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Ψ-BOUNDED SOLUTIONS FOR LINEAR DIFFERENTIAL SYSTEMS WITH LEBESGUE Ψ-INTEGRABLE FUNCTIONS ON ℝ AS RIGHT-HAND SIDES

AUREL DIAMANDESCU

Abstract. In this paper we give a characterization for the existence of Ψ-bounded solutions on $\mathbb{R}$ for the system $x' = A(t)x + f(t)$, assuming that f is a Lebesgue Ψ-integrable function on $\mathbb{R}$. In addition, we give a result in connection with the asymptotic behavior of the Ψ-bounded solutions of this system.

1. Introduction

This work is concerned with linear differential system

$$x' = A(t)x + f(t) \tag{1.1}$$

where $x(t)$, $f(t)$ are in $\mathbb{R}^d$ and A is a continuous $d \times d$ matrix-valued function. The basic problem under consideration is the determination of necessary and sufficient conditions for the existence of a solution with some specified boundedness condition. A clasic result in this type of problems is given by Coppel [4, Theorem 2, Chapter V].

The problem of Ψ-boundedness of the solutions for systems of ordinary differential equations has been studied in many papers, [1, 2, 3, 5, 6, 7, 8, 9, 10, 11]. In [5, 6, 7], the author proposes the novel concept of Ψ-boundedness of solutions, Ψ being a continuous matrix-valued function, allows a better identification of various types of asymptotic behavior of the solutions on $\mathbb{R}_+$.

Similarly, we can consider solutions of (1.1) which are Ψ-bounded not only $\mathbb{R}_+$ but on $\mathbb{R}$. In this case, the conditions for the existence of at least one Ψ-bounded solution are rather complicated, as shown in [8] and below. In [8], it is given a necessary and sufficient condition so that the system (1.1) has at least one Ψ-bounded solution on $\mathbb{R}$ for every continuous and Ψ-bounded function f on $\mathbb{R}$.

The aim of present paper is to give a necessary and sufficient condition so that the nonhomogeneous system of ordinary differential equations (1.1) has at least one Ψ-bounded solution on $\mathbb{R}$ for every Lebesgue Ψ-integrable function f on $\mathbb{R}$. The introduction of the matrix function Ψ permits to obtain a mixed asymptotic behavior of the components of the solutions. Here, Ψ is a continuous matrix-valued function on $\mathbb{R}$.

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2. Definitions, Notations and hypotheses

Let $\mathbb{R}^d$ be the Euclidean d-space. For $x = (x_1, x_2, x_3, \dots, x_d)^T \in \mathbb{R}^d$, let $\|x\| = \max\{|x_1|, |x_2|, |x_3|, \dots, |x_d|\}$ be the norm of x. For a $d \times d$ real matrix $A = (a_{ij})$, we define the norm $|A| = \sup_{\|x\| \leq 1} \|Ax\|$. It is well-known that

$$|A| = \max_{1 \leq i \leq d} \{\sum_{j=1}^{d} |a_{ij}|\}.$$

Let $\Psi_i : \mathbb{R} \to (0, \infty)$, $i = 1, 2, \dots d$, be continuous functions and

$$\Psi = \operatorname{diag}[\Psi_1, \Psi_2, \dots \Psi_d].$$

Definition. A function $\varphi : \mathbb{R} \to \mathbb{R}^d$ is said to be Ψ-bounded on $\mathbb{R}$ if $\Psi\varphi$ is bounded on $\mathbb{R}$.

Definition. A function $\varphi : \mathbb{R} \to \mathbb{R}^d$ is said to be Lebesgue Ψ-integrable on $\mathbb{R}$ if φ is measurable and $\Psi\varphi$ is Lebesgue integrable on $\mathbb{R}$.

By a solution of (1.1), we mean an absolutely continuous function satisfying (1.1) for almost all $t \in \mathbb{R}$.

Let A be a continuous $d \times d$ real matrix and let the associated linear differential system be

$$y' = A(t)y. \tag{2.1}$$

Let Y be the fundamental matrix of (2.1) for which $Y(0) = I_d$ (identity $d \times d$ matrix).

Let the vector space $\mathbb{R}^d$ be represented as a direct sum of three subspaces X_-, X_0, X_+ such that a solution $y(t)$ of (2.1) is Ψ-bounded on $\mathbb{R}$ if and only if $y(0) \in X_0$ and Ψ-bounded on $\mathbb{R}_+ = [0, \infty)$ if and only if $y(0) \in X_- \oplus X_0$. Also, let P_-, P_0, P_+ denote the corresponding projection of $\mathbb{R}^d$ onto X_-, X_0, X_+ respectively.

3. Main result

Theorem 3.1. *If A is a continuous $d \times d$ real matrix on $\mathbb{R}$, then* (1.1) *has at least one Ψ-bounded solution on $\mathbb{R}$ for every Lebesgue Ψ-integrable function $f : \mathbb{R} \to \mathbb{R}^d$ on $\mathbb{R}$ if and only if there exists a positive constant K such that*

$$\begin{gathered}
|\Psi(t)Y(t)P_-Y^{-1}(s)\Psi^{-1}(s)| \leq K \quad \textit{for } t > 0,\ s \leq 0 \\
|\Psi(t)Y(t)(P_0 + P_-)Y^{-1}(s)\Psi^{-1}(s)| \leq K \quad \textit{for } t > 0,\ s > 0, s < t \\
|\Psi(t)Y(t)P_+Y^{-1}(s)\Psi^{-1}(s)| \leq K \quad \textit{for } t > 0, s > 0,\ s \geq t \\
|\Psi(t)Y(t)P_-Y^{-1}(s)\Psi^{-1}(s)| \leq K \quad \textit{for } t \leq 0,\ s < t \\
|\Psi(t)Y(t)(P_0 + P_+)Y^{-1}(s)\Psi^{-1}(s)| \leq K \quad \textit{for } t \leq 0,\ s \geq t,\ s < 0 \\
|\Psi(t)Y(t)P_+Y^{-1}(s)\Psi^{-1}(s)| \leq K \quad \textit{for } t \leq 0,\ s \geq t,\ s \geq 0
\end{gathered} \tag{3.1}$$

Proof. First, we prove the "only if" part. Thus, suppose that the system (1.1) has at least one Ψ-bounded solution on $\mathbb{R}$ for every Lebesgue Ψ-integrable function f : $\mathbb{R} \to \mathbb{R}^d$ on $\mathbb{R}$.

We shall denote by C_Ψ the Banach space of all Ψ-bounded and continuous functions $x : \mathbb{R} \to \mathbb{R}^d$ with the norm $\|x\|_{C_\Psi} = \sup_{t \in \mathbb{R}} \|\Psi(t)x(t)\|$ and by B the Banach space of all Lebesgue Ψ-integrable functions $x : \mathbb{R} \to \mathbb{R}^d$ with the norm $\|x\|_B = \int_{-\infty}^{+\infty} \|\Psi(t)x(t)\| dt$.

We shall denote by D the set of all functions $x : \mathbb{R} \to \mathbb{R}^d$ which are absolutely continuous on all intervals $J \subset \mathbb{R}$, Ψ-bounded on $\mathbb{R}$, $x(0) \in X_- \oplus X_+$ and $x' - Ax \in B$.

Obviously, D is a vector space and $x \to \|x\|_D = \|x\|_{C_\Psi} + \|x' - Ax\|_B$ is a norm on D.

Step 1. $(D, \|\cdot\|_D)$ is a Banach space. Let $(x_n)_{n\in\mathbb{N}}$ be a fundamental sequence of elements of D. Then, it is a fundamental sequence in C_Ψ. Therefore, there exists a continuous and Ψ-bounded function $x : \mathbb{R} \to \mathbb{R}^d$ such that $\lim_{n\to\infty} \Psi(t)x_n(t) = \Psi(t)x(t)$, uniformly on $\mathbb{R}$. From the inequality

$$\|x_n(t) - x(t)\| \le |\Psi^{-1}(t)|\|\Psi(t)x_n(t) - \Psi(t)x(t)\|, \quad t \in \mathbb{R},$$

it follows that $\lim_{n\to\infty} x_n(t) = x(t)$, uniformly on every compact of $\mathbb{R}$. Thus, $x(0) \in X_- \oplus X_+$.

On the other hand, the sequence $(f_n)_{n\in\mathbb{N}}$, where $f_n(t) = x_n'(t) - A(t)x_n(t)$, is a fundamental sequence in the Banach space B. Thus, there exists $f \in B$ such that

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} \|\Psi(t)(f_n(t) - f(t))\|dt = 0.$$

For a fixed, but arbitrary, $t \in \mathbb{R}$, we have

$$\begin{aligned}
x(t) - x(0) &= \lim_{n\to\infty} \big(x_n(t) - x_n(0)\big) \\
&= \lim_{n\to\infty} \int_0^t x_n'(s)ds \\
&= \lim_{n\to\infty} \int_0^t [\Psi^{-1}(s)(\Psi(s)(f_n(s) - f(s)) + f(s) + A(s)x_n(s)]ds \\
&= \int_0^t \big(f(s) + A(s)x(s)\big)ds.
\end{aligned}$$

It follows that $x' - Ax = f \in B$ and x is absolutely continuous on all intervals $J \subset \mathbb{R}$. Thus, $x \in D$.

Now, from

$$\lim_{n\to\infty} \Psi(t)x_n(t) = \Psi(t)x(t), \quad \text{uniformly on } \mathbb{R}$$

and

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} \|\Psi(t)[(x_n'(t) - A(t)x_n(t)) - (x'(t) - A(t)x(t))]\|dt = 0,$$

it follows that $\lim_{n\to\infty} \|x_n - x\|_D = 0$. Thus, $(D, \|\cdot\|_D)$ is a Banach space.

Step 2. There exists a positive constant K such that, for every $f \in B$ and for corresponding solution $x \in D$ of (1.1), we have

$$\sup_{t\in\mathbb{R}} \|\Psi(t)x(t)\| \le K \int_{-\infty}^{+\infty} \|\Psi(t)f(t)\|dt, \tag{3.2}$$

For this, define the mapping $T : D \to B$, $Tx = x' - Ax$. This mapping is obviously linear and bounded, with $\|T\| \le 1$.

Let $Tx = 0$. Then, $x' = Ax$, $x \in D$. This shows that x is a Ψ-bounded solution on $\mathbb{R}$ of (2.1). Then, $x(0) \in X_0 \cap \big(X_- \oplus X_+\big) = \{0\}$. Thus, $x = 0$, such that the mapping T is "one-to-one" .

Now, let $f \in B$ and let x be the Ψ-bounded solution on $\mathbb{R}$ of the system (1.1) which exists by assumption. Let z be the solution of the Cauchy problem

$$x' = A(t)x + f(t), \quad z(0) = (P_- + P_+)x(0).$$

Then $u = x - z$ is a solution of (2.1) with $u(0) = x(0) - (P_- + P_+)x(0) = P_0x(0)$. From the Definition of X_0, it follows that u is Ψ-bounded on $\mathbb{R}$. Thus, z is Ψ-bounded on $\mathbb{R}$. Therefore, z belongs to D and $Tz = f$. Consequently, the mapping T is "onto" .

From a fundamental result of Banach: "If T is a bounded one-to-one linear operator of one Banach space onto another, then the inverse operator T^{-1} is also bounded" , we have $\|T^{-1}f\|_D \le \|T^{-1}\|\|f\|_B$, for all $f \in B$.

For a given $f \in B$, let $x = T^{-1}f$ be the corresponding solution $x \in D$ of (1.1). We have

$$\|x\|_D = \|x\|_{C_\Psi} + \|x' - \text{ Ax}\|_B = \|x\|_{C_\Psi} + \|f\|_B \le \|T^{-1}\|\|f\|_B$$

or

$$\|x\|_{C_\Psi} \le \big(\|T^{-1}\| - 1\big)\|f\|_B = K\|f\|_B.$$

This inequality is equivalent to (3.2).

Step 3. The end of the proof. Let $T_1 < 0 < T_2$ be a fixed points but arbitrarily, and let $f : \mathbb{R} \to \mathbb{R}^d$ a function in B which vanishes on $(-\infty, T_1] \cup [T_2, +\infty)$. It is easy to see that the function $x : \mathbb{R} \to \mathbb{R}^d$ defined by

$$x(t) = \begin{cases} -\int_{T_1}^0 Y(t)P_0Y^{-1}(s)f(s)ds - \int_{T_1}^{T_2} Y(t)P_+Y^{-1}(s)f(s)ds, & t < T_1 \\ \int_{T_1}^t Y(t)P_-Y^{-1}(s)f(s)ds + \int_0^t Y(t)P_0Y^{-1}(s)f(s)ds & \\ -\int_t^{T_2} Y(t)P_+Y^{-1}(s)f(s)ds, & T_1 \le t \le T_2 \\ \int_{T_1}^{T_2} Y(t)P_-Y^{-1}(s)f(s)ds + \int_0^{T_2} Y(t)P_0Y^{-1}(s)f(s)ds, & t > T_2 \end{cases}$$

is the solution in D of the system (1.1). Now, we put

$$G(t,s) = \begin{cases} Y(t)P_-Y^{-1}(s), & s \le 0 < t, \\ Y(t)(P_0 + P_-)Y^{-1}(s), & 0 < s < t, \\ -Y(t)P_+Y^{-1}(s), & 0 < t \le s, \\ Y(t)P_-Y^{-1}(s), & s < t \le 0, \\ -Y(t)(P_0 + P_+)Y^{-1}(s), & t \le s < 0, \\ -Y(t)P_+Y^{-1}(s), & t \le 0 \le s\,. \end{cases}$$

This function is continuous on $\mathbb{R}^2$ except on the line $t = s$, where it has a jump discontinuity. Then, we have that $x(t) = \int_{T_1}^{T_2} G(t,s)f(s)ds$, $t \in \mathbb{R}$. Indeed,

- for $t < T_1$, we have

$$\begin{aligned} &\int_{T_1}^{T_2} G(t,s)f(s)ds \\ &= -\int_{T_1}^0 Y(t)(P_0 + P_+)Y^{-1}(s)f(s)ds - \int_0^{T_2} Y(t)P_+Y^{-1}(s)f(s)ds \\ &= -\int_{T_1}^0 Y(t)P_0Y^{-1}(s)f(s)ds - \int_{T_1}^{T_2} Y(t)P_+Y^{-1}(s)f(s)ds \\ &= x(t) \end{aligned}$$

- for $t \in [T_1, 0]$, we have

$$\begin{aligned}
\int_{T_1}^{T_2} G(t,s)f(s)ds &= \int_{T_1}^{t} Y(t)P_{-}Y^{-1}(s)f(s)ds - \int_{t}^{0} Y(t)(P_0+P_+)Y^{-1}(s)f(s)ds \\
&\quad - \int_{0}^{T_2} Y(t)P_+Y^{-1}(s)f(s)ds \\
&= \int_{T_1}^{t} Y(t)P_{-}Y^{-1}(s)f(s)ds + \int_{0}^{t} Y(t)P_0Y^{-1}(s)f(s)ds \\
&\quad - \int_{t}^{T_2} Y(t)P_+Y^{-1}(s)f(s)ds \\
&= x(t),
\end{aligned}$$

- for $t \in (0, T_2]$, we have

$$\begin{aligned}
\int_{T_1}^{T_2} G(t,s)f(s)ds &= \int_{T_1}^{0} Y(t)P_{-}Y^{-1}(s)f(s)ds + \int_{0}^{t} Y(t)(P_0+P_-)Y^{-1}(s)f(s)ds \\
&\quad - \int_{t}^{T_2} Y(t)P_+Y^{-1}(s)f(s)ds \\
&= \int_{T_1}^{t} Y(t)P_{-}Y^{-1}(s)f(s)ds + \int_{0}^{t} Y(t)P_0Y^{-1}(s)f(s)ds \\
&\quad - \int_{t}^{T_2} Y(t)P_+Y^{-1}(s)f(s)ds \\
&= x(t),
\end{aligned}$$

- for $t > T_2$, we have

$$\begin{aligned}
\int_{T_1}^{T_2} G(t,s)f(s)ds &= \int_{T_1}^{0} Y(t)P_{-}Y^{-1}(s)f(s)ds + \int_{0}^{T_2} Y(t)(P_0+P_-)Y^{-1}(s)f(s)ds \\
&= \int_{T_1}^{T_2} Y(t)P_{-}Y^{-1}(s)f(s)ds + \int_{0}^{T_2} Y(t)P_0Y^{-1}(s)f(s)ds \\
&= x(t).
\end{aligned}$$

Now, the inequality (3.2) becomes

$$\sup_{t\in\mathbb{R}} \|\Psi(t)\int_{T_1}^{T_2} G(t,s)f(s)ds\| \le K\int_{T_1}^{T_2} \|\Psi(\mathrm{t})\mathrm{f}(\mathrm{t})\|dt.$$

For a fixed points $s \in \mathbb{R}$, $\delta > 0$ and $\xi \in \mathbb{R}^d$, but arbitrarily, let f the function defined by

$$f(t) = \begin{cases} \Psi^{-1}(t)\xi, & \text{for } s \le t \le s+\delta \\ 0, & \text{elsewhere.} \end{cases}$$

Clearly, $f \in B$, $\|f\|_B = \delta\|\xi\|$. The above inequality becomes

$$\|\int_{s}^{s+\delta} \Psi(t)G(t,u)\Psi^{-1}(u)\xi du\| \le K\delta\|\xi\|, \quad \text{for all t } \in \mathbb{R}.$$

Dividing by δ and letting $\delta \to 0$, we obtain for any $t \ne s$,

$$\|\Psi(t)G(t,s)\Psi^{-1}(s)\xi\| \le K\|\xi\|, \quad \text{for all } t \in \mathbb{R},\ \xi \in \mathbb{R}^d.$$

Hence, $|\Psi(t)G(t,s)\Psi^{-1}(s)| \le K$, which is equivalent to (3.1). By continuity, (3.1) remains valid also in the excepted case $t = s$.

Now, we prove the "if" part. Suppose that the fundamental matrix Y of (2.1) satisfies the condition (3.1) for some $K > 0$. Let $f : \mathbb{R} \to \mathbb{R}^d$ be a Lebesgue Ψ-integrable function on $\mathbb{R}$. We consider the function $u : \mathbb{R} \to \mathbb{R}^d$ defined by

$$\begin{aligned} u(t) = \int_{-\infty}^{t} Y(t)P_- Y^{-1}(s)f(s)ds + \int_0^t Y(t)P_0 Y^{-1}(s)f(s)ds \\ - \int_t^{\infty} Y(t)P_+ Y^{-1}(s)f(s)ds. \end{aligned} \tag{3.3}$$

Step 4. The function u is well-defined on $\mathbb{R}$. Indeed, for $v < t \le 0$, we have

$$\begin{aligned} \int_v^t \|Y(t)P_- Y^{-1}(s)f(s)\|ds &= \int_v^t \|\Psi^{-1}(t)\Psi(t)Y(t)P_- Y^{-1}(s)\Psi^{-1}(s)\Psi(s)f(s)\|ds \\ &\le |\Psi^{-1}(t)| \int_v^t |\Psi(t)Y(t)P_- Y^{-1}(s)\Psi^{-1}(s)| \|\Psi(s)f(s)\|ds \\ &\le K|\Psi^{-1}(t)| \int_v^t \|\Psi(s)f(s)\|ds, \end{aligned}$$

which shows that the integral $\int_{-\infty}^t Y(t)P_- Y^{-1}(s)f(s)ds$ is absolutely convergent. For $t > 0$, we have the same result.

Similarly, the integral $\int_t^{\infty} Y(t)P_+ Y^{-1}(s)f(s)ds$ is absolutely convergent. Thus, the function u is well-defined and is an absolutely continuous function on all intervals $J \subset \mathbb{R}$.

Step 5. The function u is a solution of (1.1). Indeed, for almost all $t \in \mathbb{R}$, we have

$$\begin{aligned} u'(t) &= \int_{-\infty}^t A(t)Y(t)P_- Y^{-1}(s)f(s)ds + Y(t)P_- Y^{-1}(t)f(t) \\ &\quad + \int_0^t A(t)Y(t)P_0 Y^{-1}(s)f(s)ds + Y(t)P_0 Y^{-1}(t)f(t) \\ &\quad - \int_t^{\infty} A(t)Y(t)P_+ Y^{-1}(s)f(s)ds + Y(t)P_+ Y^{-1}(t)f(t) \\ &= A(t)u(t) + Y(t)(P_- + P_0 + P_+)Y^{-1}(t)f(t) = A(t)u(t) + f(t). \end{aligned}$$

This shows that the function u is a solution of (1.1).

Step 6. The solution u is Ψ-bounded on $\mathbb{R}$. Indeed, for $t < 0$, we have

$$\begin{aligned} \Psi(t)u(t) &= \int_{-\infty}^t \Psi(t)Y(t)P_- Y^{-1}(s)\Psi^{-1}(s)\Psi(s)f(s)ds \\ &\quad + \int_0^t \Psi(t)Y(t)P_0 Y^{-1}(s)\Psi^{-1}(s)\Psi(s)f(s)ds \\ &\quad - \int_t^{\infty} \Psi(t)Y(t)P_+ Y^{-1}(s)\Psi^{-1}(s)\Psi(s)f(s)ds \\ &= \int_{-\infty}^t \Psi(t)Y(t)P_- Y^{-1}(s)\Psi^{-1}(s)\Psi(s)f(s)ds \\ &\quad - \int_t^0 \Psi(t)Y(t)(P_0 + P_+)Y^{-1}(s)\Psi^{-1}(s)\Psi(s)f(s)ds \end{aligned}$$

$$- \int_0^{\infty} \Psi(t)Y(t)P_+Y^{-1}(s)\Psi^{-1}(s)\Psi(s)f(s)ds\,.$$

Then

$$\|\Psi(t)u(t)\| \le K \cdot \int_{-\infty}^{\infty} \|\Psi(s)f(s)\|ds.$$

For $t \ge 0$, we have

$$\begin{aligned}
\Psi(t)u(t) &= \int_{-\infty}^{t} \Psi(t)Y(t)P_-Y^{-1}(s)\Psi^{-1}(s)\Psi(s)f(s)ds \\
&\quad + \int_0^t \Psi(t)Y(t)P_0Y^{-1}(s)\Psi^{-1}(s)\Psi(s)f(s)ds \\
&\quad - \int_t^{\infty} \Psi(t)Y(t)P_+Y^{-1}(s)\Psi^{-1}(s)\Psi(s)f(s)ds \\
&= \int_{-\infty}^{0} \Psi(t)Y(t)P_-Y^{-1}(s)\Psi^{-1}(s)\Psi(s)f(s)ds \\
&\quad + \int_0^t \Psi(t)Y(t)(P_0 + P_-)Y^{-1}(s)\Psi^{-1}(s)\Psi(s)f(s)ds \\
&\quad - \int_t^{\infty} \Psi(t)Y(t)P_+Y^{-1}(s)\Psi^{-1}(s)\Psi(s)f(s)ds\,.
\end{aligned}$$

Then

$$\|\Psi(t)u(t)\| \le K \cdot \int_{-\infty}^{\infty} \|\Psi(s)f(s)\|ds.$$

Hence,

$$\sup_{t\in\mathbb{R}} \|\Psi(t)u(t)\| \le K \cdot \int_{-\infty}^{\infty} \|\Psi(s)f(s)\|ds,$$

which shows that the solution u is Ψ-bounded on $\mathbb{R}$. The proof is now complete. □

In a particular case, we have the following result.

Theorem 3.2. *If the homogeneous equation* (2.1) *has no nontrivial* Ψ*-bounded solution on* $\mathbb{R}$*, then the* (1.1) *has a unique* Ψ*-bounded solution on* $\mathbb{R}$ *for every Lebesgue* Ψ*-integrable function* $f : \mathbb{R} \to \mathbb{R}^d$ *on* $\mathbb{R}$ *if and only if there exists a positive constant* K *such that*

$$\begin{aligned}
&|\Psi(t)Y(t)P_-Y^{-1}(s)\Psi^{-1}(s)| \le K \quad \text{for } -\infty < s < t < +\infty \\
&|\Psi(t)Y(t)P_+Y^{-1}(s)\Psi^{-1}(s)| \le K \quad \text{for } -\infty < t \le s < +\infty
\end{aligned} \tag{3.4}$$

In this case, $P_0 = 0$ and the proof is as above.

Next, we prove a theorem in which we will see that the asymptotic behavior of solutions to (1.1) is determined completely by the asymptotic behavior of the fundamental matrix Y.

Theorem 3.3. *Suppose that:*
(1) the fundamental matrix $Y(t)$ *of* (2.1) *satisfies:*
- (a) *condition* (3.1) *is satisfied for some* $K > 0$*;*
- (b) *the following conditions are satisfied:*
  - (i) $\lim_{t\to\pm\infty} |\Psi(t)Y(t)P_0| = 0$*;*
  - (ii) $\lim_{t\to-\infty} |\Psi(t)Y(t)P_+| = 0$*;*
  - (iii) $\lim_{t\to+\infty} |\Psi(t)Y(t)P_-| = 0$*;*

(2) the function $f:\mathbb{R}\to\mathbb{R}^d$ is Lebesgue Ψ-integrable on $\mathbb{R}$.

Then, every Ψ-bounded solution x of (1.1) *is such that*

$$\lim_{t\to\pm\infty}\|\Psi(t)x(t)\|=0.$$

Proof. By Theorem 3.1, for every Lebesgue Ψ-integrable function $f:\mathbb{R}\to\mathbb{R}^d$, the equation (1.1) has at least one Ψ-bounded solution on $\mathbb{R}$.

Let x be a Ψ-bounded solution on $\mathbb{R}$ of (1.1). Let u be defined by (3.3). The function u is a Ψ-bounded solution on $\mathbb{R}$ of (1.1).

Now, let the function $y(t)=x(t)-u(t)-Y(t)P_0(x(0)-u(0))$, $t\in\mathbb{R}$. Obviously, y is a solution on $\mathbb{R}$ of (2.1). Because $\Psi(t)Y(t)P_0$ is bounded on $\mathbb{R}$, y is Ψ-bounded on $\mathbb{R}$. Thus, $y(0)\in X_0$. On the other hand,

$$\begin{aligned} y(0)&=x(0)-u(0)-Y(0)P_0(x(0)-u(0))\\ &=(P_-+P_+)(x(0)-u(0))\in X_-\oplus X_+. \end{aligned}$$

Therefore, $y(0)\in X_0\cap(X_-\oplus X_+)=\{0\}$ and then, $y=0$. It follows that

$$x(t)=Y(t)P_0(x(0)-u(0))+u(t),t\in\mathbb{R}.$$

Now, we prove that $\lim_{t\to\pm\infty}\|\Psi(t)u(t)\|=0$. For $t\geq 0$, we write again

$$\begin{aligned} \Psi(t)u(t)=&\int_{-\infty}^{0}\Psi(t)Y(t)P_-Y^{-1}(s)\Psi^{-1}(s)\Psi(s)f(s)ds\\ &+\int_0^t\Psi(t)Y(t)(P_0+P_-)Y^{-1}(s)\Psi^{-1}(s)\Psi(s)f(s)ds\\ &-\int_t^{\infty}\Psi(t)Y(t)P_+Y^{-1}(s)\Psi^{-1}(s)\Psi(s)f(s)ds. \end{aligned}$$

Let $\varepsilon>0$. From the hypotheses: There exists $t_0<0$ such that

$$\int_{-\infty}^{t_0}\|\Psi(s)f(s)\|ds<\frac{\varepsilon}{5K};$$

there exists $t_1>0$ such that, for all $t\geq t_1$,

$$|\Psi(t)Y(t)P_-|<\frac{\varepsilon}{5}(1+\int_{t_0}^{0}\|Y^{-1}(s)f(s)\|ds)^{-1};$$

there exists $t_2>t_1$ such that, for all $t\geq t_2$,

$$\int_t^{\infty}\|\Psi(s)f(s)\|ds<\frac{\varepsilon}{5K};$$

there exists $t_3>t_2$ such that, for all $t\geq t_3$,

$$|\Psi(t)Y(t)(P_0+P_-)|<\frac{\varepsilon}{5}(1+\int_0^{t_2}\|Y^{-1}(s)f(s)\|ds)^{-1}.$$

Then, for $t\geq t_3$, we have

$$\begin{aligned} &\|\Psi(t)u(t)\|\\ &\leq\int_{-\infty}^{t_0}|\Psi(t)Y(t)P_-Y^{-1}(s)\Psi^{-1}(s)|\|\Psi(s)f(s)\|ds\\ &\quad+\int_{t_0}^{0}|\Psi(t)Y(t)P_-|\|Y^{-1}(s)f(s)\|ds+\int_0^{t_2}|\Psi(t)Y(t)(P_0 \end{aligned}$$

$$
\begin{aligned}
&+ P_-)|\|Y^{-1}(s)f(s)\|ds + \int_{t_2}^{t} |\Psi(t)Y(t)(P_0+P_-)Y^{-1}(s)\Psi^{-1}(s)|\|\Psi(s)f(s)\|ds \\
&+ \int_{t}^{\infty} |\Psi(t)Y(t)P_+Y^{-1}(s)\Psi^{-1}(s)|\|\Psi(s)f(s)\|ds \\
&< K\int_{-\infty}^{t_0} \|\Psi(s)f(s)\|ds + \frac{\varepsilon}{5(1+\int_{t_0}^{0}\|Y^{-1}(s)f(s)\|ds)}\int_{t_0}^{0}\|Y^{-1}(s)f(s)\|ds \\
&+ \frac{\varepsilon}{5(1+\int_{0}^{t_2}\|Y^{-1}(s)f(s)\|ds)}\int_{0}^{t_2}\|Y^{-1}(s)f(s)\|ds \\
&+ K\int_{t_2}^{t}\|\Psi(s)f(s)\|ds + K\int_{t}^{\infty}\|\Psi(s)f(s)\|ds \\
&< K\frac{\varepsilon}{5K} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + K\Big(\int_{t_2}^{t}\|\Psi(s)f(s)\|ds + \int_{t}^{\infty}\|\Psi(s)f(s)\|ds\Big) \\
&< \frac{3\varepsilon}{5} + K\frac{\varepsilon}{5K} < \varepsilon.
\end{aligned}
$$

This shows that $\lim_{t\to+\infty}\|\Psi(t)u(t)\| = 0$.

For $t < 0$, we write again

$$
\begin{aligned}
\Psi(t)u(t) = &\int_{-\infty}^{t} \Psi(t)Y(t)P_-Y^{-1}(s)\Psi^{-1}(s)\Psi(s)f(s)ds \\
&- \int_{t}^{0} \Psi(t)Y(t)(P_0+P_+)Y^{-1}(s)\Psi^{-1}(s)\Psi(s)f(s)ds \\
&- \int_{0}^{\infty} \Psi(t)Y(t)P_+Y^{-1}(s)\Psi^{-1}(s)\Psi(s)f(s)ds.
\end{aligned}
$$

Let $\varepsilon > 0$. From the hypotheses, we have: There exists $t^0 > 0$ such that

$$
\int_{t^0}^{+\infty}\|\Psi(s)f(s)\|ds < \frac{\varepsilon}{5K};
$$

there exists $t_4 < 0$ such that, for all $t < t_4$,

$$
|\Psi(t)Y(t)P_+| < \frac{\varepsilon}{5}\Big(1+\int_{0}^{t^0}\|Y^{-1}(s)f(s)\|ds\Big)^{-1};
$$

there exists $t_5 < t_4$ such that, for all t $\leq t_5$,

$$
\int_{-\infty}^{t}\|\Psi(s)f(s)\|ds < \frac{\varepsilon}{5K};
$$

there exists $t_6 < t_5$ such that, for all $t \leq t_6$,

$$
|\Psi(t)Y(t)(P_0+P_+)| < \frac{\varepsilon}{5}\Big(1+\int_{t_5}^{0}\|Y^{-1}(s)f(s)\|ds\Big)^{-1}.
$$

Then, for $t \leq t_6$, we have

$$
\begin{aligned}
&\|\Psi(t)u(t)\| \\
&\leq \int_{-\infty}^{t}|\Psi(t)Y(t)P_-Y^{-1}(s)\Psi^{-1}(s)|\|\Psi(s)f(s)\|ds \\
&\quad + \int_{t}^{t_5}|\Psi(t)Y(t)(P_0+P_+)Y^{-1}(s)\Psi^{-1}(s)|\|\Psi(s)f(s)\|ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{t_5}^{0}|\Psi(t)Y(t)(P_0+P_+)|\|Y^{-1}(s)f(s)\|ds+\int_{0}^{t^0}|\Psi(t)Y(t)P_+|\|Y^{-1}(s)f(s)\|ds\\
&+\int_{t^0}^{+\infty}|\Psi(t)Y(t)P_+Y^{-1}(s)\Psi^{-1}(s)|\|\Psi(s)f(s)\|ds\\
&< K\int_{-\infty}^{t}\|\Psi(s)f(s)\|ds+K\int_{t}^{t_5}\|\Psi(s)f(s)\|ds\\
&+\frac{\varepsilon}{5(1+\int_{t_5}^{0}\|Y^{-1}(s)f(s)\|ds)}\int_{t_5}^{0}\|Y^{-1}(s)f(s)\|ds\\
&+\frac{\varepsilon}{5(1+\int_{0}^{t^0}\|Y^{-1}(s)f(s)\|ds)}\int_{0}^{t^0}\|Y^{-1}(s)f(s)\|ds+K\int_{t^0}^{+\infty}\|\Psi(s)f(s)\|ds\\
&< K(\int_{-\infty}^{t}\|\Psi(s)f(s)\|ds+\int_{t}^{t_5}\|\Psi(s)f(s)\|ds)+\frac{\varepsilon}{5}+\frac{\varepsilon}{5}+K\frac{\varepsilon}{5K}\\
&< K\frac{\varepsilon}{5K}+\frac{3\varepsilon}{5}<\varepsilon.
\end{aligned}
$$

This shows that $\lim_{t\to-\infty}\|\Psi(t)u(t)\|=0$.

Now, it is easy to see that $\lim_{t\to\pm\infty}\|\Psi(t)x(t)\|=0$. The proof is now complete. □

The next result follows from Theorems 3.2 and 3.3.

Corollary 3.4. *Suppose that*

1. *the homogeneous equation* (2.1) *has no nontrivial* Ψ*-bounded solution on* $\mathbb{R}$*;*
2. *the fundamental matrix* $Y(t)$ *of* (2.1) *satisfies:*
   (i) the condition (3.4) *for some* $K>0$.
   (ii) $\lim_{t\to-\infty}|\Psi(t)Y(t)P_+|=0$*;*
   (iii) $\lim_{t\to+\infty}|\Psi(t)Y(t)P_-|=0$*;*
3. *the function* $f:\mathbb{R}\to\mathbb{R}^d$ *is Lebesgue* Ψ*-integrable on* $\mathbb{R}$.

Then (1.1) *has a unique solution* x *on* $\mathbb{R}$ *such that*

$$\lim_{t\to\pm\infty}\|\Psi(t)x(t)\|=0.$$

Note that Theorem 3.3 is no longer true if we require that the function f be Ψ-bounded on $\mathbb{R}$ (more, even $\lim_{t\to\pm\infty}\|\Psi(t)f(t)\|=0$), instead of the condition (2) in the above the Theorem. This is shown next.

Example. Consider (1.1) with $A(t)=O_2$ and $f(t)=(\sqrt{1+|t|},1)^T$. Then, $Y(t)=I_2$ is a fundamental matrix for (2.1). Consider

$$\Psi(t)=\begin{pmatrix}\frac{1}{1+|t|} & 0\\ 0 & \frac{1}{(1+|t|)^2}\end{pmatrix}.$$

The solutions of (2.1) are $y(t)=(c_1,c_2)^T$, where $c_1,c_2\in\mathbb{R}$. Then

$$\Psi(t)y(t)=(\frac{c_1}{1+|t|},\frac{c_2}{(1+|t|)^2})^T.$$

Therefore, $P_- = O_2$, $P_+ = O_2$ and $P_0 = I_2$. The conditions (3.1) are satisfied with $K = 1$. In addition, the hypothesis (1b) of Theorem 3.3 is satisfied. Because

$$\Psi(t)f(t) = \Big(\frac{1}{\sqrt{1+|t|}}, \frac{1}{(1+|t|)^2}\Big)^T,$$

the function f is not Lebesgue Ψ-integrable on $\mathbb{R}$, but it is Ψ-bounded on $\mathbb{R}$, with $\lim_{t\to\pm\infty} \|\Psi(t)f(t)\| = 0$. The solutions of the system (1.1) are $x(t) = (F(t) + c_1, t + c_2)^T$, where

$$F(t) = \begin{cases} -\frac{2}{3}(1-t)^{3/2} + \frac{4}{3}, & t < 0 \\ \frac{2}{3}(1+t)^{3/2}, & t \geq 0\,. \end{cases}$$

It is easy to see that $\lim_{t\to\pm\infty} \|\Psi(t)x(t)\| = +\infty$, for all $c_1, c_2 \in \mathbb{R}$. It follows that the all solutions of the system (1.1) are Ψ-unbounded on $\mathbb{R}$.

Remark. If in the above example, $f(t) = (\frac{1}{1+|t|}, 0)^T$, then $\int_{-\infty}^{+\infty} \|\Psi(t)f(t)\|dt = 2$. On the other hand, the solutions of (1.1) are $x(t) = (u(t) + c_1, c_2)^T$, where

$$u(t) = \begin{cases} -\ln(1-t), & t < 0 \\ \ln(1+t), & t \geq 0\,. \end{cases}$$

We observe that the asymptotic properties of the components of the solutions are not the same: The first component is unbounded and the second is bounded on $\mathbb{R}$. However, all solutions of (1.1) are Ψ-bounded on $\mathbb{R}$ and $\lim_{t\to\pm\infty} \|\Psi(t)x(t)\| = 0$. This shows that the asymptotic properties of the components of the solutions are the same, via the matrix function Ψ. This is obtained by using a matrix function Ψ rather than a scalar function.

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Aurel Diamandescu
University of Craiova, Department of Applied Mathematics, 13, "Al. I. Cuza" st., 200585 Craiova, Romania
E-mail address: adiamandescu@central.ucv.ro