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MAXIMIZERS FOR THE STRICHARTZ AND THE SOBOLEV-STRICHARTZ INEQUALITIES FOR THE SCHRÖDINGER EQUATION

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Abstract. In this paper, we first show that there exists a maximizer for the non-endpoint Strichartz inequalities for the Schrödinger equation in all dimensions based on the recent linear profile decomposition result. We then present a new proof of the linear profile decomposition for the Schröindger equation with initial data in the homogeneous Sobolev space; as a consequence, there exists a maximizer for the Sobolev-Strichartz inequality.

1. Introduction

We consider the free Schrödinger equation

$$i\partial_t u + \Delta u = 0, \tag{1.1}$$

with initial data $u(0,x) = u_0(x)$ where $u : \mathbb{R}\times\mathbb{R}^d \to \mathbb{C}$ is a complex-valued function and $d \geq 1$. We denote the solution u by using the Schrödinger evolution operator $e^{it\Delta}$:

$$u(t,x) := e^{it\Delta}u_0(x) := \int_{\mathbb{R}^d} e^{ix\cdot\xi - it|\xi|^2}\hat{u}_0(\xi)d\xi, \tag{1.2}$$

where $\hat{u}_0$ is the spatial Fourier transform of u_0 defined via

$$\hat{u}_0(\xi) := \int_{\mathbb{R}^d} e^{-ix\cdot\xi}u_0(x)dx, \tag{1.3}$$

where $x\cdot\xi$ (abbr. $x\xi$) denotes the Euclidean inner product of x and ξ in the spatial space $\mathbb{R}^d$. Formally the solutions to this equation have a conserved mass

$$\int_{\mathbb{R}^d} |u(t,x)|^2 dx. \tag{1.4}$$

A family of well-known inequalities, the Strichartz inequalities, is associated with (1.1), which is very useful in nonlinear dispersive equations. It asserts that, for any $u_0 \in L^2_x(\mathbb{R}^d)$, there exists a constant $C_{d,q,r} > 0$ such that

$$\|e^{it\Delta}u_0\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} \leq C_{d,q,r}\|u_0\|_{L^2_x(\mathbb{R}^d)} \tag{1.5}$$

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holds if and only if (q,r,d) is Schrödinger admissible; i.e.,

$$\frac{2}{q}+\frac{d}{r}=\frac{d}{2}, \quad (q,r,d)\neq(2,\infty,2), \quad q,r\geq 2. \tag{1.6}$$

For $(q,r)=(2,\frac{2d}{d-2})$ when $d\geq 3$ or $(q,r)=(4,\infty)$ when $d=1$, the inequality (1.5) is referred to as the "endpoint" estimate, otherwise the "non-endpoint" estimate for the rest pairs. It has a long history to establish (1.5) for all Schrödinger admissible pairs in (1.6) expect when $(q,r)=(\infty,2)$, in which case it follows from (1.4). For the symmetric exponent $q=r=2+\frac{4}{d}$, Strichartz established this inequality in [17] which in turn had precursors in [21]. The non-endpoints were established by Ginibre and Velo [8], see also [20, Theorem 2.3] for a proof; the delicate endpoints in higher dimensions were treated by Keel and Tao [10]. When $(q,r,d)=(2,\infty,2)$, it has been known to fail, see e.g., [15] and [19].

A close relative of the Strichartz inequality for the Schrödinger equation is the Sobolev-Strichartz inequality: for $2\leq q<\infty$, and $2\leq r<\infty$ and $u_0\in\dot{H}_x^{s(q,r)}(\mathbb{R}^d)$ with $s(q,r):=\frac{d}{2}-\frac{2}{q}-\frac{d}{r}>0$, there exists a constant $C_{d,q,r}>0$ such that

$$\|e^{it\Delta}u_0\|_{L_t^qL_x^r(\mathbb{R}\times\mathbb{R}^d)}\leq C_{d,q,r}\|u_0\|_{\dot{H}_x^{s(q,r)}(\mathbb{R}^d)}, \tag{1.7}$$

which can be proven by using the usual Sobolev embedding and the Strichartz inequality (1.5).

In this paper, we are interested in the existence of maximizers for the Strichartz inequality (1.5) and the Sobolev-Strichartz inequality (1.7), i.e., functions which optimize (1.5) and (1.7) in the sense that they become equal.

The answer to the former is confirmed for the non-endpoints by an application of a recent powerful result, the profile decomposition for Schrödinger equations, which was developed in [4, 14, 5, 2] and had many applications in nonlinear dispersive equations, see [12] and the reference within. The problem of the existence of maximizers and of determining them explicitly for the symmetric Strichartz inequality when $q=r=2+\frac{4}{d}$ has been intensively studied. Kunze [13] treated the $d=1$ case and showed that maximizers exist by an elaborate concentration-compactness method; when $d=1,2$, Foschi [7] explicitly determined the best constants and showed that the only maximizers are Gaussians by using the sharp Cauchy-Schwarz inequality and the space-time Fourier transform; Hundertmark and Zharnitsky [9] independently obtained this result by an interesting representation formula of the Strichartz inequalities; recently, Carneiro [6] proved a sharp Strichartz-type inequality by following the arguments in [9] and found its maximizers, which derives the results in [9] as a corollary when $d=1,2$; very recently, Bennett, Bez, Carbery and Hundertmark [3] offered a new proof to determine the best constants by using the method of heat-flow.

The answer to the latter is true as well. The proof follows almost along similar lines as in the L_x^2 case if we have an analogous profile decomposition for initial data in the homogeneous Sobolev spaces. We offer a new proof for this fact, which we have not seen in the literature.

1.1. In this subsection, we investigate the existence of maximizers for the non-endpoint Strichartz inequalities. To begin, we recall the profile decomposition result in [2] in the notation of the symmetry group which preserves the mass and the Strichartz inequalities.

Definition 1.1 (Mass-preserving symmetry group)**.** For any phase $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, scaling parameter $h_0 > 0$, frequency $\xi_0 \in \mathbb{R}^d$, space and time translation parameters $x_0, t_0 \in \mathbb{R}^d$, we define the unitary transformation $g_{\theta,h_0,\xi_0,x_0,t_0} : L_x^2(\mathbb{R}^d) \to L_x^2(\mathbb{R}^d)$ by the formula

$$[g_{\theta,h_0,\xi_0,x_0,t_0}\phi](x) = e^{i\theta} e^{ix\cdot\xi_0} e^{-it_0\Delta}[\frac{1}{h_0^{d/2}}\phi(\frac{\cdot - x_0}{h_0})](x). \tag{1.8}$$

We let G be the collection of such transformations; G forms a group.

Definition 1.2. For $j \neq k$, two sequences $\Gamma_n^j = (h_n^j, \xi_n^j, x_n^j, t_n^j)_{n\ge1}$ and $\Gamma_n^k = (h_n^k, \xi_n^k, x_k^n, t_n^k)_{n\ge1}$ in $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}$ are said to be orthogonal if one of the followings holds:

- $\lim_{n\to\infty}\left(\frac{h_n^k}{h_n^j} + \frac{h_n^j}{h_n^k} + h_n^j|\xi_n^j - \xi_n^k|\right) = \infty$,
- $\lim_{n\to\infty}\left(\frac{|t_n^j - t_n^k|}{(h_n^j)^2} + \left|\frac{x_n^j - x_n^k}{h_n^j} + \frac{t_n^k(\xi_n^k - \xi_n^j)}{h_n^j}\right|\right) = \infty$.

We rephrase the linear profile decomposition theorem in [2] by using the notation in Definition 1.1.

Theorem 1.3. *Let $\{u_n\}_{n\ge1}$ be a bounded sequence in L_x^2. Then up to passing to a subsequence of $(u_n)_{n\ge1}$, there exists a sequence of functions $\phi^j \in L_x^2$ and group elements $(g_n^j)_{n\ge1,j\ge1} = g_{0,h_n^j,\xi_n^j,x_n^j,t_n^j} \in G$ with orthogonal $(h_n^j, \xi_n^j, x_n^j, t_n^j)$ such that for any $N \ge 1$, there exists $e_n^N \in L_x^2$,*

$$u_n = \sum_{j=1}^{N} g_n^j(\phi^j) + e_n^N, \tag{1.9}$$

with the error term having the asymptotically vanishing Strichartz norm

$$\lim_{N\to\infty}\lim_{n\to\infty}\|e^{it\Delta}e_n^N\|_{L_{t,x}^{2+4/d}} = 0, \tag{1.10}$$

and the following orthogonality properties: for any $N \ge 1$,

$$\lim_{n\to\infty}\Big(\|u_n\|_{L_x^2}^2 - (\sum_{j=1}^{N}\|\phi^j\|_{L_x^2}^2 + \|e_n^N\|_{L_x^2}^2)\Big) = 0, \tag{1.11}$$

for $j \neq k$,

$$\lim_{n\to\infty}\|e^{it\Delta}g_n^j(\phi^j)e^{it\Delta}g_n^k(\phi^k)\|_{L_{t,x}^{1+2/d}} = 0, \tag{1.12}$$

$$\lim_{n\to\infty}\langle g_n^j(\phi^j), g_n^k(\phi^k)\rangle_{L_x^2} = 0, \tag{1.13}$$

for any $1 \le j \le N$,

$$\lim_{n\to\infty}\langle g_n^j(\phi^j), e_n^N\rangle_{L_x^2} = 0. \tag{1.14}$$

The first main result in this paper concerns on the existence of maximizers for the symmetric Strichartz inequality $L_x^2 \to L_{t,x}^{2+4/d}$.

Theorem 1.4. *There exists a maximizing function $\phi \in L_x^2$ such that,*

$$\|e^{it\Delta}\phi\|_{L_{t,x}^{2+\frac{4}{d}}} = S\|\phi\|_{L_x^2}$$

with $S := \sup\{\|e^{it\Delta}u_0\|_{L_{t,x}^{2+\frac{4}{d}}} : \|u_0\|_{L_x^2} = 1\}$ being the sharp constant.

The proof of this theorem uses Theorem 1.3 and the following crucial inequality in [2]: for any $N \geq 1$,

$$\lim_{n\to\infty} \| \sum_{j=1}^{N} e^{it\Delta} g_n^j(\phi^j) \|_{L^{2+4/d}_{t,x}}^{2+4/d} \leq \sum_{j=1}^{N} \lim_{n\to\infty} \| e^{it\Delta}\phi^j \|_{L^{2+4/d}_{t,x}}^{2+4/d}. \tag{1.15}$$

Remark 1.5. The inequality (1.15) is a consequence of (1.12) by an interpolation argument in [2], which we will generalize in the proof of Lemma 1.6. When $d = 1, 2$, one can actually show that (1.15) is an equality by using the fact that $2 + 4/d$ is an even integer.

The inequality (1.15) suggests a way to obtain similar claims as in Theorem 1.4 for other non-endpoint Strichartz inequalities if the following lemma were established.

Lemma 1.6. *Let (q, r, d) be non-endpoint Schrödinger admissible pairs and $N \geq 1$. If $q \geq r$,*

$$\lim_{n\to\infty} \| \sum_{j=1}^{N} e^{it\Delta} g_n^j(\phi^j) \|_{L^q_t L^r_x}^{r} \leq \sum_{j=1}^{N} \lim_{n\to\infty} \| e^{it\Delta}\phi^j \|_{L^q_t L^r_x}^{r}; \tag{1.16}$$

if $q \leq r$,

$$\lim_{n\to\infty} \| \sum_{j=1}^{N} e^{it\Delta} g_n^j(\phi^j) \|_{L^q_t L^r_x}^{q} \leq \sum_{j=1}^{N} \lim_{n\to\infty} \| e^{it\Delta}\phi^j \|_{L^q_t L^r_x}^{q}. \tag{1.17}$$

Indeed, this is the case. Together with Theorem 1.3 again, this lemma yields the following corollary.

Corollary 1.7. *Let (q, r, d) be non-endpoint Schrödinger admissible pairs. There exists a maximizing function $\phi \in L^2_x$ such that,*

$$\| e^{it\Delta}\phi \|_{L^q_t L^r_x} = S_{q,r} \|\phi\|_{L^2_x}$$

with $S_{q,r} := \sup\{ \| e^{it\Delta} u_0 \|_{L^q_t L^r_x} : \|u_0\|_{L^2_x} = 1 \}$ being the sharp constant.

The proof of this corollary is similar to that used in Theorem 1.4 and thus will be omitted. Instead, we will focus on proving Lemma 1.6.

Remark 1.8. When $(q, r) = (\infty, 2)$, from the conservation of mass (1.4), we see that every L^2_x-initial data is a maximizer for the Strichartz inequality.

1.2. In this subsection we concern on the existence of maximizers for the Sobolev-Strichartz inequality (1.7) for the Schrödinger equation.

Theorem 1.9. *Let (q, r, d) be defined as in (1.7). Then there exists a maximizing function $\phi \in \dot{H}^{s(q,r)}_x$ for (1.7) with $C_{d,q,r}$ being the sharp constant*

$$S^{q,r} := \sup\{ \| e^{it\Delta} u_0 \|_{L^q_t L^r_x} : \|u_0\|_{\dot{H}^{s(q,r)}_x} = 1 \}.$$

As we can see, it suffices to establish a profile decomposition result for initial data in $\dot{H}^{s(q,r)}_x$.

Theorem 1.10. *Let $s(q, r)$ be defined as in (1.7) and $\{u_n\}_{n\geq 1}$ be a bounded sequence in $\dot{H}^{s(q,r)}_x$. Then up to passing to a subsequence of $(u_n)_{n\geq 1}$, there exists a* *sequence of functions* $\phi^j \in \dot H^s_x$ *and a sequence of parameters* (h^j_n, x^j_n, t^j_n) *such that for any* $N \geq 1$*, there exists* $e^N_n \in \dot H^s_x$*,*

$$u_n = \sum_{j=1}^N e^{-it^j_n \Delta}\Big(\frac{1}{(h^j_n)^{d/2-s}}\phi^j(\frac{\cdot - x^j_n}{h^j_n})\Big) + e^N_n, \tag{1.18}$$

with the parameters (h^j_n, x^j_n, t^j_n) *satisfying the following constraint: for* $j \neq k$*,*

$$\lim_{n\to\infty}\Big(\frac{h^j_n}{h^k_n} + \frac{h^k_n}{h^j_n} + \frac{|t^j_n - t^k_n|}{(h^j_n)^2} + \frac{|x^j_n - x^k_n|}{h^j_n}\Big) = \infty, \tag{1.19}$$

and the error term having the asymptotically vanishing Sobolev-Strichartz norm

$$\lim_{N\to\infty}\lim_{n\to\infty}\|e^{it\Delta}e^N_n\|_{L^q_t L^r_x} = 0, \tag{1.20}$$

and the following orthogonality property: for any $N \geq 1$*,*

$$\lim_{n\to\infty}\Big(\|u_n\|^2_{\dot H^s_x} - (\sum_{j=1}^N \|\phi^j\|^2_{\dot H^s_x} + \|e^N_n\|^2_{\dot H^s_x})\Big) = 0. \tag{1.21}$$

When $s = 1$ and $d \geq 3$, Keraani [11] established Theorem 1.10 for the Schrödinger equation based on the following Besov-type improvement of the Sobolev embedding

$$\|f\|_{L^{2d/(d-2)}_x} \lesssim \|Df\|^{1-2/d}_{L^2_x}\|Df\|^{2/d}_{\dot B^0_{2,\infty}}, \tag{1.22}$$

where $\|\cdot\|_{\dot B^0_{2,\infty}}$ is the Besov norm defined via

$$\|f\|_{\dot B^0_{2,\infty}} := \sup_{k\in\mathbb{Z}}\|f_k\|_{L^2_x}$$

with f_k denoting the k-th Littlewood-Paley piece defined via the Fourier transform $\hat f_k := \hat f 1_{2^k\leq|\xi|\leq 2^{k+1}}$ [1], and D^s the fractional differentiation operator defined via the inverse Fourier transform,

$$D^s f(x) := \int_{\mathbb{R}^d} e^{ix\xi}|\xi|^s \hat f(\xi)d\xi.$$

He followed the arguments in [1] where Bahouri and Gérard established the profile-decomposition result in the context of the wave equation with initial data in $\dot H^1_x(\mathbb{R}^3)$. Recently under the same constraints on s and d, Killip and Visan [12] obtained the same result by relying on their interesting improved Sobolev embedding involving the critical $L^{2d/(d-2)}_x$-norm on the right-hand side:

$$\|f\|_{L^{2d/(d-2)}_x} \lesssim \|Df\|^{1-2/d}_{L^2_x}\sup_{k\in\mathbb{Z}}\|f_k\|^{2/d}_{L^{2d/(d-2)}_x}. \tag{1.23}$$

Note that (1.23) implies (1.22) by the usual Sobolev embedding. By following their approaches, we will generalize both Keraani's and Killip-Visan's improved $\dot H^1_x$-Sobolev embeddings to those with $\dot H^s_x$ norms where $\frac{1}{r} + \frac{s}{d} = \frac{1}{2}$ and $d \geq 1$ in the appendix of this paper. Consequently almost same approaches as in [11] or [12] would yield Theorem 1.10 without difficulties but we choose not to do it in this paper for simplicity. However, we will offer a new proof of Theorem 1.10 by taking advantage of the existing L^2_x linear profile decomposition, Theorem 1.3. The idea can be roughly explained as follows.

[1] For a rigorous definition of the Littlewood-Paley decomposition (or dyadic decomposition) in terms of smooth cut-off functions, see [16, p.241].

For $(u_n)_{n\geq 1}\in \dot H^s_x$, we regard $(D^s u_n)_{n\geq 1}$ as an L^2_x sequence and then apply Theorem 1.3 to this new sequence. Then the main task is to show how to eliminate the frequency parameter ξ^j_n from the decomposition. To do it, we have two cases according to the limits of the sequence $(h^j_n\xi^j_n)_{n\geq 1}$ for each j: when the limit of $h^j_n\xi^j_n$ is finite, we will change the profiles ϕ^j so that we can reduce to $\xi^j_n=0$; on the other hand, when it is infinite, we will group this term into the error term since one can show that its Sobolev Strichartz norm is asymptotically small.

We organize this paper as follows: in Section 1.3 we establish some notations; in Section 2 we prove Theorems 1.4, 1.6; in Section 3 we prove Theorem 1.9; finally in Appendix, we include the arguments for the general Keraani's and Killip-Visan's improved Sobolev embeddings.

1.3. **Notation.** We use $X\lesssim Y$, $Y\gtrsim X$, or $X=O(Y)$ to denote the estimate $|X|\leq CY$ for some constant $0<C<\infty$, which might depend on d, p and q but not on the functions. If $X\lesssim Y$ and $Y\lesssim X$ we will write $X\sim Y$. If the constant C depends on a special parameter, we shall denote it explicitly by subscripts.

Throughout the paper, the symbol $\lim_{n\to\infty}$ should be understood as $\limsup_{n\to\infty}$. The homogeneous Sobolev space $\dot H^s_x(\mathbb{R}^d)$ for $s\geq 0$ can be defined in terms of the fractional differentiation:

$$\dot H^s_x(\mathbb{R}^d):=\{f:\|f\|_{\dot H^s_x(\mathbb{R}^d)}:=\|D^sf\|_{L^2_x(\mathbb{R}^d)}=\||\xi|^s\hat f\|_{L^2_\xi(\mathbb{R}^d)}<\infty\}.$$

We define the space-time norm $L^q_tL^r_x$ of f on $\mathbb{R}\times\mathbb{R}^d$ by

$$\|f\|_{L^q_tL^r_x(\mathbb{R}\times\mathbb{R}^d)}:=\Big(\int_{\mathbb{R}}\Big(\int_{\mathbb{R}^d}|f(t,x)|^r\,dx\Big)^{q/r}dt\Big)^{1/q},$$

with the usual modifications when q or r are equal to infinity, or when the domain $\mathbb{R}\times\mathbb{R}^d$ is replaced by a small region. When $q=r$, we abbreviate it by $L^q_{t,x}$. Unless specified, all the space-time integration are taken over $\mathbb{R}\times\mathbb{R}^d$, and the spatial integration over $\mathbb{R}^d$.

The inner product $\langle\cdot,\cdot\rangle_{L^2_x}$ in the Hilbert space $L^2_x(\mathbb{R}^d)$ is defined via

$$\langle f,g\rangle_{L^2_x}:=\int_{\mathbb{R}^d}f(x)\bar g(x)dx,$$

where $\bar g$ denotes the usual complex conjugate of g in the complex plane $\mathbb{C}$.

2. Maximizers for the symmetric Strichartz inequalities

Proof of Theorem 1.4. We choose a maximizing sequence $(u_n)_{n\geq 1}$ with $\|u_n\|_{L^2_x}=1$, and then, up to a subsequence, decompose it into linear profiles as in Theorem 1.3. Then from the asymptotically vanishing Strichartz norm (1.10), we obtain that, for any $\epsilon>0$, there exists n_0 so that for all $N\geq n_0$ and $n\geq n_0$,

$$S-\epsilon\leq\|\sum_{j=1}^N e^{it\Delta}g^j_n(\phi^j)\|_{L^{2+4/d}_{t,x}}.$$

Thus from (1.15), there exists $n_1\geq n_0$ such that when $n,N\geq n_1$,

$$S^{2+4/d}-2\epsilon\leq\sum_{j=1}^N\|e^{it\Delta}\phi^j\|^{2+4/d}_{L^{2+4/d}_{t,x}}.$$

Choosing $j_0 \in [1,N]$ such that $e^{it\Delta}\phi^{j_0}$ has the largest $L^{2+4/d}_{t,x}$ norm among $1 \le j \le N$, we see that, by the usual Strichartz inequality,

$$S^{2+4/d} - 2\epsilon \le \|e^{it\Delta}\phi^{j_0}\|^{4/d}_{L^{2+4/d}_{t,x}} \sum_{j=1}^{N} \|e^{it\Delta}\phi^j\|^2_{L^{2+4/d}_{t,x}} \le S^{2+4/d}\|\phi^{j_0}\|^{4/d}_{L^2_x} \le S^{2+4/d} \tag{2.1}$$

since (1.11) gives

$$\sum_{j=1}^{\infty} \|\phi^j\|^2_{L^2_x} \le \lim_{n\to\infty} \|u_n\|^2_{L^2_x} = 1. \tag{2.2}$$

This latter fact also gives that $\lim_{j\to\infty} \|\phi^j\|_{L^2_x} = 0$, which together with (2.1) shows that j_0 must terminate before some fixed constant which does not depend on ϵ. Hence in (2.1) we can take ϵ to zero to obtain

$$\|\phi^{j_0}\|_{L^2_x} = 1.$$

This further shows that $\phi^j = 0$ for all but $j = j_0$ from (2.2). Therefore ϕ^{j_0} is a maximizer. Thus the proof of Theorem (1.4) is complete. $\square$

We will closely follow the approach in [2, Lemma 5.5] to prove Lemma 1.6.

Proof of Lemma 1.6. We only handle (1.16) since the proof of (1.17) is similar. By interpolating with (1.12), we see that for $j \ne k$ and non-endpoint Schrödinger admissible pair (q,r,d),

$$\lim_{n\to\infty} \|e^{it\Delta}g^j_n(\phi^j)e^{it\Delta}g^k_n(\phi^k)\|_{L^{q/2}_t L^{r/2}_x} = 0. \tag{2.3}$$

Now we expand the left hand side of (1.16) out, which is equal to

$$\begin{aligned}
&\Big(\int\Big(\int \big|\sum_{j=1}^{N} e^{it\Delta}g^j_n(\phi^j)\big|^r dx\Big)^{q/r} dt\Big)^{r/q} \\
&\le \Big(\int\Big(\int \sum_{j=1}^{N} \big|e^{it\Delta}g^j_n(\phi^j)\big|^2 \big|\sum_{l=1}^{N} e^{it\Delta}g^l_n(\phi^l)\big|^{r-2} \\
&\quad + \sum_{k\ne j} |e^{it\Delta}g^j_n(\phi^j)||e^{it\Delta}g^k_n(\phi^k)||\sum_{l=1}^{N} e^{it\Delta}g^l_n(\phi^l)|^{r-2} dx\Big)^{q/r} dt\Big)^{r/q} \\
&\le \sum_{j=1}^{N}\Big(\int\Big(\int |e^{it\Delta}g^j_n(\phi^j)|^2|\sum_{l=1}^{N} e^{it\Delta}g^l_n(\phi^l)|^{r-2}dx\Big)^{q/r} dt\Big)^{r/q} \\
&\quad + \sum_{k\ne j}\Big(\int\Big(\int |e^{it\Delta}g^j_n(\phi^j)||e^{it\Delta}g^k_n(\phi^k)||\sum_{l=1}^{N} e^{it\Delta}g^l_n(\phi^l)|^{r-2}dx\Big)^{q/r} dt\Big)^{r/q} \\
&=: A + B.
\end{aligned}$$

For B, since $\frac{r}{q} = \frac{2}{q} + \frac{r-2}{q}$, $\quad 1 = \frac{2}{r} + \frac{r-2}{r}$, the Hölder inequality yields

$$B \le \sum_{k\ne j} \|e^{it\Delta}g^j_n(\phi^j)e^{it\Delta}g^k_n(\phi^k)\|_{L^{q/2}_t L^{r/2}_x} \|\sum_{l=1}^{N} e^{it\Delta}g^l_n(\phi^l)\|^{r-2}_{L^q_t L^r_x},$$

which approaches zero by (2.3) as n approaches infinity. Hence we are left with estimating A.

For A, since (q,r,d) is in the non-endpoint region and $q \geq r$, we have $2 < r \leq 2 + 4/d$, i.e., $0 < r - 2 \leq 4/d$. We let $s := [r-2]$, the largest integer which is less than $r-2$. Then because $0 \leq r - 2 - s < 1$,

$$\begin{aligned} &\big|e^{it\Delta} g_n^j(\phi^j)\big|^2 \big|\sum_{l=1}^N e^{it\Delta} g_n^l(\phi^l)\big|^{r-2} \\ &\leq \sum_{l=1}^N \big|e^{it\Delta} g_n^j(\phi^j)\big|^2 \big|\sum_{k=1}^N e^{it\Delta} g_n^k(\phi^k)\big|^s \big|e^{it\Delta} g_n^l(\phi^l)\big|^{r-2-s}. \end{aligned} \tag{2.4}$$

We now eliminate some terms in (2.4). The first case we consider is $l \neq j$: since $r - 2 - s < 2$, we write

$$\begin{aligned} &|e^{it\Delta} g_n^j(\phi^j)|^2 |\sum_{k=1}^N e^{it\Delta} g_n^k(\phi^k)|^s |e^{it\Delta} g_n^l(\phi^l)|^{r-2-s} \\ &= |e^{it\Delta} g_n^j(\phi^j)|^{4+s-r} |\sum_{k=1}^N e^{it\Delta} g_n^k(\phi^k)|^s |e^{it\Delta} g_n^j(\phi^j) e^{it\Delta} g_n^l(\phi^l)|^{r-2-s}. \end{aligned}$$

Then the Hölder inequality and (2.3) show that the summation above goes to zero as n goes to infinity. So we may assume that $l = j$ and take out the summation in l in (2.4). The second case we consider is when the terms in the expansion of $|\sum_{k=1}^N e^{it\Delta} g_n^k(\phi^k)|^s$ contain two distinct terms:

$$\begin{aligned} &|\sum_{k=1}^N e^{it\Delta} g_n^k(\phi^k)|^s \\ &\leq \sum_{k=1}^N |e^{it\Delta} g_n^k(\phi^k)|^s + \sum_{k_1 \neq k_2,\, k_1 + \dots + k_s = s} |e^{it\Delta} g_n^{k_1}(\phi^{k_1}) e^{it\Delta} g_n^{k_2}(\phi^{k_2})| \dots |e^{it\Delta} g_n^{k_s}(\phi^{k_s})|. \end{aligned}$$

Again the interpolation argument and (2.3) show that the second term above goes to zero as n goes to infinity. Combining these two cases, we reduce (2.4) to

$$|e^{it\Delta} g_n^j(\phi^j)|^r + \sum_{k \neq j} |e^{it\Delta} g_n^j(\phi^j)|^{r-s} |e^{it\Delta} g_n^k(\phi^k)|^s.$$

For the second term above, we consider $r - s \leq s$ and $r - s \geq s$; it is not hard to see that it goes to zero as expected when n goes to infinity. Therefore the proof of Lemma 1.6 is complete. □

3. Maximizers for the Sobolev-Strichartz inequalities

Proof of Theorem 1.10. It is easy to see that $\{u_n\} \in \dot{H}_x^s(\mathbb{R}^d)$ implies that $\{D^s u_n\} \in L_x^2(\mathbb{R}^d)$. We then apply Theorem 1.3: there exists a sequence of $(\psi^j)_{j \geq 1}$ and orthogonal $\Gamma_n^j = (h_n^j, \xi_n^j, x_n^j, t_n^j)$ so that, for any $N \geq 1$

$$D^s u_n = \sum_{j=1}^N g_n^j(\psi^j) + w_n^N, \tag{3.1}$$

with $w_n^N \in L_x^2$ and all the properties in Theorem 1.3 being satisfied. Without loss of generality, we assume all ψ^j to be Schwartz functions. We then rewrite it as

$$u_n = \sum_{j=1}^{N} D^{-s} g_n^j(\psi^j) + D^{-s} w_n^N. \tag{3.2}$$

Writing $e_n^N := D^{-s} w_n^N$, we see that for q, r in (1.7), the Sobolev embedding and (1.10) together give (1.20). Next we show how to eliminate ξ_n^j in the profiles.

Case 1. Up to a subsequence, $\lim_{n\to\infty} h_n^j \xi_n^j = \xi^j \in \mathbb{R}^d$ for some $1 \le j \le N$. From the Galilean transform,

$$e^{it_0\Delta}(e^{i(\cdot)\xi_0}\phi)(x) = e^{ix\xi_0 - it_0|\xi_0|^2} e^{it_0\Delta}\phi(x - 2t\xi_0),$$

we see that

$$\begin{aligned}
&e^{-it_n^j\Delta}\Big(\frac{1}{(h_n^j)^{d/2}} e^{ix_n^j\xi_n^j + it_n^j|\xi_n^j|^2} [e^{i(\cdot)h_n^j\xi_n^j}\psi^j](\frac{\cdot - x_n^j}{h_n^j})\Big)(x - 2t_n^j\xi_n^j) \\
&= e^{it_n^j|\xi_n^j|^2} e^{-it_n^j\Delta}\Big(\frac{1}{(h_n^j)^{d/2}} e^{i(\cdot)\xi_n^j}\psi^j(\frac{\cdot - x_n^j}{h_n^j})\Big)(x - 2t_n^j\xi_n^j) \\
&= e^{ix\xi_n^j} e^{-it_n^j\Delta}\Big(\frac{1}{(h_n^j)^{d/2}}\psi^j(\frac{\cdot - x_n^j}{h_n^j})\Big)(x) = g_n^j(\psi^j)(x).
\end{aligned}$$

On the other hand, since the symmetries defined in Definition 1.1 keep the L_x^2-norm invariant,

$$\begin{aligned}
&\|e^{-it_n^j\Delta}\Big(\frac{1}{(h_n^j)^{d/2}} e^{ix_n^j\xi_n^j + it_n^j|\xi_n^j|^2} [e^{i(\cdot)h_n^j\xi_n^j}\psi^j](\frac{\cdot - x_n^j}{h_n^j})\Big)(x - 2t_n^j\xi_n^j) - \\
&- e^{-it_n^j\Delta}\Big(\frac{1}{(h_n^j)^{d/2}} e^{ix_n^j\xi_n^j + it_n^j|\xi_n^j|^2} [e^{i(\cdot)\xi^j}\psi^j](\frac{\cdot - x_n^j}{h_n^j})\Big)(x - 2t_n^j\xi_n^j)\|_{L_x^2} \\
&= \|(e^{ixh_n^j\xi_n^j} - e^{ix\xi^j})\psi^j\|_{L_x^2} \to 0
\end{aligned}$$

as n approaches infinity. Hence we can replace $g_n^j(\psi^j)$ with

$$e^{-it_n^j\Delta}\Big(\frac{1}{(h_n^j)^{d/2}} e^{i(\cdot)\xi_n^j + it_n^j|\xi_n^j|^2} [e^{i(\cdot)\xi^j}\psi^j](\frac{\cdot - x_n^j}{h_n^j})\Big)(x - 2t_n^j\xi_n^j);$$

for the differences, we put them into the error term. Thus if further regarding $e^{ix\xi^j}\psi^j$ as a new ψ^j, we can re-define

$$g_n^j(\psi^j) := e^{-it_n^j\Delta}\Big(\frac{1}{(h_n^j)^{d/2}} e^{ix_n^j\xi_n^j + it_n^j|\xi_n^j|^2}\psi^j(\frac{\cdot - x_n^j}{h_n^j})\Big)(x - 2t_n^j\xi_n^j).$$

Hence in the decomposition (3.1), we see that ξ_n^j no longer plays the role of the frequency parameter and hence we can assume that $\xi_n^j \equiv 0$ for this j term. With this assumption,

$$D^{-s} g_n^j(\psi^j) = e^{-it_n^j\Delta}\Big(\frac{1}{(h_n^j)^{d/2-s}}(D^{-s}\psi^j)(\frac{\cdot - x_n^j}{h_n^j})\Big)(x).$$

Setting $\phi^j := D^{-s}\psi^j$, we see that this case is done.

Case 2. $\lim_{n\to\infty} |h_n^j\xi_n^j| = \infty$. It is clear that

$$e^{it\Delta} g_n^j(\psi^j)(x) = e^{ix\xi_n^j - it|\xi_n^j|^2} \frac{1}{(h_n^j)^{d/2}} e^{i\frac{t - t_n^j}{(h_n^j)^2}\Delta}\psi^j(\frac{x + x_n^j - 2t_n^j\xi_n^j}{h_n^j}).$$

Hence if changing $t' = \frac{t-t_n^j}{(h_n^j)^2}$ and $x' = \frac{x+x_n^j-2t_n^j\xi_n^j}{h_n^j}$, we obtain

$$\begin{aligned}\|D^{-s}g_n^j(\psi^j)\|_{L_t^qL_x^r} &= (h_n^j)^{\frac{2}{q}+\frac{d}{r}-s-\frac{d}{2}}\|D^{-s}(e^{ixh_n^j\xi_n^j}e^{is\Delta}\psi^j)\|_{L_{t'}^qL_{x'}^r}\\ &= \|D^{-s}(e^{ixh_n^j\xi_n^j}e^{is\Delta}\psi^j)\|_{L_{t'}^qL_{x'}^r}.\end{aligned}$$

By the Hörmander-Mikhlin multiplier theorem [18, Theorm 4.4], for $2 \le r < \infty$,

$$\|D^{-s}e^{it\Delta}g_n^j(\psi^j)\|_{L_t^qL_x^r} \sim \frac{1}{(|h_n^j\xi_n^j|)^s}\|e^{is\Delta}\psi^j\|_{L_{t'}^qL_{x'}^r} \lesssim_{\psi^j} (|h_n^j\xi_n^j|)^{-s} \to 0$$

as n goes to infinity since $s > 0$ and ψ^j is assumed to be Schwartz. In view of this, we will organize $D^{-s}g_n^j(\psi^j)$ into the error term e_n^N. Hence the decomposition (1.18) is obtained. Finally the $\dot{H}_x^s$-orthogonality (1.21) follows from (1.11), and the constraint (1.19) from Definition 1.2 since $\xi_n^j \equiv 0$ for all j, n. Therefore the proof of Theorem 1.10 is complete. $\square$

4. Proof of the improved Sobolev embeddings

Here we include the arguments for the generalized Keraani's and Killip-Visan's improved Sobolev embeddings, which can be used to derive Theorem 1.10 as well. Firstly the generalization of (1.22) is as follows: for any $1 < r < \infty$, $s \ge 0$ and $\frac{1}{r} + \frac{s}{d} = \frac{1}{2}$,

$$\|f\|_{L_x^r} \lesssim \|D^sf\|_{L_x^2}^{1-2s/d}\|D^sf\|_{\dot{B}_{2,\infty}^0}^{2s/d}. \tag{4.1}$$

To prove it, we will closely follow the approach in [1] by Bahouri and Gérard.

Proof of (4.1). For every $A > 0$, we define $f_{<A}$, $f_{>A}$ via

$$\hat{f}_{<A}(\xi) := 1_{|\xi|\le A}\hat{f}(\xi), \hat{f}_{>A}(\xi) = 1_{|\xi|>A}\hat{f}(\xi).$$

From the Riemann-Lebesgue lemma,

$$\|f_{<A}\|_{L_x^\infty} \lesssim \|\hat{f}_{<A}\|_{L_\xi^1} \lesssim \sum_{k\in\mathbb{Z},\, k\le K}\|1_{2^k\le|\xi|\le2^{k+1}}\hat{f}\|_{L_\xi^1},$$

where K is the largest integer such that $2^K \le A$. Then by the Cauchy-Schwarz inequality,

$$\|1_{2^k\le|\xi|\le2^{k+1}}\hat{f}\|_{L_\xi^1} \lesssim 2^{k(\frac{d}{2}-s)}\|D^sf\|_{\dot{B}_{2,\infty}^0}.$$

Since $\frac{d}{2} - s > 0$ and $2^K \le A$,

$$\|f_{<A}\|_{L_x^\infty} \le CA^{\frac{d}{2}-s}\|D^sf\|_{\dot{B}_{2,\infty}^0}$$

for some $C > 0$. We write

$$\|f\|_{L_x^r}^r \sim \int_0^\infty \lambda^{r-1}|\{x \in \mathbb{R}^d : |f| > \lambda\}|d\lambda$$

and set

$$A(\lambda) := \Big(\frac{\lambda}{2C\|D^sf\|_{\dot{B}_{2,\infty}^0}}\Big)^{\frac{1}{d/2-s}}.$$

Thus we obtain

$$\|f_{<A}\|_{L_x^\infty} \le \frac{\lambda}{2}.$$

On the other hand, the Chebyshev inequality gives

$$|\{|f| > \lambda\}| \leq |\{f_{>A(\lambda)} > \frac{\lambda}{2}\}| \lesssim \frac{\|\hat{f}_{>A(\lambda)}\|_{L_x^2}^2}{\lambda^2}.$$

Hence we have

$$\begin{aligned}
\|f\|_{L_x^r}^r &\lesssim \int_0^\infty \lambda^{r-3} \int_{|\xi|>A(\lambda)} |\hat{f}(\xi)|^2 d\xi d\lambda \\
&\lesssim \int_0^\infty |\hat{f}(\xi)|^2 \int_0^{2C\|D^s f\|_{\dot{B}^0_{2,\infty}} |\xi|^{d/2-s}} \lambda^{r-3} d\lambda d\xi \\
&\lesssim \|D^s f\|_{\dot{B}^0_{2,\infty}}^{r-2} \int_0^\infty |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi \\
&\lesssim \|D^s f\|_{L_x^2}^2 \|D^s f\|_{\dot{B}^0_{2,\infty}}^{r-2}.
\end{aligned}$$

Therefore the proof of (4.1) is complete. □

Next we prove the generalized version of (1.23): For any $1 < r < \infty$, $s \geq 0$ and $\frac{1}{r} + \frac{s}{d} = \frac{1}{2}$,

$$\|f\|_{L_x^r} \lesssim \|D^s f\|_{L_x^2}^{1-2s/d} \sup_{k\in\mathbb{Z}} \|f_k\|_{L_x^r}^{2s/d}, \tag{4.2}$$

where f_k is defined as above. For the proof, we will closely follow the approach in [12]. We first recall the Littlewood-Paley square function estimate [16, p.267].

Lemma 4.1. *Let $1 < p < \infty$. Then for any Schwartz function f,*

$$\|f\|_{L_x^p} \sim \|(\sum_{k\in\mathbb{Z}} |f_k|^2)^{1/2}\|_{L_x^p},$$

where f_k is defined as in the introduction.

Proof of (4.2). By Lemma 4.1, we see that

$$\|f\|_{L_x^r}^r \sim \int \Big(\sum_{M\in\mathbb{Z}} |f_M|^2\Big)^{r/2} dx. \tag{4.3}$$

When $r \leq 4$, we have $d - 4s \geq 0$. Then the Hölder inequality and the Bernstein inequality yield,

$$\begin{aligned}
\|f\|_{L_x^r}^r &\sim \int \Big(\sum_{M\in\mathbb{Z}} |f_M|^2\Big)^{r/4} \Big(\sum_{N\in\mathbb{Z}} |f_N|^2\Big)^{r/4} dx \\
&\lesssim \sum_{M\leq N} \int |f_M|^{r/2} |f_N|^{r/2} dx \\
&\lesssim \sum_{M\leq N} \|f_M\|_{L_x^{\frac{2d}{d-4s}}} \||f_M|^{r/2-1}\|_{L_x^{\frac{d}{s}}} \||f_N|^{r/2-1}\|_{L_x^{\frac{d}{s}}} \|f_N\|_{L_x^2} \\
&\lesssim \Big(\sup_{k\in\mathbb{Z}} \|f_k\|_{L_x^r}\Big)^{r-2} \sum_{M\leq N} \|f_M\|_{L_x^{\frac{2d}{d-4s}}} \|f_N\|_{L_x^2} \\
&\lesssim \Big(\sup_{k\in\mathbb{Z}} \|f_k\|_{L_x^r}\Big)^{r-2} \sum_{M\leq N} N^{-s} M^s \|D^s f_M\|_{L_x^2} \|D^s f_N\|_{L_x^2}.
\end{aligned}$$

Then the Schur's test concludes the proof when $r \le 4$. On the other hand, when $r > 4$, we let $r^* = [r/2]$, the largest integer which is less than $r/2$. Still by (4.3), the Hölder inequality and the Bernstein inequality, we have

$$\begin{aligned}
&\|f\|_{L_x^r}^r \\
&\sim \int \Big(\sum_{M_1} |f_{M_1}|^2\Big)\Big(\sum_{M_2} |f_{M_2}|^2\Big)^{r^*-1}\Big(\sum_{M} |f_{M}|^2\Big)^{r/2-r^*} dx \\
&\lesssim \sum_{M_1 \le M_2 \le \dots \le M_{r^*-1} \le M} \int |f_{M_1}||f_{M_1}|f_{M_2}|^2|f_{M_3}|^2 \times \dots \times |f_{M_{r^*-1}}|^2 |f_M|^{r-2r^*} dx \\
&\lesssim \sum_{M_1 \le M_2 \le \dots \le M_{r^*-1} \le M} \|f_{M_1}\|_{L_x^r} \|f_{M_1}\|_{L_x^\infty} \|f_M\|_{L_x^r}^{r-2r^*} \|f_{M_2}\|_{L_x^{r/2}} \\
&\quad \times \|f_{M_2}\|_{L_x^r} \|f_{M_3}\|_{L_x^r}^2 \dots \|f_{M_{r^*-1}}\|_{L_x^r}^2 \\
&\lesssim \Big(\sup_{k\in\mathbb{Z}} \|f_k\|_{L_x^r}\Big)^{r-2} \sum_{M_1 \le M_2} M_1^{d/2-s} M_2^{s-d/2} \|D^s f_{M_1}\|_{L_x^2} \|D^s f_{M_2}\|_{L_x^2}.
\end{aligned}$$

Since $d/2 - s > 0$, the Schur's test again concludes the proof. Hence the proof of (4.2) is complete. □

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