i})\Big]dx_i^*=\text{const}\end{aligned}$$

and since $(u_1(t,x),\dots,u_m(t,x))\in(H^2(\Omega)\cap L^\infty(\Omega))^m$, one must get

$$\begin{aligned}&\int_{\mathbb{R}}(\mathcal{K}(t)+\frac{1}{2\varepsilon}\sum_{i=1}^{n}\int_{\mathcal{D}_i^*}\Big[p_i(x_i^{\beta_i})(\sum_{k=1}^{m}|u_k|^2)(t,x_i^{\beta_i})\\&+p_i(x_i^{\alpha_i})\big(\sum_{k=1}^{m}|u_k|^2\big)(t,x_i^{\alpha_i})\Big]dx_i^*)dt<\infty.\end{aligned}$$

It follows that the constant must be 0, which is the desired result. □

For the Dirichlet or Neumann boundary conditions, we have the integral identity given in the following theorem.

Theorem 2.2. *Let $u_1,\dots,u_m$ in $H^2(\Omega)\cap L^\infty(\Omega)$ be a solution of problems* (1.1)-(1.2) *or* (1.1)-(1.3)*. Then for each $t\in\mathbb{R}$, we have*

$$\int_{\mathcal{D}}\Big[\frac{\lambda}{2}\sum_{k=1}^{m}|\frac{\partial u_k}{\partial t}|^2+\sum_{i=1}^{n}\frac{p_i(x)}{2}\Big(\sum_{k=1}^{m}|\frac{\partial u_k}{\partial x_i}|^2\Big)+H(x,u_1,\dots,u_m)\Big]dx=0. \tag{2.3}$$

Proof. To prove (2.3) it suffices to check that the expression

$$\sum_{i=1}^{n}\int_{\mathcal{D}_i^*}\Big[p_i(x_i^{\beta_i})\big(\sum_{k=1}^{m}|u_k|^2\big)(t,x_i^{\beta_i})+p_i(x_i^{\alpha_i})\big(\sum_{k=1}^{m}|u_k|^2\big)(t,x_i^{\alpha_i})\Big]dx_i^*$$

vanishes if

$$u_1(t,s)=u_2(t,s)=\dots=u_m(t,s)=0,\ (t,s)\in\mathbb{R}\times\partial\mathcal{D} \tag{2.4}$$

or

$$\frac{\partial u_1(t,s)}{\partial n}=\frac{\partial u_2(t,s)}{\partial n}=\dots=\frac{\partial u_m(t,s)}{\partial n}=0,\ (t,s)\in\mathbb{R}\times\partial\mathcal{D}. \tag{2.5}$$

Indeed, suppose that (2.4) holds then it is known that

$$\nabla u_k=\frac{\partial u_k}{\partial n}\cdot n,\quad k=1,\dots,m;$$

i.e.,

$$\begin{bmatrix}\frac{\partial u_k}{\partial t}(t,s)\\ \frac{\partial u_k}{\partial x_1}(t,s)\\ \dots\\ \frac{\partial u_k}{\partial x_n}(t,s)\end{bmatrix}=\begin{bmatrix}0\\ n_1\frac{\partial u_k}{\partial n}(t,s)\\ \dots\\ n_n\frac{\partial u_k}{\partial n}(t,s)\end{bmatrix},\quad (t,s)\in\mathbb{R}\times\partial\mathcal{D},\quad k=1,\dots,m.$$

Consequently, for $k = 1, \dots, m$,
$$\frac{\partial u_k}{\partial t}(t, x_i^{\alpha_i}) = \frac{\partial u_k}{\partial t}(t, x_i^{\beta_i}) = 0, \quad i = 1, \dots, n.$$
Now if the boundary condition is (2.5), then for $k = 1, \dots, m$, one can write
$$0 = \frac{\partial u_k}{\partial n}(t, s) = \langle \nabla u_k, n \rangle \text{ on } \Gamma_{\alpha_1} \cup \Gamma_{\beta_1} \cup \Gamma_{\alpha_2} \cup \Gamma_{\beta_2} \dots \cup \Gamma_{\alpha_n} \cup \Gamma_{\beta_n};$$
i.e.,
$$\frac{\partial u_k}{\partial x_i}(t, x_i^{\alpha_i}) = \frac{\partial u_k}{\partial x_i}(t, x_i^{\beta_i}) = 0, \quad \text{for all } t \in \mathbb{R}, \quad i = 1, \dots, n, \ k = 1, \dots, m.$$
In both cases $\frac{d\mathcal{K}(t)}{dt} = 0$ for all $t \in \mathbb{R}$ which completes the proof. □

3. Main results

Before giving our main results, we note that the parameter λ plays, in fact, an important part as it allows (1.1) to be dealt with in two manners based on whether its value is positive or negative. Indeed, if λ is positive (resp. negative), the system (1.1) is a hyperbolic (resp. elliptic) problem.

3.1. Semi-linear hyperbolic problems.

Using identity (2.1) we obtain the following first result.

Theorem 3.1. *Let $\lambda > 0$ and $u_1, \dots, u_m \in H^2(\Omega) \cap L^\infty(\Omega)$. Assume $p_i(x) > 0$ in $\mathcal{D}$ $(i = 1, \dots, n)$ and f_k $(k = 1, \dots, m)$ satisfying*
$$H(x, u_1, \dots, u_m) \geq 0.$$
Then problems (1.1)-(1.2), (1.1)*)*-(1.3) *and* (1.1)-(1.4) *have no nontrivial solutions.*

Proof. Applying formula (2.1) (resp. (2.3)) we immediately obtain
$$\frac{\partial u_k}{\partial t}(t, x) = \frac{\partial u_k}{\partial x_i}(t, x) = 0 \quad \text{in } \Omega, \ i = 1, \dots, n, \ k = 1, \dots, m.$$
Thus $u_1, \dots, u_m$ are constant and since for $k = 1, \dots, m$,
$$\int_\Omega |u_k(t, x)|^2 dx dt \leq 0,$$
these constants are necessarily zero. □

The next theorem gives a non-existence result if the functions f_k $(k = 1, \dots, m)$ satisfy another type of non-linearity.

Theorem 3.2. *Let $\lambda > 0$ and $u_1, \dots, u_m : \Omega \to \mathbb{R}$ be a solution of problem* (1.1)-(1.4)*. Suppose that $u_1, \dots, u_m \in H^2(\Omega) \cap L^\infty(\Omega)$ and f_k $(k = 1, \dots, m)$ verify the following condition*
$$2H(x, u_1, \dots, u_m) - \sum_{k=1}^m u_k f_k(x, u_1, \dots, u_m) \geq 0. \tag{3.1}$$
Then problem (1.1)-(1.4) *has no nontrivial solutions.*

Remark 3.3. Since $u_1, \dots, u_m$ are bounded in Ω, from the Maximum Principle, the function $\mathcal{E}(t)$ is convex in $\mathbb{R}$ which implies that the solution to the problem (1.1)-(1.4) is identically equal to zero.

Proof of Theorem 3.2. It is easy to see that almost everywhere in Ω

$$(u_k\frac{\partial^2 u_k}{\partial t^2})(t,x)=\Big(\frac{1}{2}\frac{\partial^2(u_k^2)}{\partial t^2}-|\frac{\partial u_k}{\partial t}|^2\Big)(t,x),\quad k=1,\dots,m.$$

Let us multiply the k-th equation of (1.1) by $u_k/2$ and integrate over $\mathcal{D}$ we obtain

$$\begin{aligned}
&\int_{\mathcal{D}}\Big[\lambda\frac{\partial^2 u_k}{\partial t^2}\frac{u_k}{2}-\sum_{i=1}^n\frac{\partial}{\partial x_i}(p_i(x)\frac{\partial u_k}{\partial x_i})\frac{u_k}{2}+f_k(x,u_1,\dots,u_m)\frac{u_k}{2}\Big](t,x)dx\\
&=\int_{\mathcal{D}}\big[\frac{\lambda}{4}\frac{\partial^2(u_k^2)}{\partial t^2}-\frac{\lambda}{2}|\frac{\partial u_k}{\partial t}|^2\big](t,x)dx\\
&\quad+\int_{\mathcal{D}}\Big[-\sum_{i=1}^n\frac{\partial}{\partial x_i}(p_i(x)\frac{\partial u_k}{\partial x_i})\frac{u_k}{2}+f(x,u_1,\dots,u_m)\frac{u_k}{2}\Big](t,x)dx.
\end{aligned}\tag{3.2}$$

Let us transform

$$\begin{aligned}
&\int_{\mathcal{D}}\Big(-\sum_{i=1}^n\frac{\partial}{\partial x_i}(p_i(x)\frac{\partial u_k}{\partial x_i})\frac{u_k}{2}\Big)(t,x)dx\\
&=\int_{\mathcal{D}}\sum_{i=1}^n\frac{p_i(x)}{2}|\frac{\partial u_k(t,x)}{\partial x_i}|^2dx\\
&\quad-\frac{1}{2}\sum_{i=1}^n\int_{\mathcal{D}_i^*}\Big[p_i(x_i^{\beta_i})(u_k\frac{\partial u_k}{\partial x_i})(t,x_i^{\beta_i})-p_i(x_i^{\alpha_i})(u_k\frac{\partial u_k}{\partial x_i})(t,x_i^{\alpha_i})\Big]dx_i^*.
\end{aligned}$$

The substitution of this formula in (3.2) gives

$$\begin{aligned}
&\int_{\mathcal{D}}\Big[\lambda\frac{\partial^2 u_k}{\partial t^2}\frac{u_k}{2}-\sum_{i=1}^n\frac{\partial}{\partial x_i}(p_i(x)\frac{\partial u_k}{\partial x_i})\frac{u_k}{2}+f(x,u_1,\dots,u_m)\frac{u_k}{2}\Big](t,x)dx\\
&=\int_{\mathcal{D}}\Big(\frac{\lambda}{4}\frac{\partial^2(u_k^2)}{\partial t^2}-\frac{\lambda}{2}|\frac{\partial u_k}{\partial t}|^2\Big)(t,x)dx\\
&\quad+\int_{\mathcal{D}}\sum_{i}^n\frac{p_i(x)}{2}|\frac{\partial u_k(t,x)}{\partial x_i}|^2dx+\int_{\mathcal{D}}(\frac{u_k}{2}f(x,u_1,\dots,u_m))(t,x)dx\\
&\quad-\frac{1}{2}\sum_{i=1}^n\int_{\mathcal{D}_i^*}\Big[p_i(x_i^{\beta_i})(u_k\frac{\partial u_k}{\partial x_i})(t,x_i^{\beta_i})-p_i(x_i^{\alpha_i})(u_k\frac{\partial u_k}{\partial x_i})(t,x_i^{\alpha_i})\Big]dx_i^* \qquad (3.3)\\
&=\int_{\mathcal{D}}(\frac{\lambda}{4}\frac{\partial^2(u_k^2)}{\partial t^2}-\frac{\lambda}{2}|\frac{\partial u_k}{\partial t}|^2)(t,x)dx\\
&\quad+\int_{\mathcal{D}}\sum_{i=1}^n\frac{p_i(x)}{2}|\frac{\partial u_k(t,x)}{\partial x_i}|^2dx+\int_{\mathcal{D}}(\frac{u_k}{2}f(x,u_1,\dots,u_m))(t,x)dx\\
&\quad+\frac{1}{2\varepsilon}\sum_{i=1}^n\int_{\mathcal{D}_i^*}\Big[p_i(x_i^{\beta_i})|u_k(t,x_i^{\beta_i})|^2+p_i(x_i^{\alpha_i})|u_k(t,x_i^{\alpha_i})|^2\Big]\,dx_i^*.
\end{aligned}$$

Adding these identities for $k = 1, \dots, k_0$, we get

$$
\begin{aligned}
&\frac{\lambda}{4}\int_{\mathcal{D}}\Big(\sum_{k=1}^{m}\frac{\partial^2(u_k^2)}{\partial t^2}\Big)(t,x)dx - \frac{\lambda}{2}\int_{\mathcal{D}}\Big(\sum_{k=1}^{m}|\frac{\partial u_k}{\partial t}|^2\Big)(t,x)dx \\
&\quad + \int_{\mathcal{D}}\sum_{i=1}^{n}\frac{p_i(x)}{2}(\sum_{k=1}^{m}|\frac{\partial u_k}{\partial x_i}|^2)(t,x)dx + \frac{1}{2}\int_{\mathcal{D}}(\sum_{k=1}^{m}u_k f_k(x,u_1,\dots,u_m))(t,x)dx \\
&\quad + \frac{1}{2\varepsilon}\sum_{i=1}^{n}\int_{\mathcal{D}_i^*}\Big[p_i(x_i^{\beta_i})\big(\sum_{k=1}^{m}|u_k|^2\big)(t,x_i^{\beta_i}) + p_i(x_i^{\alpha_i})\big(\sum_{k=1}^{m}|u_k|^2\big)(t,x_i^{\alpha_i})\Big]dx_i^* = 0,
\end{aligned}
$$

which combined with (2.1) yields

$$
\begin{aligned}
\frac{\lambda}{4}\frac{d^2}{dt^2}\Big(\int_{\mathcal{D}}(\sum_{k=1}^{m}u_k^2)(t,x)dx\Big) &= \lambda\int_{\mathcal{D}}(\sum_{k=1}^{m}|\frac{\partial u_k}{\partial t}|^2)(t,x)dx + \int_{\mathcal{D}}\Big[H(x,u_1,\dots,u_m) \\
&\quad - \frac{1}{2}(\sum_{k=1}^{m}u_k f_k(x,u_1,\dots,u_m))(t,x,y)\Big]dx.
\end{aligned} \tag{3.4}
$$

The assumptions (3.1) and $\lambda > 0$ enable us to assert that

$$
\frac{\lambda}{4}\frac{d^2}{dt^2}\Big(\int_{\mathcal{D}}(\sum_{k=1}^{m}u_k^2)(t,x)dx\Big) \geq \lambda\int_{\mathcal{D}}\Big(\sum_{k=1}^{m}|\frac{\partial u_k}{\partial t}|^2\Big)(t,x)dx \geq 0,
$$

for all $t \in \mathbb{R}$. This completes the proof. □

Theorem 3.4. *Let $\lambda > 0$ and f_k be as described in Theorem 3.2. Assume that $u_1, \dots, u_m \in H^2(\Omega) \cap L^\infty(\Omega)$ is a solution of* (1.1)*-*(1.2) *or* (1.1)*-*(1.3)*. Then problems* (1.1)*)-*(1.2) *and* (1.1)*-*(1.3) *have no nontrivial solutions.*

Proof. By a similar arguments as in the proof of Theorem 3.2 we obtain

$$
\begin{aligned}
&\frac{\lambda}{4}\int_{\mathcal{D}}\Big(\sum_{k=1}^{m}\frac{\partial^2(u_k^2)}{\partial t^2}\Big)(t,x)dx - \frac{\lambda}{2}\int_{\mathcal{D}}\Big(\sum_{k=1}^{m}|\frac{\partial u_k}{\partial t}|^2\Big)(t,x)dx \\
&+ \int_{\mathcal{D}}\sum_{i=1}^{n}\frac{p_i(x)}{2}\Big(\sum_{k=1}^{m}|\frac{\partial u_k}{\partial x_i}|^2\Big)(t,x)dx \\
&+ \frac{1}{2}\int_{\mathcal{D}}\Big(\sum_{k=1}^{m}u_k f_k(x,u_1,\dots,u_m)\Big)(t,x)dx \\
&+ \frac{1}{2\varepsilon}\sum_{i=1}^{n}\int_{\mathcal{D}_i^*}\Big[p_i(x_i^{\beta_i})\big(\sum_{k=1}^{m}|u_k|^2\big)(t,x_i^{\beta_i}) + p_i(x_i^{\alpha_i})\big(\sum_{k=1}^{m}|u_k|^2\big)(t,x_i^{\alpha_i})\Big]dx_i^* = 0.
\end{aligned}
$$

If

$$
u_1(t,s) = \dots = u_m(t,s) = 0, \quad (t,s) \in \mathbb{R} \times \partial\mathcal{D}
$$

or

$$
\frac{\partial u_1(t,s)}{\partial n} = \dots = \frac{\partial u_m(t,s)}{\partial n} = 0,\ (t,s) \in \mathbb{R} \times \partial\mathcal{D},
$$

this formula reduces to

$$
\frac{\lambda}{4}\int_{\mathcal{D}}\Big(\sum_{k=1}^{m}\frac{\partial^2(u_k^2)}{\partial t^2}\Big)(t,x)dx - \frac{\lambda}{2}\int_{\mathcal{D}}\Big(\sum_{k=1}^{m}|\frac{\partial u_k}{\partial t}|^2\Big)(t,x)dx
$$

$$
\begin{aligned}
&+\int_{\mathcal{D}}\sum_{i=1}^{n}\frac{p_i(x)}{2}\Big(\sum_{k=1}^{m}|\frac{\partial u_k}{\partial x_i}|^2\Big)(t,x)dx\\
&+\frac{1}{2}\int_{\mathcal{D}}\Big(\sum_{k=1}^{m}u_kf_k(x,u_1,\dots,u_m)\Big)(t,x)dx=0.
\end{aligned}
$$

We can now employ (2.3) to transform this identity into the form

$$
\begin{aligned}
&\frac{\lambda}{4}\int_{\mathcal{D}}\Big(\sum_{k=1}^{m}\frac{\partial^2(u_k^2)}{\partial t^2}\Big)(t,x)dx\\
&=\lambda\int_{\mathcal{D}}\Big(\sum_{k=1}^{m}|\frac{\partial u_k}{\partial t}|^2\Big)(t,x)dx \qquad (3.5)\\
&\quad+\int_{\mathcal{D}}\Big[H(x,u_1,\dots,u_m)-\frac{1}{2}\Big(\sum_{k=1}^{m}u_kf_k(x,u_1,\dots,u_m)\Big)(t,x,y)\Big]dx\,.
\end{aligned}
$$

This completes the proof. □

3.2. **Semi-linear elliptic problems.** We shall prove that a dual result holds for $\lambda < 0$.

Theorem 3.5. *Let $(u_1,\dots,u_m)\in(H^2(\Omega)\cap L^\infty(\Omega))^m$ be a solution of* (1.1)*)*-(1.4), *$\lambda<0$ and f_k $(k=1,\dots,m)$ satisfying*

$$
2H(x,u_1,\dots,u_m)-\sum_{k=1}^{m}u_kf_k(x,u_1,\dots,u_m)\le 0. \qquad (3.6)
$$

Then problem (1.1)-(1.4) *has no nontrivial solutions.*

Proof. Formula (3.4) combined with the assumption (3.6) yields

$$
\frac{\lambda}{4}\frac{d^2}{dt^2}\Big(\int_{\mathcal{D}}(\sum_{k=1}^{m}u_k^2)(t,x)dx\Big)\le\lambda\int_{\mathcal{D}}\Big(\sum_{k=1}^{m}|\frac{\partial u_k}{\partial t}|^2\Big)(t,x)dx,\quad\text{for all } t\in\mathbb{R}
$$

and $\lambda<0$ gives the desired result. □

Theorem 3.6. *Let $\lambda<0$ and f_k $(k=1,\dots,m)$ be as described in Theorem 3.5. We assume that*

$$
u_1,\dots,u_m\in H^2(\Omega)\cap L^\infty(\Omega)
$$

is a solution of (1.1)-(1.2) *or 1.1*-(1.3)*. Then problems* (1.1)-(1.2) *and* (1.1)-(1.3) *have no nontrivial solutions.*

This theorem follows from (3.5) and (3.6) with $\lambda<0$.

4. Examples

In this section, we illustrate our theoretical results by giving some examples.

Example 1. Let $\theta : \mathcal{D} \to \mathbb{R}$ be a continuous function, the exponents $\alpha_s > 0$, $s = 1, \dots, m$ and

$$p_i(x) > 0 \quad \text{or} \quad p_i(x) < 0 \text{ in } \mathcal{D}, \quad i = 1, \dots, n.$$

Then system (1.1) with

$$f_k(x, u_1, \dots, u_m) = \theta(x)\Big[\prod_{s=1,\, s\neq k}^{m} \frac{1}{\alpha_s + 1}|u_s|^{\alpha_s+1}\Big]|u_k|^{\alpha_k - 1}u_k, \quad k = 1, \dots, m$$

subject to Dirichlet, Neumann or Robin boundary conditions, does not have nontrivial solutions. Indeed, when $\lambda > 0$ and $p_i, \theta > 0$ in $\mathcal{D}$, $(i = 1, \dots, n)$, we have

$$H(x, u_1, \dots, u_m) = \theta(x)\Big[\prod_{s=1}^{m} \frac{1}{\alpha_s + 1}|u_s|^{\alpha_s+1}\Big]$$

and Theorem 3.1 gives the desired result.

When $\lambda > 0$ (resp. $\lambda < 0$), $\theta(x) \leq 0$ (resp. $\theta(x) \geq 0$) in $\mathcal{D}$ and $p_i(x) > 0$ or $p_i(x) < 0$ in $\mathcal{D}$, $i = 1, \dots, n$, we have

$$\begin{aligned}
&2H(x, u_1, \dots, u_{k_0}) - \sum_{k=1}^{m} u_k f_k(x, u_1, \dots, u_m) \\
&= \theta(x)\frac{2 - \sum_{k=1}^{m}(\alpha_k + 1)}{\prod_{k=1}^{m}(\alpha_k + 1)} \prod_{k=1}^{m} |u_k|^{\alpha_k+1} \leq 0 \quad (\text{resp. } \geq 0)\,.
\end{aligned}$$

We conclude by using Theorem 3.2 or Theorem 3.4 (resp. Theorem 3.5 or Theorem 3.6) as the system is subject to Robin, Neumann or Dirichlet boundary conditions.

Example 2. Let us consider the system (1.1) with $m = 2$ and

$$\begin{aligned}
f_1(x, u_1, u_2) &= \rho(x)u_2(|u_1|^{\alpha-1}u_1 + \frac{1}{\beta + 1}|u_2|^{\beta-1}u_2), \\
f_2(x, u_1, u_2) &= \rho(x)u_1(\frac{1}{\alpha + 1}|u_1|^{\alpha-1}u_1 + |u_2|^{\beta-1}u_2),
\end{aligned}$$

where the continuous function $\rho(x)$ is positive (resp. negative) and α, β are positive real number. Then this problem does not have nontrivial solutions.
It suffices to remark that

$$H(x, u_1, u_2) = \rho(x)(u_2\frac{|u_1|^{\alpha+1}}{\alpha + 1} + u_1\frac{|u_2|^{\beta+1}}{\beta + 1})$$

and a simple computation gives

$$\begin{aligned}
&2H(x, u_1, u_2) - u_1 f_1(x, u_1, u_2) - u_2 f_2(x, u_1, u_2) \\
&= \rho(x)\Big[(\frac{1}{\alpha + 1} - 1)|u_1|^{\alpha+1}u_2 + (\frac{1}{\beta + 1} - 1)|u_2|^{\beta+1}u_1\Big] \leq 0 \quad (\text{resp. } \geq 0).
\end{aligned}$$

The conclusion is the same as in the previous example.

4.1. **Example3.** For the scalar case ($m = 1$), let $\theta_1, \theta_2 : \overline{\mathcal{D}} \to \mathbb{R}$ be two nonnegative continuous functions, $p, q \geq 1$ and

$$f(x,u) = \delta u + \theta_1(x)|u|^{p-1}u + \theta_2(x)|u|^{q-1}u,$$

where δ is a real constant. Then the problem

$$-\frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(p_i(x)\frac{\partial u}{\partial y_i}) + f(x,u) = 0 \quad \text{in } \Omega,$$

$$u + \varepsilon\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega,$$

does not have nontrivial solutions. A simple computation gives

$$2H(x,u) - uf(x,u) = \theta_1(x)(\frac{2}{p+1} - 1)|u|^{p+1} + \theta_2(x)(\frac{2}{q+1} - 1)|u|^{q+1} \leq 0,$$

and an application of Theorem 3.5 gives the desired result.

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